PROBLEM SET 9 (DUE ON NOV 16)

(All Exercises are references to *Introduction to Commutative Algebra* by M. Atiyah and I. Macdonald.)

Problem 1. Let $A = \mathbb{C}[t^2, t^3]$. Although A is not a Dedekind domain, we still let $\mathrm{Pic}(A)$ denote the group of invertible fractional ideals modulo principal fractional ideals, or equivalently the group of isomorphism classes of invertible A-modules. Prove that $\mathrm{Pic}(A)$ contains a subgroup isomorphic to $\mathbb{C}$ and hence is infinite. (Hint: consider the subgroup of $\mathrm{Pic}(A)$ generated by the maximal ideals $(t^2 - a^2, t^3 - a^3)$ for $a \in \mathbb{C}^*$ (you will have to check that these are invertible fractional ideals). With a little more work you can show that this subgroup is actually the full Picard group, so $\mathrm{Pic}(A) \cong \mathbb{C}$, but you aren't required to do this.)

Problem 2. Let A be a ring, complete with respect to an ideal $I \subseteq A$. (Recall that this means that the map $A \to \varprojlim A/I^i$ is an isomorphism.) Let M be an A-module satisfying $\cap_{i \geq 0} I^i M = 0$. Suppose that $x_1, \ldots, x_n \in M$ are elements whose images generate the module M/IM. Prove that $x_1, \ldots, x_n$ generate M.

Problem 3. Let A be a ring, complete with respect to a maximal ideal $\mathfrak{m} \subset A$. Prove that A is a local ring.

Problem 4. Chapter 10, Exercise 9 (Hensel's lemma, version 1. The hint given in Atiyah-Macdonald has a couple mistakes, so I'll modify it a bit here: Hint: The plan is to construct monic polynomials (for each k) $g_k, h_k \in A[x]$ of degrees $r, n-r$ such that $g_k h_k - f \in \mathfrak{m}^k A[x]$, and then use the sequences (g_k), (h_k) to construct g, h (using the fact that A is complete). For the inductive step, for each $0 \leq p < n$ we want to pick $a_p, b_p \in A[x]$ of degrees $< n-r, r$ respectively such that $a_p g_k + b_p h_k - x^p \in \mathfrak{m}A[x]$, and then construct g_{k+1}, h_{k+1} by using those a_p, b_p to modify g_k, h_k appropriately.)

Problem 5. Chapter 10, Exercise 10, part (i) (Hensel's lemma, version 2. This is the more commonly seen version. Here "simple root" means that $f(a) = 0$ but $f'(a) \neq 0$, where f' is the derivative of the polynomial f.)