THE DOUBLE RAMIFICATION CYCLE FORMULA

A. PIXTON

Abstract. The double ramification cycle $\mathrm{DR}_g(A) = \mathrm{DR}_g(\mu, \nu)$ is a cycle in the moduli space of stable curves parametrizing genus g curves admitting a map to $\mathbb{P}^1$ with specified ramification profiles μ, ν over two points. In 2016, Janda, Pandharipande, Zvonkine, and the author proved a formula expressing the double ramification cycle in terms of basic tautological classes, answering a question of Eliashberg from 2001. This formula has an intricate combinatorial shape involving an unusual way to sum divergent series using polynomial interpolation. Here we give some motivation for where this formula came from, relating it both to an older partial formula of Hain and to Givental's R-matrix action on cohomological field theories.

0. Introduction

Let g, n be nonnegative integers satisfying $2g - 2 + n > 0$, so that the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n markings is nonempty. Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a vector of n integers satisfying $a_1 + \cdots + a_n = 0$. In this paper we will be interested in a Chow cycle class

$$\mathrm{DR}_g(A) \in A^g(\overline{\mathcal{M}}_{g,n})$$

that depends on this data.

There are two main perspectives on how to think about and define $\mathrm{DR}_g(A)$, the *double ramification cycle*. The first is the source of its name; we think of it as parametrizing the genus g curves C that admit a finite map $C \to \mathbb{P}^1$ with specified ramification profiles μ, ν over two points (say 0 and ∞). These two ramification profiles are encoded in the vector A: we can take the positive and negative components of A separately to get two partitions of equal size. The marked points with nonzero a_i should then be the points in C lying above 0 and ∞, while the marked points with $a_i = 0$ are unconstrained. Ramification above points other than 0 and ∞ is unconstrained.

The above description defines a *double ramification locus* inside the moduli space of smooth curves $\mathcal{M}_{g,n}$ that is usually (but not always) of

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pure codimension g. To extend this to a codimension g class on $\overline{\mathcal{M}}_{g,n}$, we can use the virtual class in relative Gromov-Witten theory. There is a moduli space of stable (rubber) maps to $\mathbb{P}^1$ with given marked ramification over two points, $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/\{0,\infty\},\mu,\nu)^\sim$, equipped with a forgetful map $p : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/\{0,\infty\},\mu,\nu)^\sim \to \overline{\mathcal{M}}_{g,n}$, and the double ramification cycle can be taken to be the pushforward under this map of the virtual class:

$$\mathrm{DR}_g(A) = p_*[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1/\{0,\infty\},\mu,\nu)^\sim]^{\mathrm{vir}} \in A^g(\overline{\mathcal{M}}_{g,n}).$$

The second perspective on $\mathrm{DR}_g(A)$ is via Abel-Jacobi maps. Let $\mathcal{X}_g \to \mathcal{A}_g$ be the universal abelian variety of dimension g. Then the data in the vector A can be used to define a morphism $j_A : \mathcal{M}_{g,n} \to \mathcal{X}_g$ by

$$(C, p_1, \ldots, p_n) \mapsto (\mathrm{Jac}(C), \mathcal{O}_C(a_1p_1 + \cdots + a_np_n)).$$

The double ramification locus is then the inverse image under this map of the zero section $\mathcal{Z}_g$ of $\mathcal{X}_g$, since C admits a map to $\mathbb{P}^1$ with the given ramification profiles if and only if $\mathcal{O}_C(a_1p_1 + \cdots + a_np_n)$ is trivial.

This Abel-Jacobi map extends easily to $\mathcal{M}^{ct}_{g,n}$, the moduli space of curves of compact type (those with compact Jacobians), but using this perspective to define the double ramification cycle on all of $\overline{\mathcal{M}}_{g,n}$ requires more work. It also isn't obvious that constructing $\mathrm{DR}_g(A)$ in this way will give the same class as that given by relative Gromov-Witten theory, even after restriction to $\mathcal{M}^{ct}_{g,n}$. For one approach to these questions using logarithmic and tropical geometry, see the work of Marcus and Wise [13].

Eliashberg proposed the problem of giving a formula for the double ramification cycle in 2001, in the context of symplectic field theory. This problem was solved by Janda, Pandharipande, Zvonkine, and the author in 2016 [11], giving an explicit combinatorial formula for the double ramification cycle. This formula has an unexpected form – an additional integer parameter $r > 0$ is introduced, then an expression is written down that becomes polynomial in r for r sufficiently large, and finally this polynomial is specialized to $r = 0$. Subsequent papers extending or generalizing the double ramification cycle formula in various ways (e.g. [5, 12, 2]) have left the basic combinatorial structure of the formula virtually unchanged. The purpose of this paper is to discuss this structure and give some motivation for where it comes from.

In Section 1, we review the tautological classes in the Chow ring of the moduli space of stable curves. In Section 2, we discuss results leading up to the formula of [11], most notably Hain's formula for the compact type double ramification cycle. Section 3 is the heart of the paper and consists of an extended discussion motivating the shape of the double ramification cycle formula. We conclude in Section 4 by stating the formula and briefly explaining how its proof in [11] is related to some of the motivation in Section 3.

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1. Tautological classes

1.1. **Preliminaries.** In this section we review the language in which the double ramification cycle formula is written. This is the language of the tautological ring, a subring $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq A^*(\overline{\mathcal{M}}_{g,n})$ containing most classes that arise naturally in geometry.

Following Faber and Pandharipande [6], the tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ can be defined simultaneously for all $g, n \geq 0$ satisfying $2g-2+n > 0$ as the smallest subrings of the Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ closed under pushforward by forgetful maps $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ and gluing maps $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ or $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$. Our discussions of tautological classes will use a more explicit description of them. Graber and Pandharipande [8, Appendix A] gave a set of additive generators and a multiplication law satisfied by these generators.

These additive generators are formed from three ingredients: psi classes, kappa classes, and generalized gluing maps corresponding to stable graphs. The psi classes $\psi_i \in A^1(\overline{\mathcal{M}}_{g,n}), i = 1, \ldots, n$ correspond to the n marked points and are defined as the first Chern classes of the cotangent line bundles to the curves at those points. The Arbarello-Cornalba [1] kappa classes are then the pushforwards of powers of psi classes:

$$\kappa_a := \pi_*(\psi_{n+1}^{a+1}) \in A^a(\overline{\mathcal{M}}_{g,n}),$$

where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgets the last marking. The kappa classes will not appear in any of the formulas in this paper.

The tautological ring of the moduli space of smooth marked curves, $R^*(\mathcal{M}_{g,n})$, is the ring generated by these ψ_i and κ_a. To extend this to $R^*(\overline{\mathcal{M}}_{g,n})$ we need classes supported on boundary strata.

1.2. **Stable graphs.** A *stable graph* Γ is the combinatorial data of a boundary stratum in $\overline{\mathcal{M}}_{g,n}$. It consists of the following:

(1) a set of vertices $V(\Gamma)$;

(2) a genus $g_v \geq 0$ at each vertex $v \in V(\Gamma)$;
(3) a set of half-edges $H(\Gamma)$;
(4) an incidence map $H(\Gamma) \to V(\Gamma)$;
(5) a partition of $H(\Gamma)$ into sets of size 2 (called edges, the set of which is denoted $E(\Gamma)$) and sets of size 1 (called legs);
(6) a bijection between the set of legs and $\{1, \ldots, n\}$.

The underlying graph is required to be connected. The stability constraint is that $2g_v - 2 + n_v > 0$ at each vertex v, where n_v is the number of half-edges incident to v. The genera are constrained by the identity

$$2g - 2 + n = \sum_{v \in V(\Gamma)} (2g_v - 2 + n_v),$$

or equivalently that $g - \sum_v g_v = h^1(\Gamma)$, the first Betti number of the graph. Such a stable graph Γ corresponds to a generalized gluing map

$$\xi_\Gamma : \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g_v, n_v} \to \overline{\mathcal{M}}_{g,n}.$$

We can then consider classes

$$\xi_{\Gamma *}(\alpha) \in A^*(\overline{\mathcal{M}}_{g,n}),$$

where Γ is a stable graph and α is a monomial in the psi and kappa classes on the $\overline{\mathcal{M}}_{g_v,n_v}$ factors. These are the additive generators for the tautological ring considered in [8].

1.3. **Compact type.** The moduli space of curves of compact type, denoted $\mathcal{M}^{ct}_{g,n}$, is the open subscheme of $\overline{\mathcal{M}}_{g,n}$ consisting of those curves whose dual graph is a tree. Its tautological ring $R^*(\mathcal{M}^{ct}_{g,n})$ is the image of $R^*(\overline{\mathcal{M}}_{g,n})$ under restriction, so it is additively generated by classes $\xi_{\Gamma *}(\alpha)$ as above where Γ is a tree.

It will be convenient for us to have notation for the compact type boundary divisor classes when stating Hain's formula below, (2). If Γ is a stable graph with 2 vertices and 1 edge and one of the vertices is genus h and has those legs with markings in a set $S \subseteq 1, 2, \ldots, n$, let $\delta_{h,S} = \xi_{\Gamma *}(1)$ be the corresponding boundary divisor class.

2. Previous formulas and results

The first progress towards a formula for the double ramification cycle was when Faber and Pandharipande [7] proved that the double ramification cycle lies in the tautological ring, and thus in theory must be expressible in terms of the generators described in the previous section. Their proof, although in principle constructive, involves a complicated recursion and doesn't seem to yield a practical formula.

The first progress towards an explicit formula came when Hain [10] computed the double ramification cycle when restricted to the compact type locus $\mathcal{M}_{g,n}^{ct}$. On this locus the double ramification cycle is the pullback along an Abel-Jacobi map $j_A : \mathcal{M}_{g,n}^{ct} \to \mathcal{X}_{g,n}$ of the class of the zero section $\mathcal{Z}_{g,n}$ of the universal abelian variety $\mathcal{X}_{g,n} \to \mathcal{A}_{g,n}$. Hain showed that the class of this zero section is $[\mathcal{Z}_{g,n}] = \Theta^g/g!$ and computed the pullback of the theta divisor Θ as an explicit divisor on $\mathcal{M}_{g,n}^{ct}$:

$$j_A^*\Theta = \sum_{i=1}^{n} \frac{a_i^2}{2}\psi_i - \sum_{h,S} \frac{a_S^2}{4}\delta_{h,S}, \tag{1}$$

where $a_S = \sum_{i\in S} a_i$ and the second sum runs over all h, S defining boundary divisor classes.

Hain's formula for the compact type double ramification cycle is then

$$\mathrm{DR}_g^{\mathrm{ct}}(A) = \frac{1}{g!}(j_A^*\Theta)^g = \frac{1}{g!}\left(\sum_{i=1}^{n} \frac{a_i^2}{2}\psi_i - \sum_{h,S} \frac{a_S^2}{4}\delta_{h,S}\right)^g. \tag{2}$$

The divisor formula (1) is a homogeneous polynomial of degree 2 in A, so Hain's DR formula (2) is a homogeneous polynomial of degree $2g$ in A.

Grushevsky and Zakharov [9] extended Hain's computation slightly, expanding from $\mathcal{M}_{g,n}^{ct}$ to a slightly larger open subscheme of $\overline{\mathcal{M}}_{g,n}$ by adding the locus of curves whose dual graph is a tree with a single loop added at one vertex. If Γ is the stable graph with a single vertex and single loop, then their correction term is the codimension g part of

$$\xi_{\Gamma*}\left(-\prod_{i=1}^{n} \exp\left(\frac{1}{2}a_i^2\psi_i\right)\sum_{k=1}^{\infty} \frac{B_{2k}}{2^k k!}(\psi+\psi')^{k-1}\right), \tag{3}$$

where $\psi_1, \ldots, \psi_n$ are the psi classes on the legs, ψ, ψ' are the psi classes on the two half-edges of the loop, and B_{2k} is a Bernoulli number.

In particular, the double ramification cycle is no longer a homogeneous polynomial in A when computed beyond compact type. This was also seen in work of Buryak, Shadrin, Spitz, and Zvonkine [3], who showed that the top degree intersections of double ramification cycles with monomials in the psi classes are inhomogeneous polynomials of degree $2g$ in A.

3. Motivation for the formula

In this section we discuss various observations and ideas that come about when one tries to extend Hain's formula (2) to $\overline{\mathcal{M}}_{g,n}$ to obtain a full double ramification cycle formula.

3.1. **Expanding Hain's formula.** Exponentiating a boundary divisor class can be done using the multiplication laws for tautological classes [8, Appendix A]. Multiplying out Hain's formula (2) in this way gives a nice sum over trees: $\mathrm{DR}_g^{\mathrm{ct}}(A)$ is the codimension g part of

(4)
$$\sum_{T \text{ stable tree}} \frac{1}{|\operatorname{Aut}(T)|}(\xi_T)_*\Bigg[\prod_{i=1}^{n}\exp\left(\frac{1}{2}a_i^2\psi_{h_i}\right) \cdot \prod_{e=\{h,h'\}\in E(T)} \frac{1-\exp(-\frac{1}{2}w(h)w(h')(\psi_h+\psi_{h'}))}{\psi_h+\psi_{h'}}\Bigg],$$

where the function $w : H(T) \to \mathbb{Z}$ is defined here by contracting all the edges in the tree T other than the one containing h and then letting $w(h)$ be the sum of the a_i for the legs i on the same vertex as the half-edge h.

Extending this formula to $\overline{\mathcal{M}}_{g,n}$ requires us to provide a polynomial (or power series) in the psi classes for every stable graph Γ, not just every stable tree. The $w(h)$ definition above does not naturally extend to non-separating edges, so it isn't immediately clear how to do this. Moreover, we know that this power series needs to be (3) for the single-loop graph, so something quite new is going on even there.

3.2. **Cohomological field theory axioms.** A cohomological field theory (CohFT) is a collection of classes $\Omega_{g,n}(\gamma_1,\ldots,\gamma_n)$ on $\overline{\mathcal{M}}_{g,n}$ for all g and n, where the inputs γ_i belong to some finite set S (a basis for the state space of the CohFT). These classes must satisfy certain compatibility axioms relating them to each other under pullback by natural maps between the $\overline{\mathcal{M}}_{g,n}$. For one basic treatment of CohFTs and Givental's R-matrix action, see [14]. The double ramification cycle is not quite a CohFT, but it satisfies some subset of the properties of one. For example, if $j : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$ is a separating gluing map where the marked points split into sets I_1, I_2 with $|I_i| = n_i$, then we have

$$j^*\mathrm{DR}_g(a_1,\ldots,a_n) = \mathrm{DR}_{g_1}(\{a_i \mid i\in I_1\}, t)\otimes \mathrm{DR}_{g_2}(\{a_i \mid i \in I_2\}, -t),$$

where $t \in \mathbb{Z}$ is the unique insertion that makes the parameters sum to 0 in each DR term on the right.

If the double ramification cycle were a CohFT, we'd want a similar formula for the pullback along the non-separating gluing map $k : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$: the natural thing to write down would be

$$k^* \mathrm{DR}_g(a_1, \ldots, a_n) = \sum_{t\in\mathbb{Z}} \mathrm{DR}_{g-1}(a_1, \ldots, a_n, t, -t),$$

but it isn't clear how one might make sense of this infinite sum – it won't converge in any standard sense. What is going wrong here is that CohFTs are supposed to depend multilinearly on parameters from a finite-dimensional state space, but double ramification cycles take inputs in $\mathbb{Z}$ so the state space appears to be infinite-dimensional.

So the double ramification cycle behaves like a CohFT as far as separating nodes are concerned, but the state space would have to be infinite-dimensional and this makes it unclear what to do at non-separating nodes.

3.3. **Givental's R-matrix action.** Teleman [16] proved that semisimple CohFTs all have a very particular graph sum form, given by applying Givental's R-matrix action to a CohFT that lives fully in codimension zero. The rough shape of the resulting formula for a semisimple CohFT is

$$\Omega_{g,n}(\gamma_1, \ldots, \gamma_n) = \sum_{\Gamma \text{ stable graph}} \sum_{w:H(\Gamma)\to S} \frac{1}{|\operatorname{Aut}(\Gamma)|}(\xi_\Gamma)_* \Bigg[\prod_{v\in V(\Gamma)} (\textbf{vertex factor}) \prod_{i=1}^{n} (\textbf{leg factor}) \prod_{e=\{h,h'\}\in E(\Gamma)} (\textbf{edge factor}) \Bigg],$$

where the second sum is over functions w on the half-edges of the graph taking values in some set S (a basis for the state space of the CohFT) and the values of w on the legs $h_1, \ldots, h_n$ are given: $w(h_i) = \gamma_i$. The various factors are then power series (that depend on w) in the corresponding kappa and psi classes. The expanded version of Hain's compact type formula (4) is of this shape: we take $S = \mathbb{Z}$, the vertex factor is 0 unless all of the incident $w(h)$ sum to zero, and the edge factor is 0 unless the two $w(h)$ along the edge sum to zero. These vanishings effectively place the following constraints on w (to get a nonzero contribution to $\mathrm{DR}_g^{\mathrm{ct}}(A)$):

(1) $w(h_i) = a_i$ for $i = 1, 2, \ldots, n$, where h_i is the ith leg;
(2) $w(h) + w(h') = 0$ if $\{h, h'\}$ is an edge;
(3) $\sum_{h\to v} w(h) = 0$ for each vertex v.

We say w is *balanced* (with respect to A) if it satisfies these constraints. In other words, w is a flow on Γ with sources/sinks at the legs (with specified values given there by A). When Γ is a tree, there is a unique such balanced w and we recover the $w(h)$ used in (4).

From this perspective it is natural to just try to take (4) and extend it to be a Givental-type sum over arbitrary graphs (not just trees), but then there will be infinitely many choices of w and the resulting infinite sums will be nonconvergent. Moreover, careful comparison with the exact form of Givental's R-matrix action suggests that the vertex factor should contribute a total factor of something like "$|\mathbb{Z}|^{-h^1(\Gamma)}$". Note that the set of balanced w is a torsor over $H_1(\Gamma;\mathbb{Z}) \cong \mathbb{Z}^{h^1(\Gamma)}$, so this factor feels like some sort of infinite averaging procedure.

3.4. **Divergent averages.** Returning to the simplest non-tree case, the graph with one vertex and one loop, matching things up with (3) would then require making sense of the "infinite average" identity

$$\frac{1}{|\mathbb{Z}|}\sum_{c\in\mathbb{Z}} c^{2k} = B_{2k}. \tag{5}$$

This is reminiscent of the zeta regularization sum

$$\sum_{c\geq 1} c^{2k-1} = \zeta(1-2k) = -\frac{B_{2k}}{2k},$$

but there is no obvious way to make sense of this similarity. Moreover, more complicated graphs require much more complicated divergent sums; for example, a graph with two vertices, a double edge between them, and one leg on each vertex gives rise to infinite sums like

$$\frac{1}{|\mathbb{Z}|}\sum_{c+d=a} c^{2k}d^{2l} \tag{6}$$

which must be interpreted.

3.5. **Interpolating finite rank CohFTs.** The problem with writing down a double ramification cycle formula of this type is clearly that the state space is infinite-dimensional. If we replace $\mathbb{Z}$ with $\mathbb{Z}/r\mathbb{Z}$ everywhere then there is no difficulty with writing down a similar-looking finite rank CohFT. The result might be something like the following (the case of a diagonal R-matrix – for an example of a more complicated

CohFT of this general type, see [15]):

$$\sum_{\Gamma \text{ stable graph}} \sum_{\substack{w:H(\Gamma)\to\mathbb{Z}/r\mathbb{Z}\\ \text{balanced}}} \frac{1}{|\operatorname{Aut}(\Gamma)|}(\xi_\Gamma)_*\Bigg[\frac{1}{r^{h^1(\Gamma)}}\prod_{i=1}^n \exp(F_{w(h_i)}(\psi_{h_i}))$$

$$\cdot \prod_{e=\{h,h'\}\in E(\Gamma)} \frac{1-\exp(F_{w(h)}(\psi_h)+F_{w(h')}(\psi_{h'}))}{\psi_h+\psi_{h'}}\Bigg],$$

for power series $F_a(Z)$ for $a \in \mathbb{Z}/r\mathbb{Z}$ with $F_0(Z)=0$ and $F_{-a}(-Z) = -F_a(Z)$.

If we take $F_a(Z) = \frac{1}{2}a^2 Z$ for $-\frac{r}{2} < a \le \frac{r}{2}$ then this CohFT starts to look very much like the expanded version of Hain's formula, (4). In fact, if Γ is a tree then the Γ-term in this sum agrees with that in Hain's formula for all sufficiently large r. So it is tempting to try to take the limit as $r \to \infty$ of these CohFTs. But this isn't quite right: the r-version of the left side of (5) is then

$$\frac{1}{r}\sum_{-\frac{r}{2}<c\le\frac{r}{2}} c^{2k}.$$

This certainly doesn't converge as $r \to \infty$. However, if we restrict to even r then it is polynomial in r, and if we examine the coefficients of this polynomial then we see that B_{2k}, the desired value, is the constant coefficient in r.

This suggests a potential interpretation even of more complicated sums like (6):

$$\frac{1}{|\mathbb{Z}|}\sum_{c+d=a} c^{2k}d^{2l} = \frac{1}{|\mathbb{Z}/0\mathbb{Z}|}\sum_{\substack{c,d\in\mathbb{Z}/0\mathbb{Z}\\ c+d=a \pmod 0}} c^{2k}d^{2l}$$

$$:= \Bigg[\frac{1}{|\mathbb{Z}/r\mathbb{Z}|}\sum_{\substack{c,d\in\mathbb{Z}/r\mathbb{Z}\\ c+d=a \pmod r}} c^{2k}d^{2l}\Bigg]_{r=0},$$

where c and d must be interpreted inside $c^{2k}d^{2l}$ as elements of $\mathbb{Z}$ via some choice of mod r representatives (we used $-r/2+1,...,r/2$ before but $0,\ldots,r-1$ will give the same final answer) and setting $r=0$ at the end is done by polynomial interpolation.

3.6. **Geometric interpretation from (k/r)-spin structures.** An (k/r)-spin structure on a smooth curve C with marked points p_i and weights a_i is a choice of line bundle L on C such that $L^{\otimes r} \equiv \omega_C^{\otimes k}(a_1p_1 + \cdots + a_np_n)$. If we take $k=0$ and assume the weights a_i sum to zero then for any positive r any smooth curve will have such "rth root structures." But we can also interpret this construction as meaningful when $r = 0$, when we get that a curve only admits a $(0/0)$-spin structure if it is in the double ramification locus. This observation gives a vague geometric idea for what it might mean to think of the double ramification cycle as given by specializing some parameter r to 0.

4. The double ramification cycle formula

We can now state the main result of [11], the double ramification cycle formula:

Theorem 1 ([11]). $\mathrm{DR}_g(A)$ *is the codimension g part of*

$$\sum_{\Gamma \text{ stable graph}} \frac{1}{|\operatorname{Aut}(\Gamma)|}(\xi_\Gamma)_*\left[\frac{1}{|\mathbb{Z}|^{h^1(\Gamma)}} \sum_{\substack{w:H(\Gamma)\to\mathbb{Z}\\ \text{balanced}}} \prod_{i=1}^{n} \exp\left(\frac{1}{2}a_i^2\psi_{h_i}\right)\right.$$
$$\left.\cdot \prod_{e=\{h,h'\}\in E(\Gamma)} \frac{1-\exp(-\frac{1}{2}w(h)w(h')(\psi_h+\psi_{h'}))}{\psi_h+\psi_{h'}}\right],$$

where formal expressions of the form

$$\frac{1}{|\mathbb{Z}|^{h^1(\Gamma)}} \sum_{\substack{w:H(\Gamma)\to\mathbb{Z}\\ \text{balanced}}} P(\{w(h)\})$$

(for P a polynomial) are evaluated by setting $r = 0$ in the corresponding r-polynomial

$$\frac{1}{r^{h^1(\Gamma)}} \sum_{\substack{w:H(\Gamma)\to\{0,1,\dots,r-1\}\\ \text{balanced mod } r}} P(\{w(h)\}).$$

The combinatorial result (necessary for this theorem statement to make sense) that the expression in the final line is in fact a polynomial in r (for r sufficiently large) was proved in [11, Appendix A].

The proof of Theorem 1 in [11] follows some of the motivation in Section 3. We first explain the meaning of the additional r parameter. For each $r > 0$, let $\mathbb{P}^1[r]$ denote the projective line with a $B\mathbb{Z}_r$ orbifold point at 0. One can then use $\mathbb{C}^*$-localization on the moduli space of relative stable maps to $\mathbb{P}^1[r]/\{\infty\}$ to obtain complicated relations that entangle double ramification cycles, classes coming from moduli of $(0/r)$-spin curves (discussed briefly in the case of smooth curves in Section 3.6), and other basic tautological classes. The relevant $(0/r)$-spin classes were previously computed by Chiodo [4] using Grothendieck-Riemann-Roch.

These localization relations are too difficult to study effectively for specific values of r, but it turns out that they have polynomial dependence on r. Taking the constant term in r simplifies them greatly: most of the terms vanish, and the only remaining terms are a single double ramification cycle and the $r = 0$ interpolation of certain classes written in terms of the Chern characters of the pushforward of the universal rth root line bundle on the moduli space of $(0/r)$-spin curves. Chiodo's formula [4] for these Chern characters gives that these classes are CohFTs with formulas of the type described in Section 3.5. The power series in psi classes appearing in these formulas do not look exactly like those appearing in Theorem 1, but they have the same $r = 0$ interpolation. (In the language of Section 3.5, the power series $F_a(Z)$ will be congruent to $\frac{1}{2}a^2Z$ mod r.) The result is a proof of Theorem 1.

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Department of Mathematics
University of Michigan
pixton@umich.edu