PROBLEM SET 7 (DUE ON NOV 2)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Chapter 8, Exercise 3, implication (i) $\implies$ (ii). (Artinian f.g. k-algebras are finite k-algebras - you've already done the other implication.)

Problem 2. Let A be a finite ring (i.e. a ring with finitely many elements). Show that A is isomorphic to a product of rings, each of which has a prime power number of elements (i.e. p^n for prime p and $n \geq 1$).

Problem 3. Let $A = \mathbb{C}[x, y]$ and $X = \mathbb{C}^2$. For each $n \geq 0$, define

$$X^{[n]} := \{I \subseteq A \mid I \text{ is an ideal and } \dim_{\mathbb{C}}(A/I) = n\}.$$

(Here $\dim_{\mathbb{C}}$ means dimension as a $\mathbb{C}$-vector space. We say that such ideals I have *colength* n. The set $X^{[n]}$ is sometimes called the *Hilbert scheme of points* of X.)

Also let

$$S^n(X) := X^n/S_n = \{\text{multisets of size } n \text{ consisting of elements of } X\}.$$

(Here S_n is the symmetric group on n elements, acting on the n factors of the cartesian product X^n.)

Show that there exist functions

$$\phi_n : X^{[n]} \to S^n(X)$$

for each $n \geq 0$ such that

(a) ϕ_n is surjective for each $n \geq 0$.
(b) If I and J are ideals of finite colengths m and n respectively and $I \subseteq J$, then $\phi_n(J)$ is a sub-multiset of $\phi_m(I)$.
(c) If I and J are ideals of finite colengths m and n respectively, then $I \cap J$ has colength $m + n$ if and only if $\phi_m(I)$ and $\phi_n(J)$ are disjoint, and in this case $\phi_{m+n}(I \cap J) = \phi_m(I) \cup \phi_n(J)$.

Problem 4. Let k be a field. Describe all discrete valuations on the field $k(x)$ satisfying $v(f) \geq 0$ for all polynomials $f \in k[x]$ and show that the corresponding DVRs are localizations of $k[x]$ at maximal ideals.

Problem 5. Give an example of a discrete valuation on the field $\mathbb{C}(x, y)$ such that

(a) $v(f) \geq 0$ for all polynomials $f \in \mathbb{C}[x, y]$;
(b) the corresponding DVR is not equal to a localization of $\mathbb{C}[x, y]$ (both viewed as subrings of $\mathbb{C}(x, y)$).

(Hint: look for a discrete valuation such that $v(x) = v(y) = 1$.)