Logarithmic double ramification cycles

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Abstract

Let $A = (a_1, \ldots, a_n)$ be a vector of integers which sum to $k(2g-2+n)$. The double ramification cycle $\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$ on the moduli space of curves is the virtual class of an Abel-Jacobi locus of pointed curves $(C, x_1, \ldots, x_n)$ satisfying

$$\mathcal{O}_C\Big(\sum_{i=1}^{n} a_i x_i\Big) \simeq \big(\omega_C^{\mathsf{log}}\big)^k .$$

The Abel-Jacobi construction requires log blow-ups of $\overline{\mathcal{M}}_{g,n}$ to resolve the indeterminacies of the Abel-Jacobi map. Holmes [34] has shown that $\mathsf{DR}_{g,A}$ admits a canonical lift $\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ to the logarithmic Chow ring, which is the limit of the intersection theories of all such blow-ups.

The main result of the paper is an explicit formula for $\mathsf{logDR}_{g,A}$ which lifts Pixton's formula for $\mathsf{DR}_{g,A}$. The central idea is to study the universal Jacobian over the moduli space of curves (following Caporaso [14], Kass-Pagani [42], and Abreu-Pacini [3]) for certain stability conditions. Using the criterion of Holmes-Schwarz [39], the universal double ramification theory of Bae-Holmes-Pandharipande-Schmitt-Schwarz [5] applied to the universal line bundle determines the logarithmic double ramification cycle. The resulting formula, written in the language of piecewise polynomials, depends upon the stability condition (and admits a wall-crossing study). Several examples of logarithmic and higher double ramification cycles are computed.

Contents

1 Introduction

1.1 Double ramification cycles

Let $\mathcal{M}_{g,n}$ be the moduli space of nonsingular curves of genus g with n distinct marked points over $\mathbb{C}$. Given a vector of integers $A = (a_1, \ldots, a_n)$ satisfying

$$\sum_{i=1}^{n} a_i = 0\,,$$

we can define a substack of $\mathcal{M}_{g,n}$ by

$$\left\{ (C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n} \,\middle|\, \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i x_i\Big) \simeq \mathcal{O}_C \right\}. \tag{1}$$

From the point of view of relative Gromov-Witten theory, the most natural compactification of the substack (1) is the space $\overline{\mathcal{M}}^{\sim}_{g,A}$ of stable maps to *rubber*

[33, 46, 47]: stable maps to $\mathbb{CP}^1$ relative to 0 and ∞ modulo the $\mathbb{C}^*$-action on $\mathbb{CP}^1$.

The rubber moduli space carries a natural virtual fundamental class $\left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\text{vir}}$ of dimension $2g-3+n$. The pushforward via the canonical morphism

$$\epsilon : \overline{\mathcal{M}}^{\sim}_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

is the *double ramification cycle* on the moduli space of stable curves,

$$\epsilon_* \left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\text{vir}} = \mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,. \tag{2}$$

The double ramification cycle $\mathsf{DR}_{g,A}$ can also be defined via log stable maps (and was motivated in part by Symplectic Field Theory [25]).

The classical approach to the locus (1) in $\mathcal{M}_{g,n}$ is via Abel-Jacobi theory for the universal curve. However, the Abel-Jacobi map does not extend over the boundary

$$\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$$

of the moduli space of stable curves. Approaches by Marcus-Wise [48] and Holmes [34], motivated by log geometry, provide a partial resolution of the Abel-Jacobi map which is sufficient to define a double ramification cycle.

We fix for the remainder of the paper an integer k and a vector of integers $A = (a_1, \ldots, a_n)$ satisfying

$$\sum_{i=1}^{n} a_i = k(2g-2+n).$$

The Abel-Jacobi construction in fact yields a more general *k-twisted double ramification cycle* associated to the vector A,

$$\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,, \tag{3}$$

and related to the substack

$$\left\{(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n} \,\middle|\, \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i x_i\Big) \simeq (\omega_C^{\log})^{\otimes k}\right\},$$

where $\omega_C^{\log} = \omega_C\left(\sum_{i=1}^{n} x_i\right)$ is the *log canonical line bundle* of C. The cycle (3) agrees with definition (2) in the $k=0$ case.

Eliashberg posed the question of computing $\mathsf{DR}_{g,A}$ in 2001. A complete formula for $\mathsf{DR}_{g,A}$ in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ was conjectured by Pixton in 2014 and proven in [40] for $k=0$ and in [5] for general k. Pixton's formula expresses $\mathsf{DR}_{g,A}$ directly as a sum over stable graphs Γ indexing the boundary strata of $\overline{\mathcal{M}}_{g,n}$. The contribution of each stable graph Γ is the constant term of a polynomial naturally associated to the combinatorics of Γ and A.

We refer the reader to [5, Section 0], [40, Section 0], and [60, Section 5] for more leisurely introductions to the subject of double ramification cycles. For a sampling of the development and application of the theory in a variety of directions, see [4, 5, 11, 12, 13, 17, 21, 22, 23, 28, 35, 37, 38, 40, 41, 52, 53, 56, 62, 65, 67].

1.2 Logarithmic double ramification cycles

The definition of the double ramification cycle by Holmes [34] yields cycle classes on iterated blow-ups of boundary strata of the moduli space $\overline{\mathcal{M}}_{g,n}$ which push forward to $\mathsf{DR}_{g,A}$ on $\overline{\mathcal{M}}_{g,n}$. The *logarithmic Chow ring* $\mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ describes the intersection theory on all suitable blow-ups of $\overline{\mathcal{M}}_{g,n}$, and Holmes' construction naturally yields a *logarithmic double ramification cycle*

$$\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n}) .$$

The definition of $\mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ is reviewed in Sections 1.3–1.4 below. The class $\mathsf{logDR}_{g,A}$, a refinement of $\mathsf{DR}_{g,A}$, plays a fundamental role in logarithmic Gromov-Witten theory (and has also recently been applied in [19] to the study of double Hurwitz numbers and their generalizations).

The starting point of our paper is the following question: *can Pixton's formula for* $\mathsf{DR}_{g,A}$ *be lifted to a formula for* $\mathsf{logDR}_{g,A}$ *in the logarithmic cycle theory of the moduli space of curves?* Our main result is a formula for such a lift obtained by applying the universal theory of [5] to the universal Jacobian over the moduli space of curves with respect to certain stability conditions (using the criterion of [39]).

The difficulties which arise in computing $\mathsf{logDR}_{g,A}$ via the original definition of Holmes are explained in Section 1.5. Our new approach using stability conditions for line bundles on nodal curves is presented in Section 1.6. The main results about the logarithmic double ramification cycle are given in an abstract form (Theorem A) in Section 1.6 and in an explicit form (Theorem B) in Section 1.7. Pixton's double ramification cycle relations are lifted to $\mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ by Theorem C presented in Section 1.9.

We introduce the main constructions, ideas, and results of the paper in Sections 1.3-1.7. The reader should be able to obtain a full overview of our argument by reading these sections. The body of the paper contains a detailed presentation with complete proofs.

1.3 Log modifications and cone stacks

We begin by discussing *log modifications* of $\overline{\mathcal{M}}_{g,n}$,

$$\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}\,,$$

which are birational morphisms used in the definition of the logarithmic Chow ring. Log modifications generalize the notion of an iterated blow-up of boundary strata. Some background in log geometry is presented in Sections 2 and 3.

The most convenient way to describe a log modification of $\overline{\mathcal{M}}_{g,n}$ is via the *cone stack* $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ associated to the pair $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$. As in the case of toric geometry, where every normal toric variety X has an associated fan Σ_X, the cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is essentially a cone complex describing the combinatorics of the boundary stratification of $\overline{\mathcal{M}}_{g,n}$. However, as the name suggests, in contrast to the toric case, the presence of automorphisms on the stack $\overline{\mathcal{M}}_{g,n}$ forces us to work with a cone stack (in the sense of [18]) instead of a usual cone complex.[1] Just as toric modifications $\widehat{X} \to X$ of a toric variety X are in bijective correspondence with subdivisions $\widetilde{\Sigma} \to \Sigma_X$ of the associated fan, log modifications of $\overline{\mathcal{M}}_{g,n}$ are in bijective correspondence with subdivisions $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ of the cone stack.

The boundary strata of $\overline{\mathcal{M}}_{g,n}$ are indexed by *stable graphs* Γ: decorated graphs (with possible loops and multi-edges) describing the topological type of the generic stable curve $(C, x_1, \ldots, x_n)$ parameterized by the strata. For the precise definition of a stable graph, we refer the reader to [32, Appendix A]. See Figure 1 for an illustration.

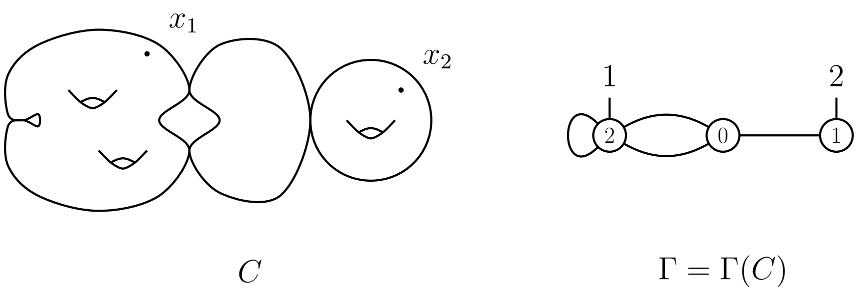


Figure 1: A curve $(C, x_1, x_2) \in \overline{\mathcal{M}}_{5,2}$ and the associated stable graph Γ

The cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is constructed from cones[2] associated to stable graphs. For stable graph Γ with edge set $E(\Gamma)$, the associated cone σ_Γ is defined by

$$\sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)} = \{\ell : E(\Gamma) \to \mathbb{R}_{\geq 0}\}\,.$$

[1]Some authors use the term *stacky fan* for the cone stack to emphasize the analogy to toric geometry, see [9, 10, 20].

[2]For us, cones are always rational polyhedral.

An element of σ_Γ is an assignment of nonnegative lengths $\ell(e)$ to the edges e of the graph Γ.

If a stable graph Γ' is obtained from Γ by *contracting* a subset $E_0 \subset E(\Gamma)$ of the edges of Γ, we view the corresponding cone $\sigma_{\Gamma'}$ as the face of σ_Γ defined by the conditions that $\ell(e_0) = 0$ for $e_0 \in E_0$. To make the face construction more flexible, recall the notion of a *morphism of stable graphs*

$$\varphi : \Gamma \to \Gamma',$$

encoding a particular way that Γ' is obtained from Γ by contracting a subset of edges.[3] Part of the data of φ is an injective map $\varphi_E : E(\Gamma') \to E(\Gamma)$ identifying the edges of Γ' as edges of Γ not contracted by φ. We then obtain a natural map of cones

$$\iota_\varphi : \sigma_{\Gamma'} \to \sigma_\Gamma\,, \quad \ell' \mapsto \left(\ell : e \mapsto \begin{cases} \ell'(\varphi_E^{-1}(e)) & \text{for } e \in \varphi_E(E(\Gamma')). \\ 0 & \text{otherwise.} \end{cases}\right)$$

representing $\sigma_{\Gamma'}$ as a face of σ_Γ.

The cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is defined as the direct limit

$$\Sigma_{\overline{\mathcal{M}}_{g,n}} = \varinjlim_{\Gamma \in \mathcal{G}_{g,n}} \sigma_\Gamma \tag{4}$$

over the category $\mathcal{G}_{g,n}$ of stable graphs (with morphisms $\varphi : \Gamma \to \Gamma'$ as above and associated morphisms $\iota_\varphi : \sigma_{\Gamma'} \to \sigma_\Gamma$ of the corresponding cones).

The limit (4) is formally defined as a cone stack in the sense of [18], see in particular [18, Sections 3.3–3.4]. However, all the additional data that we will require (subdivisions and piecewise polynomial functions on fans) will be defined on the individual cones σ_Γ. It will *not* be necessary for the reader to recall the machinery of cone stacks to follow the remainder of Section 1.

A *subdivision* $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ is a cone stack specified by a collection $(\widetilde{\Sigma}_\Gamma)_{\Gamma \in \mathcal{G}_{g,n}}$ of fans satisfying:

(i) each $\widetilde{\Sigma}_\Gamma$ is a collection of finitely many rational polyhedral cones in $(\mathbb{R}_{\geq 0})^{E(\Gamma)}$, and has total support equal to $\sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)}$,

(ii) the fans $\widetilde{\Sigma}_\Gamma$ are compatible with morphisms $\varphi : \Gamma \to \Gamma'$ in the sense that

$$\iota_\varphi^{-1}(\widetilde{\Sigma}_\Gamma) = \widetilde{\Sigma}_{\Gamma'}\,.$$

[3] See [66, Definition 2.5] or [32, Appendix A] for details. A morphism there is called a Γ'-structure on Γ.

The subdivision $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ determines a proper birational morphism $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$, which we call a *log modification*.

The essential idea is that the fan $\widetilde{\Sigma}_\Gamma$ determines, in étale local coordinates, the toric modifications near the boundary stratum associated to Γ. The compatibility of the subdivisions $\widetilde{\Sigma}_\Gamma$ with face maps ensures that these modifications glue to a global birational morphism. A detailed definition is given in Section 2.4. We illustrate a subdivision of the cone stack $\Sigma_{\overline{\mathcal{M}}_{1,2}}$ and the associated log modification, in Figure 2.

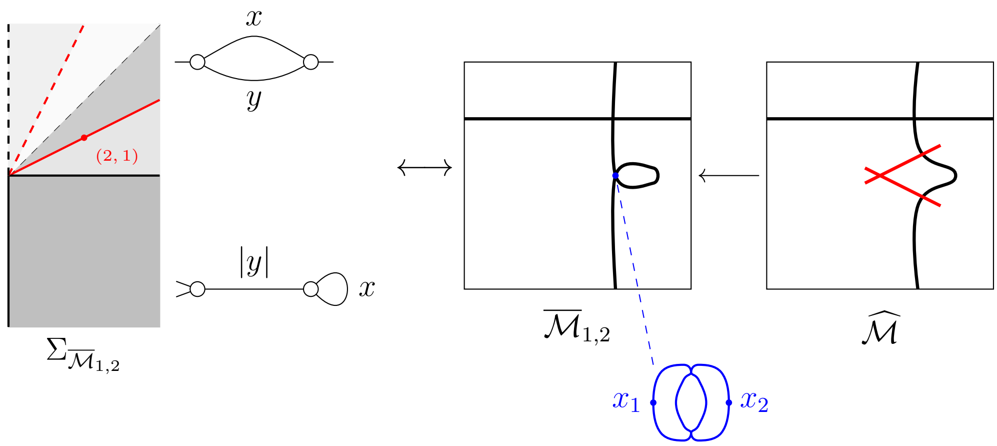


Figure 2: The cone stack $\Sigma_{\overline{\mathcal{M}}_{1,2}}$ with a subdivision (in red) and the corresponding log modification $\widehat{\mathcal{M}}$ of $\overline{\mathcal{M}}_{1,2}$, which replaces the self-intersection of the boundary divisor of irreducible curves with a chain of rational curves of length 2. Note that in the cone stack (on the left) we draw the double cover of the upper (stacky) cone, and correspondingly, the local pictures around the self-intersection of δ_{irr} on the right in fact represent an étale double cover of the neighbourhood of this point.

There is an equivalence of categories between log modifications and the subdivisions of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$. Under the equivalence, the blow-up of $\overline{\mathcal{M}}_{g,n}$ along a normal closed stratum corresponds to the star subdivision along the barycenter of the corresponding cone. The full iterated boundary blow-up of $\overline{\mathcal{M}}_{g,n}$, called the explosion in [52, Section 5], precisely corresponds to the full iterated star subdivision of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$. It is however not straightforward in general to describe the log modification corresponding to a given subdivision in terms of more familiar algebro-geometric operations.

1.4 Log intersection theory

We can now define the logarithmic Chow ring of $\overline{\mathcal{M}}_{g,n}$:

$$\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) = \varinjlim_{\widetilde{\mathcal{M}}} \mathsf{CH}^*(\widetilde{\mathcal{M}})\,. \tag{5}$$

The direct limit is taken over those log modifications $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ where the domain $\widetilde{\mathcal{M}}$ is a nonsingular Deligne-Mumford stack. There exists a morphism

$$\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$$

precisely if the associated subdivision $\widetilde{\Sigma}$ refines the subdivision $\widetilde{\Sigma}'$. Given such a morphism, there is a pullback map $\mathsf{CH}^*(\widetilde{\mathcal{M}}') \to \mathsf{CH}^*(\widetilde{\mathcal{M}})$ which is used to define the above direct limit.

A singular Deligne-Mumford stack $\mathcal{M}$ has an operational Chow ring $\mathsf{CH}^*_{\mathsf{op}}(\mathcal{M})$, as defined in [68, Section 5]. Since operational classes admit pullbacks under arbitrary maps, the limit (5) can be taken over *all* log modifications $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ using operational Chow theory:

$$\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) = \varinjlim_{\text{all } \widetilde{\mathcal{M}}} \mathsf{CH}^*_{\mathsf{op}}(\widetilde{\mathcal{M}})\,. \tag{6}$$

Definitions (5) and (6) agree since every log modification can be further modified to desingularize the domain and since operational and classical Chow groups agree for nonsingular Deligne-Mumford stacks [68, Proposition 5.6].

Viewing $\overline{\mathcal{M}}_{g,n}$ as the trivial log modification of itself, there exists a canonical algebra morphism,

$$\mathsf{CH}^*(\overline{\mathcal{M}}_{g,n}) \to \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})\,,$$

which is injective, since an inverse map of $\mathbb{Q}$-vector spaces

$$\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) \to \mathsf{CH}^*(\overline{\mathcal{M}}_{g,n})$$

is given by proper pushforward under the maps $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$. The second map fails to be a ring morphism in general. For more details on logarithmic Chow rings and intersection theory see [7, 39, 52, 53].

1.5 Resolving the Abel-Jacobi map

We describe now Holmes' construction [34] of the log double ramification cycle in $\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})$. Let

$$\mu : \mathcal{J}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

be the (non-compact) universal Jacobian of line bundles of multidegree 0 on stable curves of genus g with n marked points. Recalling that $A = (a_1, \ldots, a_n)$ is a vector of integers with sum $k(2g-2+n)$, the Abel-Jacobi map aj_A is the rational map defined by

$$\mathsf{aj}_A : \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{J}_{g,n}\,, \quad \mathsf{aj}_A([C, x_1, \ldots, x_n]) = (\omega_C^{\log})^{\otimes k}\Big(-\sum_{i=1}^{n} a_i x_i\Big)\,. \tag{7}$$

In [34], Holmes constructs a universal birational map $\mathcal{U}^{\diamond}_{g,A} \to \overline{\mathcal{M}}_{g,n}$ on which the Abel-Jacobi map (7) can be extended. The space $\mathcal{U}^{\diamond}_{g,A}$ sits as an open substack in a *non-canonical* log modification

$$\rho : \overline{\mathcal{M}}^{\diamond}_{g,A} \to \overline{\mathcal{M}}_{g,n}\,, \tag{8}$$

so we have a diagram

$$\begin{array}{ccccc} & & \rho^*\mathcal{J}_{g,n} & \longrightarrow & \mathcal{J}_{g,n} \\ & \overset{\mathsf{aj}_A}{\nearrow} & \downarrow & & \downarrow \uparrow \mathsf{aj}_A \\ \mathcal{U}^{\diamond}_{g,A} & \subseteq & \overline{\mathcal{M}}^{\diamond}_{g,A} & \xrightarrow{\rho} & \overline{\mathcal{M}}_{g,n} \end{array}\,.$$

Denoting by $e \subseteq \rho^*\mathcal{J}_{g,n}$ the preimage of the zero section of the universal Jacobian, the inverse image $\mathsf{aj}_A^{-1}(e) \subset \mathcal{U}^{\diamond}_{g,A}$ is proper (compact). The refined intersection product $\mathsf{aj}_A^*([e])$ then defines a cycle class in $\mathsf{CH}^g(\overline{\mathcal{M}}^{\diamond}_{g,n})$ which represents $\mathsf{logDR}_{g,A}$.

While the construction of the blow-up (8) is not canonical nor even explicit[4], the resulting logarithmic cycle class

$$\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})\,,$$

is well-defined by [34, Theorem 1.2]. The most basic properties are:

- If $n = 0$ and $A = \emptyset$, then $\mathsf{logDR}_{g,\emptyset} = (-1)^g \lambda_g$, where the top Chern class of the Hodge bundle λ_g is pulled back from $\mathsf{CH}^g(\overline{\mathcal{M}}_g)$, see [53].
- If $k = 0$, the class $\mathsf{logDR}_{g,A}$ pushes forward to the standard double ramification cycle $\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$ defined via the moduli space of rubber maps by [34, Theorem 1.3].

Since the $n = 0$ case is solved, we will always assume $n \geq 1$.

In order to calculate $\mathsf{logDR}_{g,A}$ using the above construction via Abel-Jacobi theory, several difficulties must be overcome:

[4] An abstract resolution of singularities is required.

(i) Since the construction of the blow-up $\rho : \overline{\mathcal{M}}^{\diamond}_{g,A} \to \overline{\mathcal{M}}_{g,n}$ in [34] is only implicit (depending upon non-canonical choices), a direct study of $\overline{\mathcal{M}}^{\diamond}_{g,A}$ is difficult.

(ii) The class $\mathsf{aj}_A^*([e])$ arises from the geometry of the Abel-Jacobi map on the open set $\mathcal{U}^{\diamond}_{g,A}$, not the global geometry of $\overline{\mathcal{M}}^{\diamond}_{g,A}$, and so is not directly accessible via intersection theory on $\overline{\mathcal{M}}^{\diamond}_{g,A}$.

(iii) Even if the above issues could be overcome, in what language would the answer be expressed?

Our solution to both (i) and (ii) is to find geometrically meaningful models for $\overline{\mathcal{M}}^{\diamond}_{g,A}$ via the moduli spaces of line bundles on quasi-stable curves. The non-canonical aspect of $\overline{\mathcal{M}}^{\diamond}_{g,A}$ is not completely lost. There is still a choice of stability condition needed to define the moduli space of line bundles, but the stability condition is the only choice. By applying the main result of [39] together with the formula of [5] for the universal double ramification cycle, we can calculate $\mathsf{logDR}_{g,A}$. For (iii), the answer is expressed in the subring of tautological classes in $\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n})$, suggested by D. Ranganathan[5] and developed in [39, 52, 53]. Tautological classes include κ_1, the cotangent line classes ψ_i, and classes coming from the algebra of piecewise polynomials on the cone stack associated to $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$.

As discussed above, the formula for $\mathsf{logDR}_{g,A}$ depends upon the choice of an appropriate[6] stability condition. The dependence can be useful: special stability conditions can be selected to simplify the formula depending upon the properties of the vector A. The wall-crossing study of the formula leads to relations in $\mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ when different stability conditions calculate the same class.

1.6 Moduli of line bundles on stable curves

A basic difficulty in writing a formula for the logarithmic double ramification cycle is the non-compactness of the universal Jacobian $\mathcal{J}_{g,n}$ of multidegree 0 line bundles on stable curves. As a consequence, the universal space $\mathcal{U}^{\diamond}_{g,A}$ is also not compact. Our approach here is to view the rational map

$$\mathsf{aj}_A : \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{J}_{g,n}$$

[5]In the lecture by D. Ranganathan in the *Algebraic Geometry and Moduli Zoominar* at ETH Zürich in April 2020.

[6]The formula is well-defined for all *nondegenerate* stability conditions, but calculates $\mathsf{logDR}_{g,A}$ only for *small* stability conditions. The precise definitions are given in Section 4.

as taking values in a *compactified* Jacobian and to resolve the indeterminacies of aj_A. Such resolutions of indeterminancies turn out to be proper and provide modular compactifications of $\mathcal{U}^{\diamond}_{g,A}$ of precisely the type needed to compute $\mathsf{logDR}_{g,A}$.

The construction of compactifications of the moduli spaces of line bundles on stable curves goes back at least to [14, 59]. In the past decades, there has been a continuous study of these spaces, see [2, 8, 15, 16, 26, 27, 42, 49, 50]. A short summary of what we need for our approach to the logarithmic double ramification cycle is presented here.

Since $\mathcal{J}_{g,n}$ is not compact, we can find 1-parameter families of nonsingular curves carrying line bundles of degree 0 degenerating to a stable nodal curve where the line bundle fails to degenerate to a line bundle of degree 0 on every irreducible component of the nodal curve. The issue is not simply about multidegrees. Searching for limits among all line bundles of total degree 0 does not suffice and, in fact, further complicates the geometry: the Picard scheme $\mathcal{P}ic\,(\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n})$ of line bundles of all multidegrees on stable curves

$$\mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

is neither separated nor universally closed.

In order to obtain a compactification of $\mathcal{J}_{g,n}$, we must include limits that are not line bundles on stable curves. There are two equivalent approaches. The first is to consider the moduli space of all rank 1 torsion free sheaves on stable curves. The second, which we will follow, is to consider the moduli space of *admissible* line bundles on *quasi-stable* curves.

A flat family of nodal curves $\mathcal{C} \to \mathcal{S}$ is *quasi-stable* if the relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{S}}$ is nef and chains of unstable components have length at most 1. A line bundle $\mathcal{L}$ on $\mathcal{C}$ is *admissible* if $\mathcal{L}$ has degree 1 on each unstable component of $\mathcal{C}$. The data of an admissible line bundle on a quasi-stable curve is equivalent to the data of a rank 1 torsion free sheaf on the associated stabilization,

$$\mathsf{st} : \mathcal{C} \to \mathcal{C}^{\mathrm{st}}\,,$$

by assigning to an admissible line bundle $\mathcal{L}$ the rank 1 torsion free sheaf $\mathsf{st}_*\mathcal{L}$.

The resulting moduli space parametrizing either admissible line bundles on quasi-stable curves or rank 1 torsion free sheaves on stable curves is the *universal Jacobian* which is universally closed over the moduli space of stable curves, but not separated.

The failure of separation of the universal Jacobian occurs because of the possibility of twisting $\mathcal{L}$ by divisors $\mathcal{T} \subset \mathcal{C}$ supported over nodal curves. Twisting defines an equivalence relation on the *universal* Jacobian by declaring

$$(C, \mathcal{L}) \sim (C, \mathcal{L} \otimes \mathcal{O}(\mathcal{T}))\,.$$

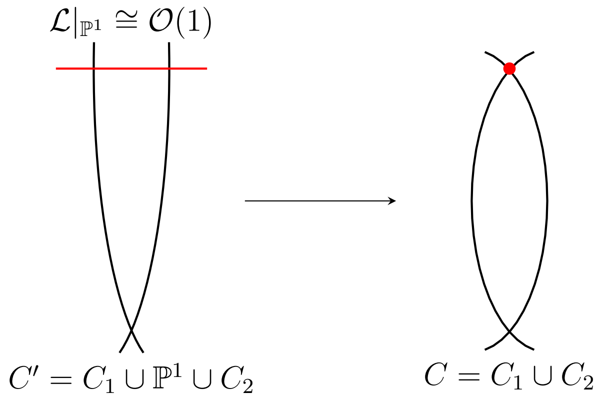


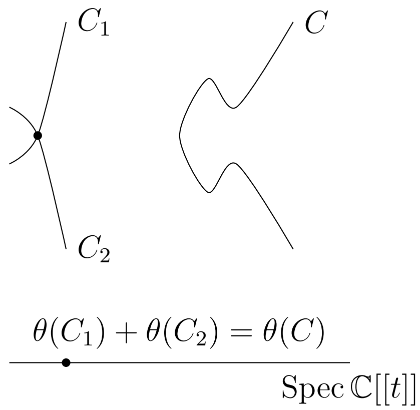
Figure 3: A quasi-stable curve C' with stabilization C, and an admissible line bundle $\mathcal{L}$ on it.

To find compactifications of the Jacobian $\mathcal{J}_{g,n}$ which are proper and separated, we must choose a unique representative in each equivalence class of twistings. Stability conditions exactly make such choices.

Following [42], a *stability condition* θ of type (g, n) and degree d is a rule which assigns a rational number to every irreducible component of every stable curve $(C, x_1, \dots, x_n)$ of genus g with n marked points and satisfies two properties:

(i) the sum of the values of θ over the irreducible components of C equals d,

(ii) θ is additive under all partial smoothings:



We may also view a stability condition θ in combinatorial terms: θ is an assignment of a rational value to every vertex of every stable graph G of type (g, n) which sums to d and which is additive with respect to contractions of edges.

There is a *trivial stability condition* θ^{tr} of degree 0 given by

$$\theta^{\mathsf{tr}}(D) = 0$$

for every irreducible component $D \subset C$. A more interesting example is the *canonical stability condition* θ^K of degree $2g - 2 + n$ defined by

$$\theta^K(D) = \mathsf{deg}\left[\omega_C\Big(\sum_{i=1}^{n} x_i\Big)\Big|_D\right]$$

for an irreducible component $D \subset C$. The value of θ^K depends on the genus of D and the number nodes and markings of C which lie on D.

Fix a stability condition θ of degree d. An admissible line bundle $\mathcal{L}$ of degree d on a quasi-stable curve C is *θ-stable* (respectively, *θ-semistable*) if and only if, for every proper subcurve[7] $\emptyset \subsetneq \underline{C} \subsetneq C$ with neither $\underline{C}$ nor its complement consisting entirely of unstable components, we have

$$\theta(\underline{C}) - \frac{E(\underline{C}, \underline{C}^c)}{2} \;<\; (\leq) \;\; \mathsf{deg}(\mathcal{L}|_{\underline{C}}) \;\;<\; (\leq) \;\;\; \theta(\underline{C}) + \frac{E(\underline{C}, \underline{C}^c)}{2}\,,$$

where $E(\underline{C}, \underline{C}^c)$ is the number of intersection points of $\underline{C}$ with the complementary subcurve $\underline{C}^c \subset C$ and

$$\theta(\underline{C}) = \sum_{D \subset \underline{C}} \theta(D)$$

is the sum of the values of θ over all irreducible components $D \subset \underline{C}$.[8] In other words, an admissible line bundle $\mathcal{L}$ is θ-semistable if the degree of $\mathcal{L}$ restricted to each subcurve $\underline{C} \subset C$ is close enough to the assigned value $\theta(\underline{C})$.

The connection with the above discussion about compactifications of the Jacobian can be seen as follows. A non-trivial twist by $\mathcal{O}(\mathcal{T})$ in families changes the degree on $\underline{C}$ by at least $E(\underline{C}, \underline{C}^c)$ for some subcurve $\underline{C} \subset C$. Thus, there can be at most one representative of the equivalence class of a line bundle which is θ-stable. A stability condition θ is *nondegenerate* if every θ-semistable line bundle on every quasi-stable curve C is stable.

Let θ be a nondegenerate stability condition of type (g, n). By a result of [42], there exists a moduli stack $\mathcal{P}^{\theta}_{g,n}$ of θ-stable admissible line bundles on quasi-stable curves, satisfying the following properties:

(i) The stack $\mathcal{P}^{\theta}_{g,n}$ is proper, nonsingular, and of dimension $4g - 3 + n$.

[7]We will later phrase these inequalities in combinatorial terms using the dual graph of C.

[8]The stability condition θ extends to quasi-stable curves by assigning weight 0 to all unstable components.

(ii) There is natural morphism $\mu : \mathcal{P}^{\theta}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.

(iii) The stack $\mathcal{P}^{\theta}_{g,n}$ carries a universal quasi-stable curve with an universal admissible line bundle.[9]

In general, neither the trivial stability condition θ^{tr} nor the canonical stability condition θ^K is nondegenerate. However, in case $n \geq 1$, explicit perturbations can be easily constructed in both cases and seen to be nondegenerate (as explained in [42] and reviewed in Section 4).

A stability condition θ of degree 0 is *small* if line bundles of multidegree 0 on the universal curve $\mathcal{C}_{g,n}$ are θ-stable. Nondegenerate perturbations of θ^{tr} sufficiently close to θ^{tr} are small. For our formula for the logarithmic double ramification cycle, we will require θ to be a small nondegenerate stability condition of degree 0.

Recall that $A = (a_1, \ldots, a_n)$ is a vector of integers summing to $k(2g - 2 + n)$. Given a small nondegenerate stability condition θ of degree 0 and type (g, n), there is a rational Abel-Jacobi section of the morphism μ,

$$\mathsf{aj}_A : \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{P}^{\theta}_{g,n} \,,$$

defined (as before) by

$$\mathsf{aj}_A([C, x_1, \ldots, x_n]) = (\omega_C^{\log})^{\otimes k}\Big(-\sum_{i=1}^{n} a_i x_i\Big) \,.$$

As the stability condition θ is nondegenerate, the space $\mathcal{P}^{\theta}_{g,n}$ is proper. The stability condition θ then canonically[10] determines a log modification

$$\rho : \overline{\mathcal{M}}^{\theta}_{g,A} \to \overline{\mathcal{M}}_{g,n} \tag{9}$$

which resolves the Abel-Jacobi map,

$$\mathsf{aj} : \overline{\mathcal{M}}^{\theta}_{g,A} \to \mathcal{P}^{\theta}_{g,n} \,, \qquad \mathsf{aj} = \mathsf{aj}_A \circ \rho \,.$$

Over $\overline{\mathcal{M}}^{\theta}_{g,A}$, we have a universal family

$$\pi : \mathcal{C}^{\theta} \to \overline{\mathcal{M}}^{\theta}_{g,A}$$

[9] The condition $n \geq 1$ is used for the existence of the universal line bundle.

[10] The subdivision $\widetilde{\Sigma}^{\theta} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ which determines the log modification ρ is defined in Section 1.7.2. The resolution of the Abel-Jacobi map $\mathsf{aj} : \overline{\mathcal{M}}^{\theta}_{g,A} \to \mathcal{P}^{\theta}_{g,n}$ is constructed in Section 4, see Definition 26.

which is a quasi-stable model of the pullback $\overline{\mathcal{M}}^{\theta}_{g,A} \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{C}$ of the universal stable curve, and an admissible line bundle

$$\mathcal{L}^{\theta} \to \mathcal{C}^{\theta}$$

obtained from the universal admissible line bundle on the moduli stack of θ-stable sheaves. Since θ is small, $\mathcal{J}_{g,n} \subset \mathcal{P}^{\theta}_{g,n}$ is an open substack. In fact, $\mathcal{U}^{\diamond}_{g,A}$ is an open substack of $\overline{\mathcal{M}}^{\theta}_{g,A}$, and $\overline{\mathcal{M}}^{\theta}_{g,A}$ can play the role of $\overline{\mathcal{M}}^{\diamond}_{g,A}$ in the definition of [34]. The class

$$\mathsf{aj}_A^*[e] \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{M}}^{\theta}_{g,A})$$

therefore represents $\mathsf{logDR}_{g,A}$.

The geometry here is much more favorable than for the abstractly defined spaces $\overline{\mathcal{M}}^{\diamond}_{g,A}$. After the small nondegenerate stability condition θ has been chosen, there are *no* further choices, and we have complete understanding of the points of $\overline{\mathcal{M}}^{\theta}_{g,A}$. A modular interpretation of $\overline{\mathcal{M}}^{\theta}_{g,A}$ is described in Definition 22 of Section 4.2. Furthermore, $\overline{\mathcal{M}}^{\theta}_{g,A}$ naturally carries a universal double ramification cycle class [5],

$$\mathsf{DR}^{\mathsf{op}}_{g,\emptyset,\mathcal{L}^{\theta}} \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{M}}^{\theta}_{g,A})\,,$$

associated to the universal admissible line bundle $\mathcal{L}^{\theta}$ on $\mathcal{C}^{\theta}$. By applying the theory of almost-twistable families developed in [39], we obtain the first form of our main result (which simply states that the two natural classes on $\overline{\mathcal{M}}^{\theta}_{g,A}$ are equal).

Theorem A. *Let θ be a small nondegenerate stability condition. The universal double ramification cycle associated to the line bundle $\mathcal{L}^{\theta}$ on $\mathcal{C}^{\theta} \to \overline{\mathcal{M}}^{\theta}_{g,A}$,*

$$\mathsf{DR}^{\mathsf{op}}_{g,\emptyset,\mathcal{L}^{\theta}} \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{M}}^{\theta}_{g,A})\,,$$

provides a representative for $\mathsf{logDR}_{g,A}$.

While the claim of Theorem A is conceptual, the statement can be transformed into an explicit formula for $\mathsf{logDR}_{g,A}$ since the universal double ramification cycle has been explicitly computed in [5]. Theorem A is proven in Section 5.

1.7 Formula for logDR

The final step is to translate Theorem A into an explicit class in the logarithmic Chow ring using the formula for the universal double ramification cycle in [5].

1.7.1 Tautological classes in the logarithmic Chow ring

As we have seen, the logarithmic modifications $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ appearing in the definition of the logarithmic Chow ring are described by subdivisions $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ of the cone stack. Following [39, 52, 53], we can use the same convex-geometric language to describe natural cycle classes on $\widetilde{\mathcal{M}}$ which define elements of $\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})$.[11]

Let Σ be a fan in $\mathbb{R}^m$ with maximal cones all of dimension m (the dimension of the ambient vector space). A *strict piecewise polynomial* $f \in \mathsf{sPP}(\Sigma)$ is a continuous function $f : |\Sigma| \to \mathbb{R}$ on the support $|\Sigma| \subseteq \mathbb{R}^m$ of Σ which is given by a polynomial with rational coefficients on each maximal cone of Σ. A *piecewise polynomial function* $f \in \mathsf{PP}(\Sigma)$ is a continuous function $f : |\Sigma| \to \mathbb{R}$ which is a strict piecewise polynomial on *some* subdivision Σ' of Σ.

Given a subdivision $\widetilde{\Sigma} = (\widetilde{\Sigma}_\Gamma)_{\Gamma \in \mathcal{G}_{g,n}}$ of the cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$, a strict piecewise polynomial function

$$f = (f_\Gamma)_{\Gamma \in \mathcal{G}_{g,n}} \in \mathsf{sPP}(\widetilde{\Sigma})$$

is a collection of strict piecewise polynomials f_Γ on the fans $\widetilde{\Sigma}_\Gamma$, which are compatible with the face maps ι_φ. More precisely, for $\varphi : \Gamma \to \Gamma'$, we have $f_\Gamma \circ \iota_\varphi = f_{\Gamma'}$. A piecewise polynomial on $\widetilde{\Sigma}$ is then similarly a collection $f = (f_\Gamma)_\Gamma$ of piecewise polynomials f_Γ satisfying the same compatibility.

The (strict) piecewise polynomials on $\widetilde{\Sigma}$ form a $\mathbb{Q}$-algebra. For the log modification $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ associated to $\widetilde{\Sigma}$, there are natural ring morphisms

$$\Phi : \mathsf{sPP}(\widetilde{\Sigma}) \to \mathsf{CH}^*_{\mathrm{op}}(\widetilde{\mathcal{M}}) \tag{10}$$

constructed in [39, Section 3.3].[12] When $\widetilde{\Sigma}$ is the cone over a simplicial complex the construction proceeds in two steps:

(i) There exists a natural map from piecewise *linear* functions on $\widetilde{\Sigma}$ to divisor classes on $\widetilde{\mathcal{M}}$ using the fact that rays of $\widetilde{\Sigma}$ correspond to boundary divisors of $\widetilde{\mathcal{M}}$.

(ii) The piecewise linear functions generate $\mathsf{sPP}(\widetilde{\Sigma})$ as $\mathbb{Q}$-algebra. The unique extension of (i) defines Φ.

In general, we can subdivide $\widetilde{\Sigma}$ until it is the cone over a simplicial complex and then push forward the resulting class. A more abstract construction of Φ is given in [52, 53].

---

[11] The language of log structures and ghost sheaves is used in [39]. Our presentation avoids ghosts and follows instead the language of fans and cones from toric geometry. The comparison of terminology is straightforward and discussed in more detail in Remark 8.

[12] Following our analogy with toric varieties, the equivariant Chow ring of a toric variety is isomorphic to the ring of strict piecewise polynomials on the associated fan, see [61].

The maps Φ are compatible under subdivision of $\widetilde{\Sigma}$ and the associated birational modification of $\widetilde{\mathcal{M}}$, and thus give rise to an algebra morphism

$$\Phi : \mathsf{PP}(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \to \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})\,. \tag{11}$$

Definition 1. [39, Definition 3.18] The *logarithmic tautological ring*

$$\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})$$

is the subring generated by the image of Φ and the usual tautological ring[13]

$$\mathsf{R}^*(\overline{\mathcal{M}}_{g,n}) \subseteq \mathsf{CH}^*(\overline{\mathcal{M}}_{g,n}) \subseteq \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})\,.$$

$\diamondsuit$

In particular, the above definition gives a natural surjection

$$\mathsf{R}^*(\overline{\mathcal{M}}_{g,n}) \otimes_{\mathbb{Q}} \mathsf{PP}(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \to \mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})\,. \tag{12}$$

By the results of [39, 53], we know $\mathsf{logDR}_{g,A} \in \mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$.

Our goal is to describe an explicit element of the left side of (12) which maps to $\mathsf{logDR}_{g,A}$. As we will see in Section 1.7.3, the formulas describing the element naturally involve taking exponentials of certain piecewise linear functions. Exponentiation leaves the realm of piecewise polynomials and yields *piecewise power series*. To make sense of the formulas below, we simply observe that the map Φ above naturally extends to such piecewise power series by first truncating the power series to degree (at most) $3g-3+n$ to obtain a piecewise polynomial.

1.7.2 The subdivision associated to a stability condition

Let θ be a nondegenerate stability condition of type (g,n) and degree 0 (as defined in Section 1.6). The data of the vector $A=(a_1,\ldots,a_n)$ and θ together determine a canonical subdivision

$$\widetilde{\Sigma}^\theta \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$$

which we construct here. The subdivision $\widetilde{\Sigma}^\theta$ defines the log modification

$$\rho : \overline{\mathcal{M}}^\theta_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

from (9). We will require the following notation for the construction:

• For a quasi-stable graph Γ, let $\vec{E}(\Gamma)$ be the set of *oriented edges* of Γ. The natural map

$$\vec{E}(\Gamma) \to E(\Gamma)$$

[13] We refer the reader to [60] for an introduction to the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

has fibers $\{\vec{e}, \overleftarrow{e}\}$ of size two, the two orientations on a given edge e. For $\vec{e} \in \vec{E}(\Gamma)$, we write

$$\vec{e} \to v \in V(\Gamma)$$

if $\vec{e}$ is incident to and points *towards* v. A *cycle* γ on Γ is a collection of oriented edges forming a cycle without self-intersection.

Given a family $(C, \mathcal{L})$ of curves and line bundles, the change of the multidegree of $\mathcal{L}$ on a special fibre C_0 by twisting with a vertical divisor $\mathcal{T}$ (as in Section 1.6) is described by an acyclic flow on Γ_{C_0}. Acyclic flows appear under the name of *twists* in [29].

• A *flow* on Γ is a function $f : \vec{E}(\Gamma) \to \mathbb{Z}$ satisfying $f(\overleftarrow{e}) = -f(\vec{e})$ for all edges $e \in E(\Gamma)$. By choosing some fixed orientation on Γ, we can non-canonically identify the group $\mathrm{Flow}(\Gamma)$ of flows on Γ with $\mathbb{Z}^{E(\Gamma)}$. A flow f is called *acyclic* if there exists no cycle γ of Γ satisfying the following conditions:

$$\forall \vec{e} \in \gamma\,,\ f(\vec{e}) \geq 0 \quad \text{and} \quad \exists \vec{e} \in \gamma\,,\ f(\vec{e}) > 0\,.$$

• A *divisor* on Γ is an element of $\mathbb{Z}^{V(\Gamma)}$. For $f \in \mathrm{Flow}(\Gamma)$, we write

$$\mathrm{div}(f) = \left(\sum_{\vec{e} \to v} f(\vec{e}) \right)_{v \in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)} \tag{13}$$

for the divisor of f. The first homology group $H_1(\Gamma)$ is precisely the kernel of the group homomorphism $\mathrm{div}\colon \mathrm{Flow}(\Gamma) \to \mathbb{Z}^{V(\Gamma)}$. There is a straightforward generalization of flows and divisors allowing values in an arbitrary abelian group instead of $\mathbb{Z}$.

• Given a length assignment $\ell : E(\Gamma) \to \mathbb{R}_{\geq 0}$, we have a *pairing*[14]

$$\langle f, g \rangle_\ell = \frac{1}{2} \sum_{\vec{e} \in \vec{E}(\Gamma)} \ell(e) f(\vec{e}) g(\vec{e}) \tag{14}$$

for $f, g \in \mathrm{Flow}(\Gamma)$.

• The multidegree of the line bundle $(\omega_C^{\log})^{\otimes k}(-\sum_i a_i x_i)$ on a quasi-stable curve C with graph Γ is

$$\underline{\deg}_{k,A} = \left(k(2g(v) - 2 + n(v)) - \sum_{i \text{ at } v} a_i \right)_{v \in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)}\,,$$

[14]The importance of the pairing will be seen in Lemma 16: a flow arises as the slopes of a *piecewise linear function* (Definition 15) on the metric graph (Γ, ℓ) if and only if the pairing with every element of $H_1(\Gamma)$ vanishes.

where $n(v)$ is the number of half-edges attached to v (the number of markings at v plus the number of ends of edges at v).

We can now describe the subdivision $\widetilde{\Sigma}^\theta = \{\widetilde{\Sigma}^\theta_\Gamma\}_{\Gamma\in\mathcal{G}_{g,n}}$ associated to the non-degenerate stability condition θ. Fix a stable graph Γ with associated cone

$$\sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)} \quad \text{in} \quad \Sigma_{\overline{\mathcal{M}}_{g,n}}\,.$$

The fan $\widetilde{\Sigma}^\theta_\Gamma$ of the subdivision has interior cones[15] in bijective correspondence with tuples $(\widehat{\Gamma}, D, I)$ where

(i) $\widehat{\Gamma}$ is a quasi-stable graph with stabilization $\widehat{\Gamma}^s = \Gamma$,

(ii) $D \in \mathbb{Z}^{V(\widehat{\Gamma})}$ is a θ-stable multidegree with total degree 0,

(iii) I is an acyclic flow on $\widehat{\Gamma}$ satisfying

$$\mathrm{div}(I) = \underline{\deg}_{k,A} - D\,. \tag{15}$$

For each tuple $(\widehat{\Gamma}, D, I)$ as above, we will define a cone $\sigma_{\widehat{\Gamma},I} \subseteq \sigma_\Gamma$. The collection of cones $\{\sigma_{\widehat{\Gamma},I}\}$ together with their faces defines the subdivision $\widetilde{\Sigma}^\theta_\Gamma$.

Let $\sigma_{\widehat{\Gamma}} = (\mathbb{R}_{\geq 0})^{E(\widehat{\Gamma})}$ be the cone associated to the quasi-stable graph $\widehat{\Gamma}$. There is a natural map

$$\mathsf{pr} : \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma\,, \quad \widehat{\ell} \mapsto \ell : e \mapsto \begin{cases} \widehat{\ell}(e_1) + \widehat{\ell}(e_2) & \text{if } \widehat{\Gamma} \text{ subdivides edge } e \text{ into } e_1, e_2, \\ \widehat{\ell}(e) & \text{otherwise.} \end{cases}$$

We define a subcone $\tau_{\widehat{\Gamma},I} \subseteq \sigma_{\widehat{\Gamma}}$ by the condition

$$\tau_{\widehat{\Gamma},I} = \left\{ \widehat{\ell} \in \sigma_{\widehat{\Gamma}} \;\middle|\; \langle \gamma, I \rangle_{\widehat{\ell}} = 0 \text{ for all } \gamma \in H_1(\widehat{\Gamma}) \right\}\,,$$

where the pairing is (14) and $H_1(\widehat{\Gamma})$ is identified with a subgroup of $\mathrm{Flow}(\widehat{\Gamma})$ as explained above. The following crucial claim is proven in Lemma 27 of Section 4.

Claim 1. *The map* $\mathsf{pr}\colon \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma$ *induces an isomorphism from the cone* $\tau_{\widehat{\Gamma},I}$ *to the image*

$$\sigma_{\widehat{\Gamma},I} = \mathsf{pr}(\tau_{\widehat{\Gamma},I}) \subseteq \sigma_\Gamma\,.$$

[15]An *interior cone* of $\widetilde{\Sigma}^\theta_\Gamma$ is a cone of the fan intersecting the interior $(\mathbb{R}_{>0})^{E(\Gamma)}$ of σ_Γ. The interior cones uniquely determine the fan $\widetilde{\Sigma}^\theta_\Gamma$: the fan is the collection of these interior cones together with all of their faces.

Moreover, the collection of cones $\{\sigma_{\widehat{\Gamma},I}\}$ together with their faces forms a fan $\widetilde{\Sigma}^\theta_\Gamma$ with support σ_Γ.

We record an important consequence of Claim 1. Let $\ell_e, \widehat{\ell}_{\widehat{e}}$ be the coordinate functions on $\sigma_\Gamma, \sigma_{\widehat{\Gamma}}$ giving the lengths of edges $e, \widehat{e}$, viewed now as polynomials on these cones.

Claim 2. *On the cone $\sigma_{\widehat{\Gamma},I}$, there exist unique linear functions $\widehat{\ell}_{\widehat{e}} = \widehat{\ell}_{\widehat{e}}(\ell)$ in the variables ℓ_e which define a section of the map $\mathsf{pr} : \tau_{\widehat{\Gamma},I} \to \sigma_{\widehat{\Gamma},I}$.*

The above definition of the cones $\sigma_{\widehat{\Gamma},I}$ is rather formal. Geometric intuition for the construction can be found in the theory developed in Sections 3 and 4, see Remark 28.

1.7.3 The formula for logDR

Let θ be a small nondegenerate stability condition. We present here an explicit formula for the cycle

$$\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n}) \,.$$

To write the formula, we must first define two special strict piecewise power series $\mathfrak{P}, \mathfrak{L}$ on $\widetilde{\Sigma}^\theta$. The functions $\mathfrak{P}, \mathfrak{L}$ are uniquely determined by their restriction to the interior cones of $\widetilde{\Sigma}^\theta$. As discussed in Section 1.7.2, the interior cones correspond to a stable graph $\Gamma \in \mathcal{G}_{g,n}$ together with a tuple $(\widehat{\Gamma}, D, I)$. We will define the functions $\mathfrak{P}, \mathfrak{L}$ on the associated cone $\sigma_{\widehat{\Gamma},I} \in \widetilde{\Sigma}^\theta_\Gamma$.

• The definition of $\mathfrak{P}$ requires a sum over weightings: for a positive integer r, an *admissible weighting mod* r on $\widehat{\Gamma}$ is a flow w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

$$\mathrm{div}(w) = D \in (\mathbb{Z}/r\mathbb{Z})^{V(\widehat{\Gamma})} \,.$$

We define

$$\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)} = \sum_w r^{-h^1(\widehat{\Gamma})} \prod_{e \in E(\widehat{\Gamma})} \exp\left(\frac{\overline{w}(\vec{e}) \cdot \overline{w}(\overleftarrow{e})}{2} \widehat{\ell}_e \right) \in \mathbb{Q}[[\widehat{\ell}_e : e \in E(\widehat{\Gamma})]] \,,$$

where the sum runs over admissible weightings w mod r. Inside the exponential, $\overline{w}(\vec{e})$ and $\overline{w}(\overleftarrow{e})$ denote the unique representative of $w(\vec{e}) \in \mathbb{Z}/r\mathbb{Z}$ and $w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ in $\{0, \ldots, r-1\}$.

As in [40, Appendix], for each fixed degree in the variables $\widehat{\ell}_e$, the element $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$ is polynomial in r for sufficiently large r. We denote by $\mathrm{Cont}_{(\widehat{\Gamma},D,I)}$ the polynomial in the variables $\widehat{\ell}_e$ obtained by substituting $r = 0$ into the polynomial expression for $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$. We define

$$\mathfrak{P}\big|_{\sigma_{\widehat{\Gamma},I}} = \mathrm{Cont}_{(\widehat{\Gamma},D,I)}\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \in \mathbb{Q}[[\ell_e : e \in E(\Gamma)]] \,,$$

where we use the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$ associated to $\sigma_{\widehat{\Gamma},I}$ from Claim 2. We claim that these functions fit together to give a well-defined strict piecewise power series $\mathfrak{P}$ on $\widetilde{\Sigma}^\theta$.

• To define $\mathfrak{L}$ on $\widetilde{\Sigma}^\theta$, we fix a vertex $v_0 \in V(\widehat{\Gamma})$. For every length assignment $\widehat{\ell}$ in the cone $\tau_{\widehat{\Gamma},I}$ and any vertex $v \in V(\widehat{\Gamma})$, let $\gamma_{v_0 \to v}$ be a path from v_0 to v in $\widehat{\Gamma}$. We define

$$\alpha(v) = \sum_{\vec{e} \in \gamma_{v_0 \to v}} I(\vec{e}) \cdot \widehat{\ell}_e \,, \tag{16}$$

where the sum is over the oriented edges $\vec{e}$ constituting the path $\gamma_{v_0 \to v}$. The defining equations of $\tau_{\widehat{\Gamma},I}$ imply that for $\widehat{\ell} \in \tau_{\widehat{\Gamma},I}$ the expression (16) is independent of the chosen path $\gamma_{v_0 \to v}$. We define

$$\mathfrak{L} = \sum_{v \in V(\widehat{\Gamma})} (D + \underline{\deg}_{k,A})(v) \cdot \alpha(v)\big|_{\widehat{\ell} = \widehat{\ell}(\ell)} \in \mathbb{Q}[\ell_e : e \in E(\Gamma)] \,. \tag{17}$$

The substitution of variables $\widehat{\ell} = \widehat{\ell}(\ell)$, which give the inverse of the isomorphism $\tau_{\widehat{\Gamma},I} \to \sigma_{\widehat{\Gamma},I}$ and thus have image in $\tau_{\widehat{\Gamma},I}$, ensure that the expression is independent of the choice of the paths $\gamma_{v_0 \to v}$. The expression is also independent of the base vertex v_0, which follows from the fact that the divisor $D + \underline{\deg}_{k,A}$ has total degree 0 on $\widehat{\Gamma}$.

For the $\mathsf{logDR}_{g,A}$ formula, in addition to $\mathfrak{P}$ and $\mathfrak{L}$, we will also require the tautological class

$$\eta = k^2 \kappa_1 - \sum_{i=1}^n a_i^2 \psi_i \in \mathsf{R}^*(\overline{\mathcal{M}}_{g,n}) \,. \tag{18}$$

Define the mixed degree logarithmic class

$$\mathbf{P}^\theta_{g,A} = \exp\left(-\frac{1}{2}(\eta + \Phi(\mathfrak{L}))\right) \cdot \Phi(\mathfrak{P}) \in \mathsf{logR}^*(\overline{\mathcal{M}}_{g,n}) \,, \tag{19}$$

where Φ is the extension of the map (11) to piecewise power series as described at the end of Section 1.7.1.

Theorem B. *Let θ be a small nondegenerate stability condition. The log double ramification cycle is the degree g part of $\mathbf{P}^\theta_{g,A}$,*

$$\mathsf{logDR}_{g,A} = \mathbf{P}^{g,\theta}_{g,A} \in \mathsf{logR}^g(\overline{\mathcal{M}}_{g,n}) \,.$$

As a consequence of Theorem B, the class

$$\mathbf{P}^{g,\theta}_{g,A} \in \mathsf{logR}^g(\overline{\mathcal{M}}_{g,n})$$

is *independent* of θ (for all small nondegenerate stability conditions θ). The definition of $\mathbf{P}_{g,A}^{g,\theta}$, however, only requires nondegeneracy of θ. The interpretation and wall-crossing analysis of $\mathbf{P}_{g,A}^{g,\theta}$ for large stability conditions θ is an interesting direction of study.

The calculation of $\mathsf{logDR}_{g,A}$ plays a fundamental role in the log Gromov-Witten theory of toric varieties: by an elegant argument of Ranganathan and Urundolil Kumaran [64], the log Gromov-Witten theory of all nonsingular projective toric varieties (with log structure determined by the full toric boundary) can be reduced to $\mathsf{logDR}_{g,A}$. After a discussion of Pixton's formula in the language of piecewise polynomials in Section 6, Theorem B is proven in Section 7.

1.8 Example in genus 1

We illustrate Theorem B for $\mathsf{logDR}_{1,(3,-3)}$ in Figure 4 (giving a similar illustration of the regular DR cycle $\mathsf{DR}_{1,(3,-3)}$ for comparison). The general genus 1 computation is worked out in Section 8. A genus 2 calculation using the Sage package `logtaut` is presented in Section 9.1.

1.9 Relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$

The logarithmic double ramification cycle provides two nontrivial paths to tautological relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$. The first path uses the choice of stability condition in Theorem B, and the second path is via Pixton's relations for the universal double ramification cycle [5].

Theorem C. *The following two constructions yield relations in* $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$*:*

(i) Let θ *and* $\widehat{\theta}$ *be two small nondegenerate stability conditions of type* (g,n)*. Then,*

$$\mathbf{P}_{g,A}^{g,\theta} = \mathbf{P}_{g,A}^{g,\widehat{\theta}} \in \mathsf{logR}^g(\overline{\mathcal{M}}_{g,n}).$$

(ii) For every nondegenerate condition θ *of type* (g,n)*,*

$$\mathbf{P}_{g,A}^{h,\theta} = 0 \in \mathsf{logR}^h(\overline{\mathcal{M}}_{g,n})$$

for all $h > g$*.*

Theorem C is proven together with Theorem B in Section 7.

Higher double ramification cycles are discussed in Section 9. A third construction of relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$ is presented in Section 9.2 via the *double-double ramification cycle*. The general study of $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$ will be taken up in a future paper.

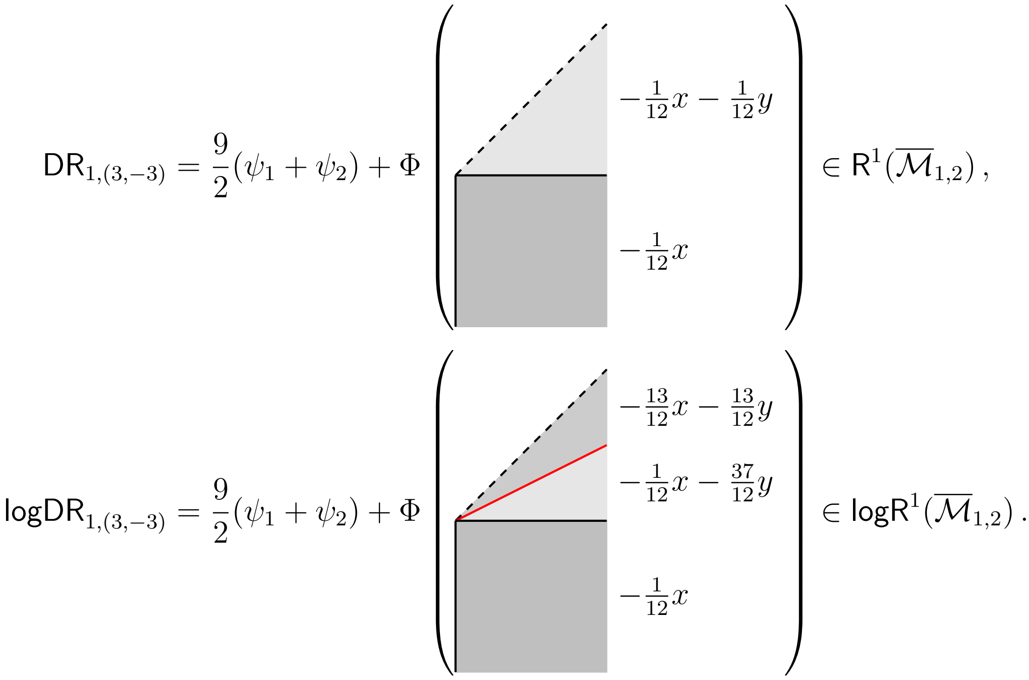


Figure 4: Formulas for the cycles $\mathsf{DR}_{1,(3,-3)} = \frac{9}{2}(\psi_1 + \psi_2) - \frac{1}{12}\delta_0$ and $\mathsf{logDR}_{1,(3,-3)}$, each consisting of a linear combination of classes ψ_1, ψ_2 from $\overline{\mathcal{M}}_{1,2}$ and a contribution from a piecewise linear function on a subdivision $\widetilde{\Sigma}$ of $\Sigma_{\overline{\mathcal{M}}_{1,2}}$ (the trivial subdivision in the case of the regular DR cycle). See Figure 2 for an explanation of the coordinates x, y on the cone stack, where we now only draw the coarse space of the stacky cone.

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2 Logarithmic Geometry

2.1 Overview

We recall here constructions from log geometry which will be needed later: tropicalization, Artin fans, and subdivisions. We will work throughout with fine saturated log schemes (in the sense of [44, 57]) over a point with trivial log structure of characteristic 0. The main case of interest for us is the moduli space $\overline{\mathcal{M}}_{g,n}$ equipped with the divisorial log structure obtained from $\partial\overline{\mathcal{M}}_{g,n}$.

2.2 Tropicalization

Let $X = (X, \mathsf{M}_X)$ be a log smooth algebraic stack with a logarithmic structure. To X, we can assign a combinatorial shadow of the *characteristic monoid*

$$\overline{\mathsf{M}}_X = \mathsf{M}_X/\mathcal{O}_X^*,$$

called the *tropicalization* of X, and denoted by Σ_X. The tropicalization Σ_X is a stack over the category of rational polyhedral cones[16], but we can think of Σ_X as simply a collection of rational polyhedral cones glued along common (not necessarily proper) faces. The theory of *cone stacks*, stacks over rational polyhedral cones, is developed at length in [18], where the details of the construction of Σ_X can be found.

For the definition of Σ_X, we recall that $\overline{\mathsf{M}}_X$ is constructible and therefore stratifies X. The tropicalization is easy to describe when X is "sufficiently local" and, in general, is constructed by gluing such local constructions together. Local here is with respect to the log structure. The most local case as far as the characteristic monoid $\overline{\mathsf{M}}_X$ is concerned is when X is an *atomic* log scheme: X has a unique closed stratum, and for every point x in the closed stratum,

$$\Gamma(X, \overline{\mathsf{M}}_X) = \overline{\mathsf{M}}_{X,x}\,.$$

Atomic neighborhoods exist on a log smooth algebraic stack with a log structure smooth-locally. In the atomic case, Σ_X is simply the rational polyhedral cone

$$\sigma_{X,x} = (\mathrm{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{R}_{\geq 0}), \mathrm{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{N}))\,,$$

[16]The latter modification is necessary to capture both the stack structure of X and the monodromy in $\overline{\mathsf{M}}_X$.

where we view a rational polyhedral cone as a usual cone in a real vector space together with an integral structure (the intersection of the cone with a lattice that spans the vector space). In general, we present X as a colimit

$$\varinjlim X_i \cong X \tag{20}$$

with each X_i atomic, and we define

$$\Sigma_X = \varinjlim \Sigma_{X_i}\,. \tag{21}$$

For a map $f : X_i \to X_i'$ in the colimit (20), the associated map $\Sigma_{X_i} \to \Sigma_{X_i'}$ is induced by the map

$$\overline{\mathsf{M}}_{X_i',x_i'} \cong \Gamma(X_i', \overline{\mathsf{M}}_{X_i'}) \xrightarrow{f^*} \Gamma(X_i, \overline{\mathsf{M}}_{X_i}) \cong \overline{\mathsf{M}}_{X_i,x_i}$$

of monoids. The colimit (21) is taken in the 2-category of stacks over rational polyhedral cones.

To understand the above construction, there are two examples to keep in mind. The first is when X is a toric variety. Then, Σ_X is the fan of X (though Σ_X is not embedded in the cocharacter lattice of the torus[17]).

The second example is when X is the log scheme associated to a nonsingular variety V with a normal crossings divisor D. When D is normal crossings in the Zariski topology, Σ_X is the cone over the dual intersection complex of the irreducible components D_i of D. Concretely, each stratum of X, which corresponds to a connected component of a intersection

$$D_{i_1} \cap D_{i_2} \cdots \cap D_{i_n}$$

of irreducible components, contributes the cone

$$(\mathbb{R}^n_{\geq 0}, \mathbb{N}^n)$$

in Σ_X. These rational polyhedral cones are glued together in the obvious way: a cone becomes a face of every cone corresponding to a deeper stratum (further non-empty intersections of the D_i). In the Zariski normal crossings case, Σ_X does not have non-trivial stack structure.

In general, D may only be normal crossings étale locally: some D_i may self-intersect. Then, for integers $k > 1$, there can be strata which étale locally are the intersections of k distinct divisors $D_{i,1}, \ldots, D_{i,k}$ which are globally the k branches of a single divisor D_i coming together. Thus, intersections of the form

$$\bigcap_{j=1}^{n} D_{i_j,1} \cap \cdots \cap D_{i_j,k_j}$$

[17]For example, if the toric variety has torus factors, the cocharacter lattice cannot be recovered from Σ_X.

can also be strata of X. The monodromy group G of the branches (which permutes the branches upon transversing loops in the stratum) acts on

$$\mathbb{R}_{\geq 0}^{k_1+\cdots+k_n},$$

and the stack quotient has the same interior as the corresponding stacky cone of Σ_X.

Convention. It is cumbersome to carry both the cones and the integral structures around everywhere in the notation. Thus, we will simply indicate the cones σ, which should be always understood as coming with an implicit integral structure. If σ is a cone with integral structure $P \subset \sigma$, we define

$$\overline{\mathsf{M}}_\sigma = \mathsf{Hom}(P, \mathbb{N}).$$

In particular, for a sharp monoid[18] P with associated cone

$$\sigma = (\mathsf{Hom}(P, \mathbb{R}_{\geq 0}), \mathsf{Hom}(P, \mathbb{N}))$$

we have $\overline{\mathsf{M}}_\sigma = P$.

2.3 Artin fans

While the tropicalization Σ_X is a purely combinatorial object, it can be promoted to a functor on log schemes. For an atomic log scheme X, there exists a unique algebraic stack $\mathcal{A}_X$ ([1, 58]) characterized by the property

$$\mathsf{Hom}(S, \mathcal{A}_X) = \mathsf{Hom}_{\text{Cone Stacks}}(\Sigma_S, \Sigma_X).$$

The stack $\mathcal{A}_X$ is simply representable by

$$\mathcal{A}_X = [\operatorname{Spec} \mathbb{C}[\overline{\mathsf{M}}_{X,x}] / \operatorname{Spec} \mathbb{C}[\overline{\mathsf{M}}_{X,x}^{\mathsf{gp}}]],$$

We can then lift this construction to all algebraic stacks with log structure by sheafifying the resulting functor with respect to the strict smooth topology. Doing so, one finds that the functor determined by Σ_X is representable by an algebraic stack with a log structure, called the *Artin fan* of X. The Artin fan can also be constructed directly from X in essentially the same way as Σ_X, except in the category of algebraic stacks with a log structure instead of the category of stacks over rational polyhedral cones. Namely, we present a general log algebraic stack X as a colimit

$$X = \varinjlim X_i$$

[18] A monoid where the only invertible element is the identity.

by atomic log schemes, and define

$$\mathcal{A}_X = \varinjlim \mathcal{A}_{X_i}$$

in the category of algebraic stacks with a log structure.

More formally, an Artin fan is any algebraic stack with a log structure which is logarithmically étale over a point (in particular, every Artin fan is 0-dimensional). It is shown in [18] that the category of Artin fans is equivalent to the category of stacks over rational polyhedral cones. We do not review this, but mention that it is a straightforward consequence of the following simple fact: if X and Y are atomic, then

$$\mathrm{Hom}_{\mathrm{LogStacks}}(\mathcal{A}_X, \mathcal{A}_Y) = \mathrm{Hom}_{\mathrm{Cone\ Complexes}}(\Sigma_X, \Sigma_Y)\,.$$

Here, $\mathcal{A}_X$ is the Artin fan corresponding to Σ_X under the equivalence of categories above (hence the notions $\mathcal{A}_X$ and Σ_X are equivalent).

The reason to consider both $\mathcal{A}_X$ and Σ_X is that the combinatorial constructions we perform are most easily understood in terms of the cone stack Σ_X. On the other hand, there is a canonical map

$$X \to \mathcal{A}_X\,.$$

This map, together with the equivalence between Σ_X and $\mathcal{A}_X$, allows us to lift our combinatorial constructions to algebraic constructions on X.

2.4 Subdivisions

The tropicalization Σ_X of a logarithmic algebraic stack X provides access to one of the most important operations in the subject.

Definition 2. A *subdivision* of a rational polyhedral cone σ is a fan $F \subseteq \sigma$ whose support $\mathsf{Supp}(F)$ is equal to that of σ. ◊

Definition 3. A *subdivision* of a cone stack Σ_X is a map $\widetilde{\Sigma}_X \to \Sigma_X$ from a cone stack $\widetilde{\Sigma}_X$ such that for every map $\sigma \to \Sigma_X$ from a rational polyhedral cone σ, the fiber product $\widetilde{\Sigma}_X \times_{\Sigma_X} \sigma \to \sigma$ is a subdivision. ◊

Remark 4. Alternatively, a subdivision of a cone stack Σ_X can be defined by descent. The stack Σ_X comes with a presentation as a colimit

$$\Sigma_X = \varinjlim_{i \in \mathcal{C}} \Sigma_{X_i}$$

with Σ_{X_i} rational polyhedral cones, $\mathcal{C}$ a poset, and where all maps $\Sigma_{X_i} \to \Sigma_{X_j}$ in the colimit are isomorphisms to a (non-necessarily proper) face. Every map from a cone $\sigma \to \Sigma_X$ necessarily factors through one of the Σ_{X_i}, and so subdivisions of Σ_X are the same as subdivisions of each Σ_{X_i} that respect all maps in the diagram. In particular, subdivisions respect identifications of faces and automorphisms in Σ_X. ♢

A subdivision $\widetilde{\Sigma}_X \to \Sigma_X$ can be pulled back to X via the Artin fan. Under the equivalence of categories between cone stacks and Artin fans, the map $\widetilde{\Sigma}_X \to \Sigma_X$ induces a proper, representable and birational map of Artin fans $\widetilde{\mathcal{A}}_X \to \mathcal{A}_X$ which we can pull back to X:

$$\widetilde{X} = X \times_{\mathcal{A}_X} \widetilde{\mathcal{A}}_X \to X\,.$$

The map $\widetilde{X} \to X$ is proper and representable. When X is log smooth, the map $X \to \mathcal{A}_X$ is smooth, and $\widetilde{X} \to X$ is also birational.

Definition 5. A map $\widetilde{X} \to X$ of the form $X \times_{\mathcal{A}_X} \widetilde{\mathcal{A}}_X \to X$ is a *log modification* of X. ♢

The functor of points of the log modification is characterized by the following result, which is essentially standard [45, 55]. As we could not find a precise reference covering the required level of generality, we include a proof for completeness.

Lemma 6. *Let $p : \widetilde{X} \to X$ be a log modification. Then, the following two properties hold:*

(i) p is a representable monomorphism of logarithmic algebraic stacks,

(ii) a log map $S \to X$ lifts to $\widetilde{X}$ if and only if, étale locally on S, the map[19] *$\Sigma_S \to \Sigma_X$ lifts through $\widetilde{\Sigma}_X$.*

Proof. Since $\widetilde{X} = X \times_{\mathcal{A}_X} \widetilde{\mathcal{A}}_X$, it suffices to fix a map $f : S \to \mathcal{A}_X$ and prove the properties for

$$q : \widetilde{\mathcal{A}}_X \to \mathcal{A}_X$$

instead of p.

For property (i), we can work étale locally on S. In particular, we may assume throughout that S and X are atomic, with global charts given by sharp f.s. monoids Q and P respectively. Let

$$\sigma = \mathsf{Hom}(P, \mathbb{R}_{\geq 0})$$
$$\tau = \mathsf{Hom}(Q, \mathbb{R}_{\geq 0})\,,$$

[19]The map $\Sigma_S \to \Sigma_X$ may only exist étale locally on S.

and let $N = \mathsf{Hom}(P, \mathbb{Z})$, $L = \mathsf{Hom}(Q, \mathbb{Z})$. Then, Σ_S is the single cone τ in the lattice L, and Σ_X is the single cone σ in the lattice N. The toric variety associated to σ in N is $\operatorname{Spec} k[P] = V(\sigma, N)$, the Artin fan is

$$\mathcal{A}_X = [\operatorname{Spec} k[P]/\operatorname{Spec} k[P^{\mathrm{gp}}]] = [V(\sigma, N)/T]\,,$$

the global quotient of $V(\sigma, N)$ by its dense torus, and

$$\widetilde{\mathcal{A}}_X = [V(\widetilde{\sigma}_X, N)/T]$$

is the quotient of the toric variety defined by the fan $\widetilde{\Sigma}_X$ in N by its dense torus. Logarithmic maps $S \to \mathcal{A}_X$ are then very simple, with

$$\mathsf{Hom}(S, \mathcal{A}_X) = \mathsf{Hom}_{\mathrm{Mon}}(P, Q) = \mathsf{Hom}_{\mathrm{Cones}}(\Sigma_S, \Sigma_X)\,.$$

Furthermore, $\widetilde{\mathcal{A}}_X$ has an open cover by the global quotients

$$\widetilde{\mathcal{A}}_i = [V(\sigma_i, N)/T]\,,$$

where the σ_i are the cones of $\widetilde{\Sigma}_X$. If $\sigma_i < \sigma_j$ then $\widetilde{\mathcal{A}}_i \subset \widetilde{\mathcal{A}}_j$.

Suppose now that we are given two maps $g, h : S \to \widetilde{\mathcal{A}}_X$ lifting f. As we are working locally on S, we can assume that g factors through $\widetilde{\mathcal{A}}_i$ and h through $\widetilde{\mathcal{A}}_j$ for two cones σ_i, σ_j. Since

$$\mathsf{Hom}(S, \widetilde{\mathcal{A}}_i) = \mathsf{Hom}(\Sigma_S, \sigma_i)$$

and similarly for $\widetilde{\mathcal{A}}_j$ and the interiors of the cones σ_i partition σ, we see that the only way

$$q \circ g = q \circ h$$

can agree is if the maps $S \to \widetilde{\mathcal{A}}_i$ and $S \to \widetilde{\mathcal{A}}_j$ factor through $\widetilde{\mathcal{A}}_k$ for $k < i$ and $k < j$. But since $\sigma_k \to \sigma$ is a monomorphism, we must have $g = h$. Thus q is a monomorphism.

Once property (i) is established, to prove property (ii), we need only study the lifting of f étale locally on S. If

$$g : S \to \widetilde{\mathcal{A}}_X$$

is a lift f, then g will factor through one of the $\widetilde{\mathcal{A}}_i$, and so the map $\Sigma_S \to \Sigma_X$ will factor through $\widetilde{\Sigma}_X$. Conversely, suppose $\Sigma_S \to \Sigma_X$ factors through $\widetilde{\Sigma}_X$. The factorization determines maps $S \to \widetilde{\mathcal{A}}_i \to \widetilde{\mathcal{A}}_X$. ⧫

Remark 7. When $X = (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$, the notions discussed here precisely coincide with the discussion of Section 1.3 of the Introduction. In particular, the definition of log Chow rings there also generalizes in a straightforward way to an arbitrary log smooth X:

$$\mathsf{logCH}(X) = \varinjlim_{X' \to X} \mathsf{CH}^*_{\mathrm{op}}(X') .$$

where $X' \to X$ ranges over all log modifications of X. $\Diamond$

2.5 Piecewise linear functions

If X is a logarithmic algebraic stack, we define a piecewise linear function on X to be a global section

$$\alpha \in H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) .$$

There is a natural interpretation of α as a piecewise linear function on the cone stack Σ_X: for $x \in X$, restriction gives $\alpha_x \in \overline{\mathsf{M}}_{X,x}^{\mathsf{gp}}$ which induces an evaluation map

$$\mathsf{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{R}_{\geq 0}) \to \mathbb{R} , \quad \varphi \mapsto \varphi(\alpha_x) ,$$

where φ is extended in the obvious way from $\overline{\mathsf{M}}_{X,x}$ to $\overline{\mathsf{M}}_{X,x}^{\mathsf{gp}}$. Recall from Remark 13 that the cone $\mathsf{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{R}_{\geq 0})$ covers the cone σ_x in Σ_X. The above functions then descend to a piecewise linear function on Σ_X.

The construction induces an isomorphism between the group of (combinatorial) PL functions on Σ_X and the group

$$H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

Remark 8. In [39], the ring of strict piecewise polynomials on X was defined as the ring of global sections of the sheaf

$$\mathrm{Sym}^\bullet_{\mathbb{Q}} \overline{\mathsf{M}}_X^{\mathsf{gp}} .$$

In particular the degree 1 part is given by

$$H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q} ,$$

the same as the group of linear functions defined above. Locally on X the ring $\mathrm{Sym}^\bullet_{\mathbb{Q}} \overline{\mathsf{M}}_X^{\mathsf{gp}}$ is just the ring of polynomials in linear functions, and the same holds for the combinatorially-defined ring of piecewise polynomial functions, so in fact the definition from [39] coincides with that of the present paper in all degrees. We will not make use of this comparison in the present paper, but include it for completeness. $\Diamond$

If $\alpha \in H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}})$ is a PL function, we define the $\mathcal{O}_X^\times$-torsor $\mathcal{O}_X^\times(\alpha)$ to be the preimage of α under the natural map $\mathsf{M}_X^{\mathsf{gp}} \to \overline{\mathsf{M}}_X^{\mathsf{gp}}$. The preimage is an $\mathcal{O}_X^\times$-torsor because of the exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathsf{M}_X^{\mathsf{gp}} \to \overline{\mathsf{M}}_X^{\mathsf{gp}} \to 0\,.$$

Definition 9. Let α be a PL function on X. We define $\mathcal{O}_X(\alpha)$ to be the line bundle obtained from $\mathcal{O}_X^\times(\alpha)$ by glueing in the *infinity* section. ◊

If $\alpha \leq \alpha'$, in the sense that

$$\alpha' - \alpha \in H^0(X, \overline{\mathsf{M}}_X) \subset H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}})\,,$$

then we have a natural map of line bundles

$$\mathcal{O}_X(\alpha) \to \mathcal{O}_X(\alpha')\,.$$

More generally, if $\mathcal{L}$ is a line bundle on X, we define $\mathcal{L}(\alpha) = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha)$.

Remark 10. The induced homomorphism

$$H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) \to \mathsf{Pic}(X) \to \mathsf{CH}^1(X)$$

coincides with the degree 1 part of the homomorphism

$$\Phi\colon \mathsf{sPP}(\Sigma_X) \to \mathsf{CH}(X)$$

from (10). ◊

3 Logarithmic Curves

3.1 Definitions

Up to now, we have been discussing *absolute* logarithmic geometry. We now turn our attention to the simplest relative situation: *logarithmic curves*.

Definition 11. Let S be a log scheme. A *logarithmic curve* $C \to S$ over S is a log smooth, proper, connected, integral, and vertical morphism with reduced geometric fibers which are of pure dimension 1. ◊

Briefly, the adjective *integral* concerns the flatness of $C \to S$, and the adjective *vertical* means that we do not put additional log structure on the marked points of the family. However, we do not formally review the relevant terminology, and refer the reader to [43] for a discussion. We only will use the most basic consequences of Definition 11: the map of underlying schemes of every log curve is a prestable curve, and the log structures on C and S give us access to a family of tropical curves.

More precisely, over every geometric point $s \in S$, the characteristic monoid $\overline{\mathsf{M}}_{C_s}$ of the fiber C_s satisfies:

- at the generic point c of an irreducible component, $\overline{\mathsf{M}}_{C_s,c} = \overline{\mathsf{M}}_{S,s}$,
- at a node q, there exists a unique (non-zero) element $\ell_q \in \overline{\mathsf{M}}_{S,s}$ such that
$$\overline{\mathsf{M}}_{C_s,q} = \overline{\mathsf{M}}_{S,s} \oplus_{\mathbb{N}} \mathbb{N}^2 \,,$$
where the homomorphism $\mathbb{N} \to \mathbb{N}^2$ is the diagonal, and the homomorphism $\mathbb{N} \to \overline{\mathsf{M}}_{S,s}$ sends 1 to ℓ_q.

We encode the above data as a *tropical curve over S*.

Definition 12. Let M be a sharp monoid. A *tropical curve* metrized by M is the data of a connected graph Γ together with elements $\ell(e) \in M - 0$ for each edge of e. We call $\ell(e)$ the *length* of e. ♢

For every log curve C/S and geometric point $s \in S$, we obtain a *tropical curve metrized by* $\overline{\mathsf{M}}_{S,s}$ consisting of the dual graph Γ_s of C_s with each edge e of Γ_s corresponding to a node $q \in C_s$ assigned the *length*

$$\ell(e) = \ell_q \in \overline{\mathsf{M}}_{S,s} \,.$$

Since the above data comes from a log map $C \to S$, it is severely constrained. First of all, the tropical curve $(\Gamma_s, \ell(e))$ is étale locally constant along strata of S. Furthermore, for each étale specialization $\zeta : t \rightsquigarrow s$, there is an induced map

$$p_\zeta^\sharp \colon \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{S,t}$$

and an induced *edge contraction* map

$$f_\zeta : \Gamma_s \to \Gamma_t$$

which is compatible with $p_\zeta^\sharp$. If an edge e of Γ_s has length mapping to 0 then the edge is contracted, otherwise it is sent to an edge with length $p_{s,t}^\sharp(\ell(e))$.

Remark 13. The connection of Definition 12 with the constructions in Section 2.2 is as follows. The strata of S are in bijection with the cones of the cone stack Σ_S. For a geometric point $s \in S$ in a given stratum O_s, the corresponding stacky cone σ_s in Σ_S has a cover by the rational polyhedral cone $\mathsf{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0})$. The étale specialisations $\zeta\colon t \rightsquigarrow s$ correspond to the gluing relations and automorphisms that produce σ_s out of the cover $\mathsf{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0})$.

The graphs Γ also appear naturally in two ways: as the dual graphs of the various fibers of $C \to S$ and as the fibers of the induced map of cone stacks $\Sigma_C \to \Sigma_S$. The latter fibers have the same combinatorial type in the interior of each cone of Σ_S, but their edge lengths vary. The parameter $\ell(e) \in \overline{\mathsf{M}}_{S,s}$ associated to an edge of a dual graph induces by evaluation a homomorphism

$$\ell(e)\colon \mathsf{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0}\,.$$

The compatibility of the $\ell(e)$ with étale specializations means that these homomorphisms descend to Σ_S, and under this interpretation, they measure the length of the edge in the fibers of $\Sigma_C \to \Sigma_S$ as x moves in Σ_S. $\diamondsuit$

3.2 Subdivisions of log curves

Let $C \to S$ be a logarithmic curve. As C is a logarithmic scheme, we may perform subdivisions on C, but the composition of a subdivision $C' \to C$ with $C \to S$ may no longer be a logarithmic curve (as $C' \to S$ may fail to be integral or have non-reduced fibers[20]).

The subdivisions $C' \to C$ which are logarithmic curves over S are special, and are essentially those for which the fibers of $\Sigma_{C'} \to \Sigma_S$ retain the same combinatorial type over interiors of cones of Σ_S. More precisely, in terms of the tropicalizations $(\Gamma_s, \ell(e) \in \overline{\mathsf{M}}_{S,s}, f_\zeta)$, the subdivisons $C' \to C$ which are logarithmic curves over S are exactly the subdivisions of the following form:

- (Subdivision of each fiber) For each geometric point $s \in S$, a subdivision of each edge e_s of Γ_s into a union of edges $e(i)_s$ of length $\ell(e(i)_s) \in \overline{\mathsf{M}}_{S,s} - 0$, with
$$\ell(e_s) = \sum \ell(e(i)_s)\,.$$
- (Compatibility with generization) For each étale specialisation $\zeta : t \rightsquigarrow s$ and edge e_t mapping to e_s under $E(\Gamma_t) \to E(\Gamma_s)$, a bijection φ between the edges $e(i)_s$ for which $\ell(e(i)_s)$ does not map to 0 under the map $p_\zeta^\sharp : \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{S,t}$ and the edges $e(j)_t$ subdividing e_t, so that $p_\zeta^\sharp(\ell(e(i)_s)) = \ell(\varphi(e(i)_s)) \in \overline{\mathsf{M}}_{S,t}$.

Even though $C' \to S$ is not a log curve for every subdivision $C' \to C$, we have:

[20]In good cases, when S is log smooth, $C' \to C$ is essentially a blow-up, and integrality is equivalent to flatness. Then, the pathology can be rephrased as *even though $C \to S$ is flat, the blow-up C' may fail to be flat over S*. Moreover, even if $C' \to S$ is flat, the fibers need not be reduced.

(i) $C \times_S S' \to S'$ is a log curve for every subdivision $S' \to S$ (as the definition of a log curve is stable under base change with respect to any log map $S' \to S$).

(ii) Subdivisions by log curves are in a sense cofinal among subdivisions of the total space C: for any subdivision $C' \to C$, there exists a log alteration[21] $S' \to S$ and a further subdivision $C'' \to C' \times_S S'$ such that $C'' \to S'$ is a log curve.

We will not use (ii) in the paper.

A special class of subdivisions of log curves $C \to S$ will be central in what follows.

Definition 14 (Quasi-stable models)**.** A quasi-stable model of a log curve $C \to S$ is a subdivision $\widehat{C} \to C$ such that $\widehat{C} \to S$ is a log curve and, fiberwise, the exceptional locus of $\widehat{C} \to C$ consists of isolated rational curves (every connected component of the exceptional locus is a nonsingular rational curve). $\Diamond$

In particular, on the level of the dual graphs Γ_s, a quasi-stable model has at most one exceptional vertex on each edge. We have chosen the terminology quasi-stable model because, when the underlying family $C \to S$ is a family of stable curves, the curve $\widehat{C} \to S$ is called quasi-stable in the literature.

3.3 Abel-Jacobi theory on a tropical curve

The main notions of classical Abel-Jacobi theory have tropical analogues. Fix a tropical curve Γ metrized by a sharp monoid M. A *divisor* on a tropical curve Γ is an element of $\mathbb{Z}^{V(\Gamma)}$. Following (13), a flow on Γ induces a divisor by taking outgoing slopes,

$$\mathrm{div} : \mathsf{Flow}(\Gamma) \to \mathbb{Z}^{V(\Gamma)} .$$

Flows and divisors are essentially combinatorial notions: they depend only on the graph Γ and not the metric structure of Γ. On the other hand, the analogue of a rational function on a curve is a *piecewise linear* function, which requires the metric structure.

Definition 15. A *piecewise linear* function on Γ is a function

$$F \colon V(\Gamma) \to M^{\mathsf{gp}}$$

[21] A log alteration is a composition of a log modification with a root stack along the log structure.

satisfying the following property: for every directed edge $\vec{e}$ from vertex u to vertex v, there is an integer $s(\vec{e}) \in \mathbb{Z}$, the *slope of F along $\vec{e}$*, such that

$$F(v) - F(u) = s(\vec{e})\ell(e). \tag{22}$$

We will use the abbreviation PL for piecewise linear, and we write $\mathsf{PL}(\Gamma)$ for the group of PL functions on Γ. ♢

By taking slopes, we obtain a homomorphism

$$\mathsf{PL}(\Gamma) \to \mathsf{Flow}(\Gamma).$$

Composing with the homomorphism div yields a homomorphism

$$\mathsf{PL}(\Gamma) \to \mathsf{Div}(\Gamma)$$

which we also denote by div. Explicitly, if $s(\vec{e})$ is the slope of F along the oriented edge $\vec{e}$, we have

$$\operatorname{div}(F) = \left(\sum_{\vec{e}\to v} s(\vec{e})\right)_{v\in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)}.$$

We say a divisor is *principal* if it is the divisor of some PL function. Two divisors are *linearly equivalent* if their difference is principal. Whether a divisor is principal depends strongly on the monoid M (via the edge lengths), and *not* just on the underlying graph Γ.

If two PL functions have the same slopes, then they differ by addition of a global constant. The following result determines when an (acyclic) flow arises as the slopes of a PL function.

Lemma 16. *Let f be a flow on Γ. Then f arises as the slopes of a PL function if and only if, for every cycle γ in Γ, we have*

$$\langle \gamma, f\rangle_\ell = 0,$$

where we induce a flow from γ by identifying $H_1(\Gamma)$ as the kernel of div *as in* (13)*, and where $\langle -,-\rangle_\ell$ is the pairing defined*[22] *defined by* (14).

[22]The definition

$$\langle f, g\rangle_\ell = \frac{1}{2}\sum_{\vec{e}\in\vec{E}(\Gamma)} \ell(e)f(\vec{e})g(\vec{e}) \tag{23}$$

now takes values in M^{gp}. The prefactor $\frac{1}{2}$ is placed to count every edge contribution only once.

Proof. The statement is well-known. That the pairing vanishes on the slopes of a PL function, we leave to the reader. For the converse, start by choosing any value for F at a particular vertex. By moving around the graph, the condition (22) determines the value of F at every other vertex. The method can fail if two different ways of reaching a vertex were to yield different answers, which is precisely precluded by the vanishing of the pairing. ♦

In particular, if a flow f arises as the slopes of a PL function, then f must be acyclic: starting at a vertex v in the cycle, the values $F(w)$ of a PL function F must strictly increase[23] when traversing edges with positive slope, but must return to $F(v)$ when traversing the whole cycle.

3.4 Algebraizing tropical Abel-Jacobi theory

Logarithmic geometry allows us to lift the combinatorics of tropical Abel-Jacobi theory to algebraic geometry.

Following Section 2.5, a *piecewise linear* function on a log curve $C \to S$ is a global section of $\overline{\mathsf{M}}_C^{\mathsf{gp}}$. Over a geometric point s of S, such a PL function

$$\alpha \in H^0(C, \overline{\mathsf{M}}_C^{\mathsf{gp}})$$

induces a combinatorial PL function (in the sense of Definition 15) on the tropical curve, by sending a vertex v of Γ_s to the value of α at the generic point of the irreducible component of C_s corresponding to v.

When S is a geometric point, this gives a bijection between combinatorial PL functions and $H^0(C, \overline{\mathsf{M}}_C^{\mathsf{gp}})$. On the other hand, suppose that we are given, for each geometric point s of S, a combinatorial PL function

$$F_s \colon V(\Gamma_S) \to \overline{\mathsf{M}}_{S,s}^{\mathsf{gp}}\,.$$

Suppose, moreover, that the F_s are compatible with étale specialisation: whenever $\zeta\colon t \rightsquigarrow s$ is an étale specialisation inducing

$$p_\zeta^\# \colon \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{S,t}$$

and

$$f_\zeta \colon \Gamma_s \to \Gamma_t\,,$$

we have

$$p_\zeta^\# F_s(v) = F_t(f_\zeta(v))\,.$$

[23]Here the order on M^{gp} is induced by the inclusion $M \subset M^{\mathsf{gp}}$: given $x, y \in M^{\mathsf{gp}}$ we say $x \le y \in M^{\mathsf{gp}}$ if $y - x \in M \subset M^{\mathsf{gp}}$.

Then there is a unique global section

$$\alpha \in H^0(C, \overline{\mathsf{M}}_C^{\mathsf{gp}})$$

inducing the combinatorial PL functions F_s.

The geometric connection of the two constructions is the following: The PL function α induces a line bundle $\mathcal{O}(\alpha)$ on C, as in Definition 9. Then, the multi-degree of $\mathcal{O}(\alpha)$ on C_s is opposite to the tropical divisor associated to F_s:

$$\underline{\deg}\mathcal{O}(\alpha)|_{C_s} = -\mathsf{div}(F_s) \tag{24}$$

This equality is in particular responsible for our convention to glue the infinity rather than the zero section to the torsor $\mathcal{O}_C^\times(\alpha)$.

4 Stability conditions

4.1 Definitions

The definitions related to stability of line bundles here are adapted from [51]. Let

$$\pi : C \to S$$

be a logarithmic curve.

Definition 17. A *stability condition θ of degree d for π* consists of a function

$$\theta : V(\Gamma_s) \to \mathbb{R}$$

for each geometric point $s \in S$, which satisfies:

(i) For $s \in S$, we have $\sum_{v \in V(\Gamma_s)} \theta(v) = d$.

(ii) For every étale specialisation $\zeta \colon t \rightsquigarrow s$, the stability condition θ respects the induced map $\Gamma_s \to \Gamma_t$, in the sense that

$$\theta(w) = \sum_{v_i} \theta(v_i)$$

whenever $v_1, \cdots, v_n$ are the vertices of Γ_s mapping to $w \in \Gamma_t$. ◊

For the universal family over the moduli space of stable curves,

$$\pi : C \to S = \overline{\mathcal{M}}_{g,n} ,$$

a stability condition of the form of Definition 17 is exactly a stability condition in the sense of Section 1.6.

If $T \to S$ is a logarithmic map, and $\widehat{C} \to C \times_S T = C_T$ is a quasi-stable model for C_T/T, then we can lift a degree-d stability condition θ for C/S canonically to a degree-d stability condition $\widehat{\theta}$ for $\widehat{C}/T$ by setting the value of $\widehat{\theta}$ to zero on every exceptional vertex of the dual graph of every fiber $\widehat{C}_t$ over every geometric point $t \in T$ (and setting the value of $\widehat{\theta}$ to equal the value of θ on all other vertices).

Definition 18. Let $T \to S$ be a logarithmic map. A line bundle $\mathcal{L}$ on a quasi-stable model $\widehat{C} \to C_T$ is *admissible* if for any geometric point $t \in T$ and exceptional component E of $\widehat{C}_t \to (C_T)_t$, the degree of $\mathcal{L}$ on E is 1. ♢

The line bundle $\mathcal{L}$ tropicalizes to a divisor on the fibers of $\widehat{C} \to C_T$: for each geometric point $t \in T$, the divisor is given by the multidegree

$$\underline{\deg}(\mathcal{L}|_{\widehat{C}_t}) \in \mathbb{Z}^{V(\widehat{\Gamma}_t)} = \mathrm{Div}(\widehat{\Gamma}_t)\,.$$

A line bundle $\mathcal{L}$ is then admissible if and only if the associated tropical divisor has degree 1 on every exceptional vertex.

Definition 19. Let $T \to S$ be a logarithmic map, and let $\mathcal{L}$ be an admissible line bundle on a quasi-stable model $\widehat{C} \to C_T$ of C_T/T. Let θ be a stability condition for C/S with lift $\widehat{\theta}$ to $\widehat{C}/T$. Then, $\mathcal{L}$ is *θ-semistable* if, for every geometric point $t \in T$, the associated tropical divisor $\underline{\deg}(\mathcal{L}|_{\widehat{C}_t})$ on the graph $\widehat{\Gamma}_t$ of $\widehat{C}_t/t$ satisfies the condition:

$$\widehat{\theta}(G) - \frac{\mathrm{val}(G)}{2} \leq \underline{\deg}(\mathcal{L}|_{\widehat{C}_t})(G) \leq \widehat{\theta}(G) + \frac{\mathrm{val}(G)}{2}$$

for every subset $G \subseteq V(\widehat{\Gamma}_t)$, where

$$\widehat{\theta}(G) = \sum_{v \in G} \widehat{\theta}(v)\,, \quad \underline{\deg}(\mathcal{L}|_{\widehat{C}_t})(G) = \sum_{v \in G} \underline{\deg}(\mathcal{L}|_{\widehat{C}_t})(v)$$

and $\mathrm{val}(G)$ is the valence of G, defined as the number of edges connecting G and its complement.

An admissible line bundle $\mathcal{L}$ on $\widehat{C}/T$ is *θ-stable* if the inequalities of Definition 19 are strict except possibly when G or its complement are unions of exceptional vertices. ♢

In Section 1.6, we cast the stability condition in terms of sub-curves which correspond here exactly to subsets of the vertices.

Definition 20. A stability condition θ for C/S is

- *nondegenerate* if, for every logarithmic map $T \to S$ and quasi-stable model $\widehat{C} \to C_T$, every θ-semistable line bundle $\mathcal{L}$ on $\widehat{C}/T$ is θ-stable,

- *small* if the trivial bundle is θ-semistable on C_T.

$\Diamond$

The category fibered in groupoids over $\mathbf{LogSch}/S$ which assigns to $T \to S$ the groupoid consisting of pairs of

(i) a quasi-stable model $\widehat{C} \to C_T$,

(ii) an admissible line bundle $\mathcal{L}$ on $\widehat{C} \to T$

forms an algebraic stack. Imposing the condition that $\mathcal{L}$ is θ-semistable defines open substack which we denote by $\mathbf{P}^\theta_\pi$, or $\mathbf{P}^\theta$ when $\pi : C \to S$ is understood. The relative inertia of $\mathbf{P}^\theta_\pi$ over S is given by $\mathbb{G}_m$. The associated rigidification over S, denoted by $\mathcal{P}^\theta_\pi$, is a proper algebraic space over S. Both $\mathbf{P}^\theta_\pi$ and $\mathcal{P}^\theta_\pi$ are log smooth over S, and their tropicalizations parametrize θ-semistable tropical divisors on the tropicalization $\Sigma_C \to \Sigma_S$ (see [51] for details).

Remark 21. According to the definition of θ-semistability, the trivial bundle is *not* semistable on quasi-stable models $\widehat{C} \to C_T$ that have exceptional components, as its degree fails to be 1 on them. Instability is forced here as otherwise there would be no chance of getting a separated space $\mathcal{P}^\theta_\pi$: over a discrete valuation ring T with generic point η, both $(C_T, \mathcal{O}_{C_T})$ and $(\widehat{C}, \mathcal{O}_{\widehat{C}})$ would be limits of $(C_\eta, \mathcal{O}_{C_\eta})$. $\Diamond$

4.2 Nondegenerate stability conditions

4.2.1 Moduli of θ-stable bundles

Let $\pi : C \to S$ be a logarithmic curve together with a section x_1. Let

$$\mathcal{L} \to C$$

be a line bundle on C of degree[24] d. Let θ be a nondegenerate stability condition of degree d for C/S. We present here a generalization of the construction of $\overline{\mathcal{M}}^\theta_{g,A}$ in Section 1.7.2. The main ideas are already contained in [3], but in a setting less general than we require.

[24] Degree here means the degree of $\mathcal{L}$ on the fibers of π.

Definition 22. Let $S^\theta_\mathcal{L}$ be the fibered category over $\mathbf{LogSch}/S$ with objects tuples

$$(T/S, \widehat{C} \to C_T, \alpha) \tag{25}$$

where T is a log scheme over S, $\widehat{C} \to C_T$ is a quasi-stable model of C_T/T, and α is as sPL function on $\widehat{C}/T$ vanishing on x_1 and for which $\mathcal{L}_T(\alpha)$ is θ-stable. $\Diamond$

Standard arguments show that $S^\theta_\mathcal{L}$ is a stack in groupoids for the strict étale topology: a quasi-stable model of C_T is associated to a subdivision of Σ_{C_T}, and α is a piecewise linear function on Σ_{C_T} (both of these are by definition étale local on T).

We denote the universal quasi-stable model by

$$C^\theta \to C_{S^\theta_\mathcal{L}}$$

and the universal PL function on C^θ by α^θ.

Remark 23. Let $\mathcal{P}^\theta_\pi$ be the moduli space of θ-stable line bundles over S trivialized along the section x_1. There is a natural Abel-Jacobi map

$$\mathsf{aj}\colon S^\theta_\mathcal{L} \to \mathcal{P}^\theta_\pi \tag{26}$$

sending $(T/S, \widehat{C} \to C_T, \alpha)$ to the line bundle $\mathcal{L}_T(\alpha)$ on $\widehat{C}$. $\Diamond$

The definition of $S^\theta_\mathcal{L}$ is natural from the point of view of logarithmic geometry. While standard arguments show that $S^\theta_\mathcal{L}$ is a stack over $\mathbf{LogSch}/S$, what is not standard is whether $S^\theta_\mathcal{L}$ *can be represented by an algebraic stack with a log structure.* Theorem 24 gives an explicit description of $S^\theta_\mathcal{L}$ as a log modification of S, thus showing $S^\theta_\mathcal{L}$ is an algebraic stack with log structure.

Theorem 24. *The structure morphism $\rho : S^\theta_\mathcal{L} \to S$ is a log modification (obtained from a subdivision). In particular, ρ is proper, log étale, and relatively representable by logarithmic schemes. If S is log smooth then ρ is also birational.*

Remark 25. In Definition 22, the notation $S^\theta_\mathcal{L}$ is used to highlight the fact that $S^\theta_\mathcal{L}$ depends both on the stability condition θ and the line bundle $\mathcal{L}$. The proof of Theorem 24 will however show that $S^\theta_\mathcal{L}$ only depends on $\mathcal{L}$ through the associated multidegree $\underline{\deg}(\mathcal{L})$, which we can view as a tropical divisor on Σ_C. We could have thus equally written $S^\theta_{\underline{\deg}(\mathcal{L})}$. $\Diamond$

Theorem 24 generalizes the construction of

$$\rho : \overline{\mathcal{M}}^\theta_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

promised in (9). The following definition makes the latter claim precise.

Definition 26. Let $S = \overline{\mathcal{M}}_{g,n}$ with universal curve $\pi : C \to \overline{\mathcal{M}}_{g,n}$ carrying the standard log structure. Let

$$\mathcal{L} = (\omega_C^{\mathsf{log}})^{\otimes k}\left(-\sum_{i=1}^{n} a_i x_i\right)$$

be a line bundle on C with markings $x_1, \ldots, x_n$. For a nondegenerate stability condition θ for π, we define

$$\overline{\mathcal{M}}^{\theta}_{g,A} = (\overline{\mathcal{M}}_{g,n})^{\theta}_{\mathcal{L}}\,, \tag{27}$$

with Abel-Jacobi map

$$\mathsf{aj}\colon \overline{\mathcal{M}}^{\theta}_{g,A} \to \mathcal{P}^{\theta}_{g,n} \tag{28}$$

as in Remark 23. ♢

4.2.2 Proof of Theorem 24

We present an explicit construction of the subdivision of Σ_S which defines ρ. In the special case of Definition 26, where $S = \overline{\mathcal{M}}_{g,n}$, the subdivision reduces exactly to the construction in Section 1.7.2.

Let s be a geometric point of S and let $\sigma_s = \mathsf{Hom}(\overline{M}_{S,s}, \mathbb{R}_{\geq 0})$ be the corresponding cover of the stacky cone in Σ_S. By definition, a subdivision of Σ_S is a subdivision of the various cones σ_s as s ranges through S, which is compatible with all identifications of faces and automorphisms required in Σ_S. Let $\Gamma = \Gamma_s$ be the dual graph of C_s. We will write σ_Γ for σ_s to highlight the connection of the constructions below with Γ.

The logarithmic map $C \to S$ induces the structure of a tropical curve on Γ metrized by

$$\overline{\mathsf{M}}_{\sigma_\Gamma} = \overline{\mathsf{M}}_{S,s}$$

via the lengths $\ell(e) \in \overline{\mathsf{M}}_{S,s}$, as in Section 3. All together, the lengths $\ell(e)$ can be thought of as a homomorphism

$$\ell\colon \sigma_\Gamma \to \mathbb{R}^{E(\Gamma)}_{\geq 0}\,,$$

so we can write $(\ell(p))_e = \ell_e(p) = p(\ell_e)$. Let $\underline{\deg}(\mathcal{L})$ be the multidegree of $\mathcal{L}$ viewed as a tropical divisor on Γ. The cones of our subdivision in σ_Γ will be indexed by triples $(\widehat{\Gamma}, D, I)$ where

(i) $\widehat{\Gamma}$ is a quasi-stable graph with stabilization Γ,

(ii) D is a θ-stable divisor on $\widehat{\Gamma}$,

(iii) I an acyclic flow on $\widehat{\Gamma}$ satisfying $\operatorname{div}(I) = \underline{\deg}(\mathcal{L}) - D$.

We define the cone $\sigma_{\widehat{\Gamma}}$ by

$$\sigma_{\widehat{\Gamma}} = \sigma_\Gamma \times_{\mathbb{R}^{E(\Gamma)}_{\geq 0}} \mathbb{R}^{E(\widehat{\Gamma})}_{\geq 0} .$$

Concretely, a point of the cone $\sigma_{\widehat{\Gamma}}$ is given by

- a point $p \in \sigma_\Gamma$,
- for every edge e of Γ that is subdivided into edges e_1 and e_2 in $\widehat{\Gamma}$, a pair of non-negative real numbers $\widehat{\ell}(e_1)$ and $\widehat{\ell}(e_2)$ such that $\ell(p)_e = \widehat{\ell}(e_1) + \widehat{\ell}(e_2)$.

The cone $\sigma_{\widehat{\Gamma}}$ has relative dimension over σ_Γ equal to the number of exceptional vertices of $\widehat{\Gamma}$.

In the universal case, when $S = \overline{\mathcal{M}}_{g,n}$, the map ℓ is an isomorphism, $\sigma_{\widehat{\Gamma}} = \mathbb{R}^{E(\widehat{\Gamma})}_{\geq 0}$, and the projection $\sigma_{\widehat{\Gamma}} \to \sigma_\Gamma$ is the map

$$\mathsf{pr}\colon \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma\,, \quad \widehat{p} \mapsto p : e \mapsto \begin{cases} \widehat{p}(e_1) + \widehat{p}(e_2) & \text{if } \widehat{\Gamma} \text{ subdivides edge } e \text{ into } e_1, e_2, \\ \widehat{p}(e) & \text{otherwise,} \end{cases}$$

from Section 1.7.2.

We define a subcone $\tau_{\widehat{\Gamma},I} \subseteq \sigma_{\widehat{\Gamma}}$ by the condition

$$\tau_{\widehat{\Gamma},I} = \left\{ \widehat{p} \in \sigma_{\widehat{\Gamma}} \;\middle|\; \langle \gamma, I \rangle_{\widehat{p}} = 0 \text{ for all } \gamma \in H_1(\widehat{\Gamma}) \right\}\,, \tag{29}$$

where the pairing is (23) evaluated at the point $\widehat{p}$.

Lemma 27. *The map* $\mathsf{pr} : \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma$ *induces an isomorphism from the cone* $\tau_{\widehat{\Gamma},I}$ *to the image*

$$\sigma_{\widehat{\Gamma},I} = \mathsf{pr}(\tau_{\widehat{\Gamma},I}) \subseteq \sigma.$$

Moreover, the collection of cones $\{\sigma_{\widehat{\Gamma},I}\}$ *together with their faces forms a fan* $\widetilde{\Sigma}^\theta_\Gamma$ *with support* σ_Γ*.*

Proof. The claim is purely combinatorial and is proven by Abreu and Pacini [3] in a slightly more restricted setting (over the moduli stack of stable 1-marked curves with a particular stability condition). The arguments of [3] go through essentially unchanged in our more general setting.

The notation of [3] can be translated to ours. The cones $C_{\Gamma,\mathcal{E},\varphi}$ of [3, Definition 3.4] correspond to our cones $\tau_{\widehat{\Gamma},I}$, and the cones $K_{\Gamma,\mathcal{E},\varphi}$ are our $\sigma_{\widehat{\Gamma},I}$. By [3, Proposition 3.7], the map

$$\mathsf{pr} : \tau_{\widehat{\Gamma},I} \to \sigma_{\widehat{\Gamma},I}$$

is an isomorphism. That these cones fit together to a complete fan is proven in [3, Theorem 3.9]. The content of the proof of completeness is [2, Theorem 5.6]: every degree d tropical divisor is linearly equivalent to a unique θ-stable divisor on a unique quasi-stable subdivision. ♦

To show that the construction of Lemma 27 defines a global subdivision $\widetilde{\Sigma}^\theta$ of Σ_S, we need furthermore to check that the cones $\sigma_{\widehat{\Gamma},I}$ descend to Σ_S: the cones must fit together along all identifications of faces and automorphisms of the various cones σ_Γ which define Σ_S. The proof of [3, Theorem 3.9] again goes through unchanged in our setting to show that a cone $\sigma_{\widehat{\Gamma}',I'}$ is a face of $\sigma_{\widehat{\Gamma},I}$ whenever

- $\sigma_{\Gamma'}$ is a face of σ_Γ,
- $\widehat{\Gamma}'$ is obtained from $\widehat{\Gamma}$ by contracting some edges,
- $I'(e) = I(e)$ for all edges $e \in E(\widehat{\Gamma}') \subset E(\widehat{\Gamma})$.

Thus, the cones $\sigma_{\widehat{\Gamma},I}$ form a polyhedral complex and define a global subdivision of Σ_S. For the gluing, it is necessary that the data $\widehat{\Gamma}, I$ respects the monodromy and identifications of faces of σ_Γ. But this is precisely implied by condition (ii) in the definition of a stability condition.

Remark 28. The intuition behind the construction of the cones $\sigma_{\widehat{\Gamma},I}$ is as follows. Given I on $\widehat{\Gamma}$ and p in the interior of σ_Γ, it is in general impossible to both

(i) assign lengths to the edges of $\widehat{\Gamma}$ summing to the lengths on edges of Γ associated to $\ell(p)$,

(ii) find a piecewise linear function on $\widehat{\Gamma}$ whose slopes are given by I.

Indeed, by Lemma 16, the existence of such a lifting imposes several conditions on the lengths of the edges of $\widehat{\Gamma}$. The cone $\sigma_{\widehat{\Gamma}}$ is the universal cone over which the pullback of Γ admits a quasi-stable model of combinatorial type $\widehat{\Gamma}$. The subcones $\tau_{\widehat{\Gamma},I}$ are cut out inside $\sigma_{\widehat{\Gamma}}$ precisely by the linear conditions necessary to lift I into a piecewise linear function. Thus, the content of Lemma 27 is that the locus of points $p \in \sigma$ which support a quasi-stable model $\widehat{\Gamma}$ of Γ with a PL function with slopes I is a cone $\sigma_{\widehat{\Gamma},I}$, isomorphic to $\tau_{\widehat{\Gamma},I}$.

Furthermore, the cones for various choices $\widehat{\Gamma}, I$ that come from the stability condition θ subdivide σ_Γ: each p in the interior of σ_Γ belongs to the interior of precisely one such cone. Points p in the boundary of σ_Γ similarly belong to the interior of cones $\sigma_{\widehat{\Gamma}',I'}$, where Γ' is the contraction of Γ lying over p. Thus, the property of supporting a quasi-stable model with piecewise linear function with slopes I is cut out by inequalities on the lengths of the edges of Γ. ♢

We will show below in Lemma 29 that the log modification

$$\tilde{\rho} : \widetilde{S}^{\theta}_{\mathcal{L}} \to S$$

corresponding to the subdivision $\widetilde{\Sigma}^{\theta}$ of Σ_S is isomorphic over S to $S^{\theta}_{\mathcal{L}}$.

Let $f \colon T \to S$ be any logarithmic scheme over S, and let $t \in T$ be a geometric point mapping to $s \in S$. The log curve $C \to S$ pulls back to a log curve $C_T \to T$. The induced tropical curve around t has underlying graph $\Gamma = \Gamma_s$ metrized by the lengths $f^{\sharp}(\ell(e)) \in \overline{\mathsf{M}}_{T,t}$, where

$$f^{\sharp} : \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{T,t}$$

is the induced homomorphism. Equivalently, the tropical curve is determined by the composition

$$\tau \longrightarrow \sigma_{\Gamma} \xrightarrow{\ \ell\ } \mathbb{R}^{E(\Gamma)}_{\geq 0} ,$$

where τ is the cone dual to $\overline{\mathsf{M}}_{T,t}$. As in Section 3.2, to give a quasi-stable model $\widehat{C} \to C_T$, we must specify, for every geometric point $t \in T$, the structure of a quasi-stable model of Γ metrized by $\overline{\mathsf{M}}_{T,t}$, compatibly with étale specializations. In other words, to give a quasi-stable model with specified dual graph $\widehat{\Gamma}$, we must lift the homomorphism $\tau \to \mathbb{R}^{E(\Gamma)}_{\geq 0}$ to a homomorphism

$$\tau \to \mathbb{R}^{E(\widehat{\Gamma})}_{\geq 0}$$

for the various cones τ dual to the $\overline{\mathsf{M}}_{T,t}$, compatibly with étale specializations. By construction, the cone $\sigma_{\widehat{\Gamma}}$ is the universal cone over which the morphism ℓ lifts. Thus, $\widehat{\Gamma}$ inherits the structure of a tropical curve over $\sigma_{\widehat{\Gamma}}$, and also over its subcones $\tau_{\widehat{\Gamma},I}$. Since the $\sigma_{\widehat{\Gamma},I}$ are isomorphic to $\tau_{\widehat{\Gamma},I}$, we see that $\widehat{\Gamma}$ admits the structure of a tropical curve over $\sigma_{\widehat{\Gamma},I}$. For any face $\sigma_{\Gamma'} < \sigma_{\Gamma}$, the tropical curve Γ' over $\sigma_{\Gamma'}$ induced by $C \to S$ is obtained from Γ by contracting some edges. If over $\sigma_{\Gamma'}$ an edge e of Γ is contracted, both corresponding edges of $\widehat{\Gamma}$ are contracted as well, since their lengths must add up to that of e. Thus $\widehat{\Gamma}$ naturally contracts to a quasi-stable model of Γ' over $\sigma_{\Gamma'}$. Therefore, the various $\widehat{\Gamma}$ glue and, as described in Section 3.2, determine a quasi-stable subdivision

$$\widehat{C} \to C \times_S \widetilde{S}^{\theta}_{\mathcal{L}} .$$

Just as the cone $\sigma_{\widehat{\Gamma}}$ tautologically carries a quasi-stable subdivision of $\Gamma \times_{\sigma_{\Gamma}} \sigma_{\widehat{\Gamma}}$, the subcone $\tau_{\widehat{\Gamma},I}$ is the universal cone on which the flow I can be realized as the slopes of a piecewise linear function unique up to pullback from the base. Hence the curve $\widehat{C}$ carries a unique PL function

$$\alpha \colon V(\widehat{\Gamma}) \to \overline{\mathsf{M}}^{\mathsf{gp}}_{\sigma_{\widehat{\Gamma},I}} \tag{30}$$

vanishing along x_1.

For each vertex $v \in \widehat{\Gamma}$, the PL function α assigns an element of $\overline{\mathsf{M}}^{\mathsf{gp}}_{\sigma_{\widehat{\Gamma},I}}$, or equivalently, a homomorphism

$$\alpha(v)\colon \sigma_{\widehat{\Gamma},I} \to \mathbb{R}\,.$$

By construction, the function α has the following explicit description. Denote by v_0 the vertex in $\widehat{\Gamma}$ corresponding to the component that contains the marking x_1. Then, for any other $v \in V(\widehat{\Gamma})$, we have

$$\alpha(v) = \sum_{\vec{e} \in \gamma_{v_0 \to v}} I(\vec{e}) \cdot \widehat{\ell}_e\,,$$

where $\gamma_{v_0 \to v}$ is *any* oriented path from $v_0 \to v$, $I(\vec{e})$ is the integer slope of the flow I along $\vec{e}$, and

$$\widehat{\ell}_e : \sigma_{\widehat{\Gamma},I} \to \mathbb{R}$$

is the homomorphism determined by the length of the edge e. We thus obtain equation (16) of Section 1.7.3.

The definition of $\alpha(v)$ does not depend on the choice of a path between v_0 and v, If γ and γ' are path from $v_0 \to v$, $\gamma - \gamma'$ is an oriented cycle, and

$$\langle \gamma - \gamma', I \rangle_{\widehat{\ell}} = 0$$

on $\sigma_{\widehat{\Gamma},I}$ by definition.[25] On the other hand, α does depend on the choice of v_0.

We have defined a combinatorial PL function on the tropical curve $\widehat{\Gamma}$. It is easy to see that these PL functions over the various cones $\sigma_{\widehat{\Gamma},I}$ are compatible and so, as explained in Section 3.4, they give rise to a global PL function α on $\widehat{C}$.

The role of α is to twist the line bundle $\mathcal{L}$ into a θ-stable bundle. By applying (24) and part (iii) of our defining conditions for the tuples $(\widehat{\Gamma}, D, I)$, we obtain

$$\underline{\deg}\mathcal{L}(\alpha) = \underline{\deg}\mathcal{L} - \mathsf{div}(\alpha) = \underline{\deg}\mathcal{L} - \mathsf{div}I = D\,.$$

Therefore, the curve $\widehat{C}$ and the PL function α together induce a map of stacks

$$\varphi\colon \widetilde{S}^{\theta}_{\mathcal{L}} \to S^{\theta}_{\mathcal{L}}$$

over S.

Lemma 29. *The map of stacks $\varphi\colon \widetilde{S}^{\theta}_{\mathcal{L}} \to S^{\theta}_{\mathcal{L}}$ is an isomorphism.*

[25]A similar argument is used in the proof of Lemma 16.

Proof. We may check the isomorphism strict-étale locally. Let C/T be a nuclear log curve (in the sense of [36]) with a line bundle $\mathcal{L}$ of degree d. We must show the following two conditions are equivalent:

(i) There exists a quasi-stable subdivision $\widehat{C} \to C$ and a PL function α on $\widehat{C}/T$ such that $\mathcal{L}(\alpha)$ is θ-stable.

(ii) There exists a quasi-stable subdivision $\widehat{\Gamma} \to \Gamma$ and a flow I on $\widehat{\Gamma}$ such that the condition $\langle \gamma, I \rangle_\ell = 0 \in \overline{\mathsf{M}}_T^{\mathsf{gp}}$ holds for every $\gamma \in H_1(\widehat{\Gamma})$.

The condition (i) characterizes the functor of points of $S_{\mathcal{L}}^\theta$, and condition (ii) characterizes the functor of points of $\widetilde{S}_{\mathcal{L}}^\theta$ by Lemma 6. The correspondence between subdivisions of the graph and of the curve is clear. If we start with a PL function α, we define the flow I to be the associated slopes. Conversely given a flow I, we know I can be realised as the slopes of a PL function by Lemma 16. ♦

Remark 30. During the course of the proof of Lemma 27, we have noted that while the subdivision of [3] is not constructed in the level of generality that we require (since the results of [3] are for $S = \overline{\mathcal{M}}_{g,n}$ and a stability condition pulled back from $\overline{\mathcal{M}}_{g,1}$), a slight modification of their argument suffices to construct $\widetilde{S}_{\mathcal{L}}^\theta$ in general. In fact, more can be said. From [2], for their specific choice of stability condition, there is a cone stack $\Sigma_{\mathcal{P}_{g,n}^\theta}$ subdividing the universal tropical Jacobian $\mathrm{Tro}\mathcal{P}\mathrm{ic}_{g,n}$ (see for example [54]), and a *tropical Abel-Jacobi section*

$$\Sigma_{\overline{\mathcal{M}}_{g,n}} \to \mathrm{Tro}\mathcal{P}\mathrm{ic}_{g,n}\,.$$

The fan $\widetilde{\Sigma}_\Gamma^\theta$ of [3] (and of Lemma 27 for their particular θ) is nothing but the pullback of the subdivision

$$\Sigma_{\mathcal{P}_{g,n}}^\theta \to \mathrm{Tro}\mathcal{P}\mathrm{ic}_{g,n} \tag{31}$$

to the various cones σ of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$. In the proof of [2] that (31) is a subdivision, nothing about the specific θ is used, except nondegeneracy. Their proof essentially produces a subdivision (31) for any stability condition, as discussed in [51, Theorem 5.5]. By pulling-back the Abel-Jacobi section, we obtain the same subdivision $\widetilde{\Sigma}_{\overline{\mathcal{M}}_{g,n}}^\theta$ that we construct in Lemma 27 for the case $S = \overline{\mathcal{M}}_{g,n}$.

The choice of [3] to work with stability conditions coming from 1-pointed curves is not about the subdivision $\overline{\mathcal{M}}_{g,A}^\theta$, but rather about the proof that the Abel-Jacobi section extends to a map

$$\mathsf{aj} : \overline{\mathcal{M}}_{g,A}^\theta \to \mathcal{P}_{g,n}^\theta$$

The claim is proven in [3] by writing a map in formal local coordinates and checking explicitly the required extension to a well-defined global map. The argument uses an explicit description of formal local charts of $\mathcal{P}_{g,1}^\theta$ for their particular choice of stability condition. Our main observation here is that in the category of log schemes, regardless of stability condition, the functor of points of $\widetilde{S}^\theta_{\mathcal{L}}$ is easy to describe and is represented by $S^\theta_{\mathcal{L}}$. The latter has an evident Abel-Jacobi section. So the formal local analysis is *not* needed to extend the Abel-Jacobi section.

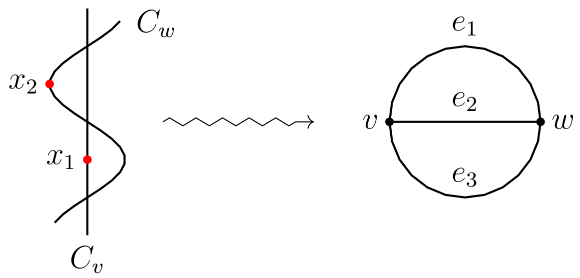
A more geometrical argument can also be given. The line bundle $\mathcal{L}$ always determines an Abel-Jacobi section S to the logarithmic Jacobian Log$\mathcal{P}$ic and a natural (logarithmic) fiber product diagram

$$\begin{array}{ccc} S^\theta_{\mathcal{L}} & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \mathcal{L}} \\ \mathcal{P}^\theta & \longrightarrow & \mathrm{Log}\mathcal{P}\mathrm{ic}, \end{array}$$

where the bottom map is a subdivision. Granting the theory of the logarithmic Jacobian, the above diagram shows that $S^\theta_{\mathcal{L}}$ is a subdivision and that the Abel-Jacobi section extends automatically to $S^\theta_{\mathcal{L}} \to \mathcal{P}^\theta$. The fiber product description readily leads to the definition of the functor of points of $S^\theta_{\mathcal{L}}$. This perspective will be explained carefully in the sequel to [51]. $\diamondsuit$

4.3 A graph of genus 2

A simple example in genus 2 is rich enough to contain most of the phenomena discussed. Let C_0 denote the *dollar sign* curve, the union of two rational curves C_v and C_w joined at 3 nodes. The dual graph Γ of C_0 has two vertices v and w and three edges e_1, e_2, e_3:



We place two markings on C_0. The first marking is on the component corresponding to v, and the second is on the component corresponding to w.

Let (S, s) be a nonsingular 3-dimensional base of a versal deformation of C_0,

$$\pi : C \to S\,, \quad \pi^{-1}(s) = C_0\,,$$

together with sections $x_1, x_2 : S \to C$ meeting the fibre C_0 as above. Both C and S carry canonical log structures. The tropicalization of S is the cone

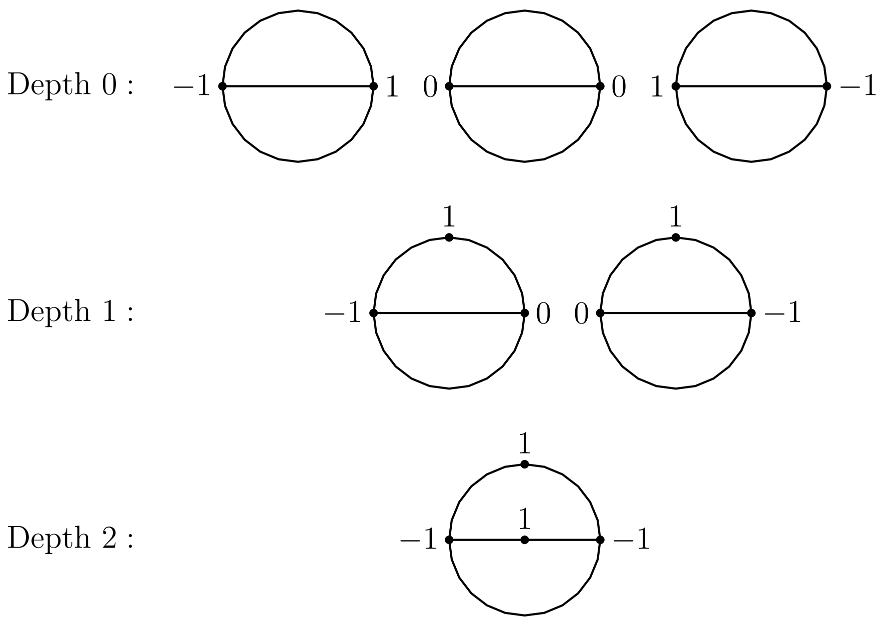
$$\Sigma_S = \mathbb{R}^3_{\geq 0}\,,$$

and the tropicalization of C/S is the fibration $\Sigma_C \to \Sigma_S$, which, over a point $(\delta_1, \delta_2, \delta_3) \in \Sigma_S$, assigns the graph Γ with edge e_i having length δ_i (with the understanding that when $\delta_i = 0$, the edge e_i is contracted in the fiber).

For every nondegenerate degree 0 stability condition θ, we find that θ-stable divisors D must satisfy the inequalities

$$\theta(v) - \frac{3}{2} < D(v) < \theta(v) + \frac{3}{2}, \qquad \theta(w) - \frac{3}{2} < D(v) < \theta(w) + \frac{3}{2}.$$

Therefore, for a small stability condition (close enough to $\theta = 0$), θ-stable divisors must satisfy $-1 \leq D(v), D(w) \leq 1$. Picking such a θ, we find that the list of all admissible θ-stable divisors (up to isomorphism of the graph) is:



Each graph of depth 1 and 2 stands for three graphs (after including those related by symmetry). There are 12 divisors in total on the list

Consider the vector $A = (-3, 3)$ of Abel-Jacobi data for the double ramification cycle $\mathsf{DR}_{2,A}$. The associated tropical divisor $\underline{\deg}_{k=0,A}$ is given by $\underline{\deg}_{0,A} = 3v - 3w$, where in the following we denote divisors on graphs as $\mathbb{Z}$-linear combinations of their vertices. To determine the subdivision

$$\widetilde{\Sigma}_S^{\theta} \to \Sigma_S\,,$$

we must solve the tropical problem

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D\,,$$

where D is an admissible θ-semistable divisor on the above list, and f is a PL function on Γ (or on $\widehat{\Gamma}$ in the depth 1 and 2 cases).

• Depth 0 cases:

For the divisor $D = 0v + 0w$, there is a unique ray of solutions in Σ_S. To build a piecewise linear function on Γ, we must choose 3 slopes $s_1, s_2, s_3 \in \mathbb{Z}$ on the edges of Γ. If we orient all three edges from w to v (with the convention that s_i is positive if the function increases from w to v), then, in order to have a PL function on Γ, the condition

$$s_1\delta_1 = s_2\delta_2 = s_3\delta_3$$

on the lengths of the edges of Γ must be satisfied. In particular, the s_i must either all be 0, all positive, or all negative. If, in addition, we demand

$$\mathsf{div}(f) = (s_1 + s_2 + s_3)v + (-s_1 - s_2 - s_3)w = \underline{\deg}_{0,A} - D = 3v - 3w\,,$$

we find a unique possible solution

$$s_1 = s_2 = s_3 = 1$$

along the ray $\delta_1 = \delta_2 = \delta_3$, and no other solutions.

The divisor $D = -v + w$ can be analyzed similarly. The equation

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D = 4v - 4w$$

yields three possible solutions

$$(s_1, s_2, s_3) = (2, 1, 1)\,,\ (1, 2, 1)\,,\ (1, 1, 2)$$

along the the three rays

$$\begin{aligned} 2\delta_1 &= \delta_2 &&= \delta_3\,, \\ \delta_1 &= 2\delta_2 &&= \delta_3\,, \\ \delta_1 &= \delta_2 &&= 2\delta_3\,. \end{aligned}$$

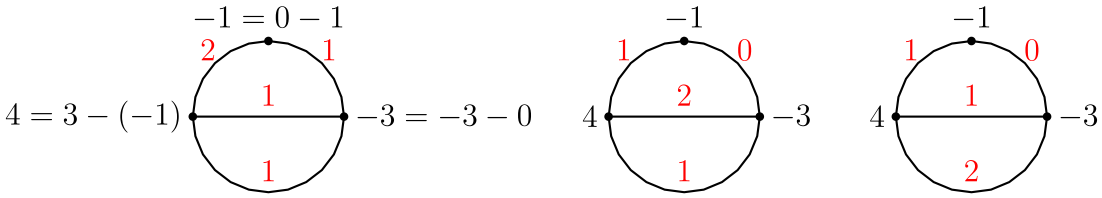
And, for the divisor $D = v - w$, there are no solutions at all to

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D = 2v - 2w\,.$$

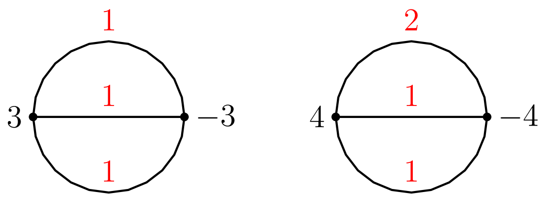
• Depth 1 cases:

The graph Γ is replaced here with a corresponding quasi-stable model $\widehat{\Gamma}$. Instead of studying the PL function (as we did in the depth 0 cases), we simply specify the possible underlying slopes of such functions corresponding to acyclic flows I with associated divisor $\underline{\deg}_{0,A} - D$.

The first depth 1 divisor D on the list contributes three possible acyclic flows (with all edges oriented from right to left and slopes of I in red):



The first of these flows, called $(\{2,1\},1,1)$ for brevity, has two specializations obtained by letting the exceptional vertex specialize to either of the original vertices v or w:



The flow $(\{2,1\},1,1)$ contributes a 2-dimensional cone of solutions to

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D$$

defined by the convex hull of the solutions of the two specializations (given by the rays $\delta_1 = \delta_2 = \delta_3$ and $2\delta_1 = \delta_2 = \delta_3$). Taken together, the depth 1 θ-stable divisors contribute 12 two dimensional cones: 9 from the three possible flows on the first divisor (and the those related by symmetry) and 3 from the second.

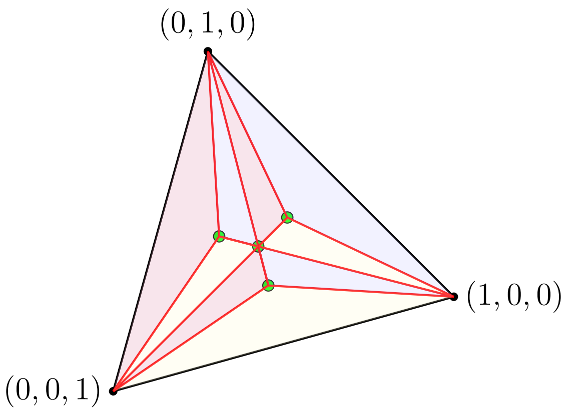
- Depth 2 cases:

A similar analysis can be applied to the three depth 2 divisors. (which contain two exceptional vertices). Each divisor D contributes three 3-dimensional cones, for a total of 9.

- Full subdivision:

Altogether, the subdivision $\widetilde{\Sigma}_S^\theta \to \Sigma_S = \mathbb{R}^3_{\geq 0}$ has 4 new rays (corresponding to the depth 0 divisors), 12 new 2-dimensional cones (corresponding to depth 1 divisors), and 9 maximal cones (corresponding to the depth 2 divisors). The 2-dimensional intersection of the subdivision $\widetilde{\Sigma}_S^\theta$ with the hyperplane $\delta_1 + \delta_2 + \delta_3 = 1$

can be drawn as:



The vertices in green correspond to the depth 0 divisors, the red lines to the depth 1 divisors. The maximal cells which correspond to different flows realizing the same underlying divisor are shaded in the same color.

Remark 31. The geometrical meaning of the subdivision is as follows. There is the point $s \in S$ (of codimension 3) over which the curve $C \to S$ is a dollar sign curve. We blow-up S at s, which replaces s with $\mathbb{P}^2$. We then blow-up the 3 fixed points of $\mathbb{P}^2$ to obtain $S^\theta_\mathcal{L}$. $\Diamond$

5 Theorem A

5.1 Universal constructions

Let $\mathfrak{M}_g$ be the stack of log curves, and let $\mathfrak{Pic}$ be the universal Picard stack of degree 0 line bundles on the universal curve over $\mathfrak{M}_g$, which we equip with the strict log structure. By the main result of [5], there is an operational class[26]

$$\mathsf{DR} \in \mathsf{CH}^g(\mathfrak{Pic})\,. \tag{32}$$

By [39], there is a natural lift of (32) to a logarithmic class

$$\mathsf{logDR} \in \mathsf{logCH}^g(\mathfrak{Pic})\,.$$

For a stack S, prestable curve C/S of genus g, and line bundle $\mathcal{L}$ on C of degree 0, we have a classifying map

$$\varphi_\mathcal{L} \colon S \to \mathfrak{Pic}\,.$$

[26]In Section 1 and the reference [5], the more precise notation $\mathsf{CH}^*_{\mathsf{op}}$ is used for operational Chow. We will drop the subscript op to simplify the notation (unless needed for emphasis).

The double ramification cycle is defined in [5] as an operational class on S by

$$\mathsf{DR}(\mathcal{L}) = \varphi_{\mathcal{L}}^* \mathsf{DR} \in \mathsf{CH}^g(S)\,.$$

If S is a log smooth log stack, C/S a log curve of genus g, and $\mathcal{L}$ a line bundle on C of degree 0, we have a classifying map (of log stacks)

$$\varphi_{\mathcal{L}} \colon S \to \mathfrak{Pic}\,,$$

and we can pull back logDR. Following [39], we define

$$\mathsf{logDR}(\mathcal{L}) = \varphi_{\mathcal{L}}^* \mathsf{logDR} \in \mathsf{logCH}^g(S).$$

Recall that $A = (a_1, \ldots, a_n)$ is a vector of integers summing to $k(2g-2+n)$. Let C/S be the universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$ over the moduli space of stable curves with markings $x_1, \ldots, x_n$, and let

$$\mathcal{L} = (\omega_C^{\mathsf{log}})^{\otimes k}\left(-\sum_{i=1}^n a_i x_i\right)\,.$$

By [5] and [39], the above universal constructions are *both* compatible with the previous definitions of the double ramification cycle:

$$\begin{aligned} \mathsf{DR}(\mathcal{L}) &= \mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,, \\ \mathsf{logDR}(\mathcal{L}) &= \mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})\,. \end{aligned}$$

5.2 Almost twistability

Let $\widehat{S}$ be a log smooth log algebraic stack, let $\widehat{C}/\widehat{S}$ be a log curve of genus g, and let $\widehat{\mathcal{L}}$ be a line bundle on $\widehat{C}$ of degree 0. Let

$$\mathcal{J}_{\widehat{C}/\widehat{S}} \to \widehat{S}$$

be the multidegree $\underline{0}$ part of the relative Picard stack of $\widehat{C}/\widehat{S}$.

Definition 32. The pair $(\widehat{C}/\widehat{S}, \widehat{\mathcal{L}})$ is *almost twistable* if there exists a dense open $i \colon U \hookrightarrow \widehat{S}$ which satisfies the following two conditions:

(i) the line bundle $\widehat{\mathcal{L}}$ has multidegree $\underline{0}$ over U,

(ii) the map $U \xrightarrow{\varphi_{\mathcal{L}} \circ i} \mathcal{J}_{\widehat{C}/\widehat{S}}$ is a closed immersion (or, equivalently, the image of $\varphi_{\mathcal{L}} \circ i$ is closed in $\mathcal{J}_{\widehat{C}/\widehat{S}}$).

◊

Definition 32 is the specialization of [39, Definition 4.10] obtained by setting the piecewise linear function α there to be zero. The motivation for Definition 32 lies in the following result.

Proposition 33. *Let $(\widehat{C}/\widehat{S}, \widehat{\mathcal{L}})$ be almost twistable. Let $\varphi_{\widehat{\mathcal{L}}}\colon \widehat{S} \to \mathfrak{Pic}$ be the map induced by $\mathcal{L}$. Then,*

$$\varphi_{\widehat{\mathcal{L}}}^{*}\mathsf{logDR} = \varphi_{\widehat{\mathcal{L}}}^{*}\mathsf{DR}$$

in $\mathsf{logCH}^{g}(\widehat{S})$.

Proof. When $\widehat{S}$ is smooth, the claim is a special case of [39, Lemma 4.13]. However, the proof given there does not use smoothness as we work throughout with operational classes (the smoothness assumption is present in [39] only as a running assumption which simplifies some other parts of the paper). ◆

5.3 Category of twists

Let S be a log smooth log algebraic stack, let C/S a log curve of genus g, and let $\mathcal{L}$ be a line bundle on C of degree 0. We consider the category $\mathbf{Twist}(S)$ of tuples

$$(T/S, \widehat{C} \to C_T, \alpha)$$

where T/S is a log scheme, $\widehat{C} \to C_T$ is a quasi-stable model of C_T/T, and α is a PL function on $\widehat{C}$. Let

$$\mathbf{Twist}^{\underline{0}}(S, \mathcal{L}) \hookrightarrow \mathbf{Twist}(S)$$

be the open substack consisting of those objects where

$$\widehat{C} \xrightarrow{\sim} C_T$$

and $\mathcal{L}_T(\alpha)$ has multidegree $\underline{0}$ on C_T/T.

If T is a trait (the spectrum of a discrete valuation ring) with generic point $j\colon \eta \to T$, the log structure on T is *maximal* if the natural map

$$\mathsf{M}_T \to \mathcal{O}_T \times_{j_*\mathcal{O}_\eta} j_*\mathsf{M}_\eta$$

is an isomorphism. For example, if $\mathsf{M}_\eta = \mathcal{O}_\eta^{*}$, then $\overline{\mathsf{M}}_T(T) = \mathbb{N}$, and $\mathsf{sPL}(C)$ is canonically isomorphic to the group of Cartier divisors on C supported over the closed point of T.

When checking the valuative criterion for properness in the logarithmic setting, test valuation rings should be restricted to those with maximal log structure, see [48, §5.3].

Lemma 34. *Let T be a trait with generic point η and maximal log structure. Let C/T be a log curve with line bundles $\mathcal{F}$, $\mathcal{F}'$ on C, and $\alpha \in \overline{\mathsf{M}}_C^{\mathsf{gp}}(C_\eta)$ an sPL function, together with an isomorphism*

$$\psi\colon \mathcal{F}_\eta(\alpha) \to \mathcal{F}'_\eta\,.$$

Then there exists an sPL function $\bar{\alpha} \in \overline{\mathsf{M}}_C^{\mathsf{gp}}(C)$ on C restricting to α over η and such that ψ extends to an isomorphism $\mathcal{F}(\bar{\alpha}) \to \mathcal{F}'$.

Proof. The line bundles $\mathcal{F}, \mathcal{F}'$ naturally correspond to $\mathcal{O}_C^*$-torsors on C. By extending the structure group, we obtain two $\mathsf{M}_C^{\mathsf{gp}}$-torsors $\mathcal{F}^{\log}, \mathcal{F}'^{\log}$. By the exact sequence

$$\pi_*\overline{\mathsf{M}}_C^{\mathsf{gp}} \to R^1\pi_*\mathcal{O}_C \to R^1\pi_*M_C^{\mathsf{gp}}$$

these agree on the generic point of T, and hence they agree on T by [54, Theorem 4.10.1]. By another application of the same exact sequence we see that the group of line bundles on C whose associated $\mathsf{M}_C^{\mathsf{gp}}$ torsor is trivial is the image of $\overline{\mathsf{M}}_C^{\mathsf{gp}}(C)$, hence there exists $\beta \in \overline{\mathsf{M}}_C^{\mathsf{gp}}(C)$ and an isomorphism

$$\mathcal{F}(\beta) \xrightarrow{\sim} \mathcal{F}'.$$

Then, $\mathcal{O}_{C_\eta}(\alpha - \beta_\eta)$ is trivial, hence $\alpha - \beta_\eta$ is constant, so there exists $\gamma \in \overline{\mathsf{M}}_T^{\mathsf{gp}}(\eta)$ with

$$\alpha - \beta_\eta = \pi^*\gamma\,.$$

By maximality of the log structure, there exists $\bar{\gamma} \in \overline{\mathsf{M}}_T^{\mathsf{gp}}(T)$ restricting to γ. Setting $\bar{\alpha} = \beta + \pi^*\bar{\gamma}$ gives the result. ♦

Proposition 35. *Let $X \hookrightarrow$ **Twist**(S) be an open substack containing* **Twist**$^{\underline{0}}(S,\mathcal{L})$ *as a dense open. If X is separated, then $(\widehat{C}/X, \mathcal{L}(\alpha))$ is an almost twistable family.*

Proof. Note that

$$\mathcal{J}_{\widehat{C}/X} = \mathcal{J}_{C_X/X}\,.$$

Let $U =$ **Twist**$^{\underline{0}}(S,\mathcal{L}) \hookrightarrow X$, so $\mathcal{L}(\alpha)$ defines a monomorphism

$$U \to \mathcal{J}_{C/X}\,,$$

which we will show to be proper by the valuative criterion.

Let T be a trait with generic point η, and fix a map

$$t\colon T \to \mathcal{J}_{C_X/X}$$

such that η lands in the image of U. The map t corresponds to a line bundle $\mathcal{F}'$ on C_T/T of multidegree $\underline{0}$. Since $\mathcal{F}'_\eta$ lies in the image of **Twist**$^{\underline{0}}(S,\mathcal{L})$, there exist an sPL function α on C_η and an isomorphism

$$\psi\colon \mathcal{L}_\eta(\alpha) \to \mathcal{F}'.$$

By Lemma 34, we can extend the PL function α and isomorphism ψ over the whole of T, defining an object

$$(T/S, C_T \xrightarrow{\sim} C_T, \alpha)$$

of $\mathbf{Twist}^{\underline{0}}(S, \mathcal{L})$. The line bundle $\mathcal{L}(\alpha)$ induces a map $t' \colon T \to \mathcal{J}_{C_T/T}$, which agrees with t at η, and hence agrees with t by the separatedness of X. ♦

5.4 Proof of Theorem A

To prove Theorem A, we will apply Propositions 33 and 35 under the following specialization of the geometry:

- C/S is the universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$ over the moduli space of stable curves,
- $\mathcal{L}$ is the line bundle of total degree 0 on $\mathcal{C}/\overline{\mathcal{M}}_{g,n}$ defined by

$$\mathcal{L} = (\omega_C^{\mathsf{log}})^{\otimes k} \left(-\sum_{i=1}^{n} a_i x_i \right) ,$$

- $\widehat{S} = \overline{\mathcal{M}}_{g,A}^{\theta} \xrightarrow{\rho} \overline{\mathcal{M}}_{g,n}$ for a small nondegenerate stability condition θ,
- $\widehat{\mathcal{L}} = \mathcal{L}^{\theta}$ is the universal line bundle on the universal quasi-stable curve

$$\widehat{C} = \mathcal{C}^{\theta} \to \overline{\mathcal{M}}_{g,A}^{\theta} ,$$

- $\alpha = \alpha^{\theta}$ is the universal PL function vanishing on x_1 on $\mathcal{C}^{\theta}$ and satisfying

$$\mathcal{L}^{\theta} = \mathcal{L}(\alpha^{\theta}) .$$

Proposition 36. *For a small nondegenerate stability condition θ, the line bundle $\mathcal{L}^{\theta}$ on $\mathcal{C}^{\theta} \to \overline{\mathcal{M}}_{g,A}^{\theta}$ is almost twistable.*

Proof. There is a natural open immersion

$$\overline{\mathcal{M}}_{g,A}^{\theta} \to \mathbf{Twist}(\overline{\mathcal{M}}_{g,n})$$

induced by the subdivision $\mathcal{C}^{\theta}$ and the PL function α^{θ}, identifying $\overline{\mathcal{M}}_{g,A}^{\theta}$ with the open substack X of $(\widehat{C}, \alpha) \in \mathbf{Twist}(\overline{\mathcal{M}}_{g,n})$ where $\mathcal{L}|_{\widehat{C}}(\alpha)$ is θ-stable. We must verify the conditions required by Proposition 35 for the open set X:

- Since multidegree $\underline{0}$ line bundles on stable models are θ-stable, X contains $\mathbf{Twist}^{\underline{0}}(\overline{\mathcal{M}}_{g,n}, \mathcal{L})$ as a dense open.

- Separatedness of X follows from Theorem 24: since $X \to \overline{\mathcal{M}}_{g,n}$ is proper, X is also proper. The key input is the nondegeneracy of θ.

Since the conditions of Proposition 35 are satisfied, we conclude that $(\mathcal{C}^\theta/\overline{\mathcal{M}}^\theta_{g,A}, \mathcal{L}^\theta)$ is an almost twistable family. ♦

To complete the proof of Theorem A, the class

$$\varphi^*_{\mathcal{L}^\theta}\mathsf{DR} = \mathsf{DR}^{\mathsf{op}}_{g,\emptyset,\mathcal{L}^\theta} \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{M}}^\theta_{g,A})$$

must be shown to represent $\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$. We apply Proposition 33 to the family $(\mathcal{C}^\theta/\overline{\mathcal{M}}^\theta_{g,A}, \mathcal{L}^\theta)$ to conclude

$$\varphi^*_{\mathcal{L}^\theta}\mathsf{logDR} = \varphi^*_{\mathcal{L}^\theta}\mathsf{DR}$$

The last step is to prove that

$$\varphi^*_{\mathcal{L}^\theta}\mathsf{logDR} \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{M}}^\theta_{g,A})$$

represents $\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$.

The claim of the last step is true because of invariance properties of the logarithmic double ramification cycle [39]:

(i) If $f : C' \to C$ over S is a subdivision of log curves, then

$$\mathsf{logDR}(f^*\mathcal{L}) = \mathsf{logDR}(\mathcal{L})\,,$$

(ii) If β is a PL function on C then

$$\mathsf{logDR}(\mathcal{L}(\beta)) = \mathsf{logDR}(L)\,,$$

(iii) If $f : S' \to S$ is a log modification, then $\mathsf{logCH}^*(S') = \mathsf{logCH}^*(S)$ and

$$\mathsf{logDR}(f^*\mathcal{L}) = \mathsf{logDR}(L)\,.$$

Invariances (i) and (ii) are proven in [39, Lemma 4.13] and [39, Theorem 4.18]. Invariance (iii) is by the construction of logDR.

6 Piecewise polynomials and strata classes

6.1 Overview

The $\mathbb{Q}$-algebras of piecewise polynomials and strict piecewise polynomials associated to $\overline{\mathcal{M}}_{g,n}$ were defined in Section 1.7.1 and used there to define the logarithmic tautological ring

$$\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) \,.$$

In order to derive the formulas of Theorem B from Theorem A, an explicit correspondence between the normally decorated strata classes of [52] and strict piecewise polynomials[27] is required. The explicit correspondence is established in Proposition 43 of Section 6.2. In Section 6.3, we rewrite the formula from [5] for the universal double ramification cycle in the language of piecewise polynomials in preparation for the proof of Theorem B in Section 7.

6.2 Relating piecewise polynomials and graph sums

6.2.1 Normally decorated strata classes

Let (X, D) be a nonsingular algebraic stack equipped with a normal crossings divisor, viewed as a log stack. We begin by briefly recalling the definition of *normally decorated strata classes* for (X, D) from [52, §5.1].

Let S be a codimension k stratum in X, and let B_S be the set of branches of D through S (so $|B_S| = k$). Let

$$\epsilon : \widetilde{S} \to X$$

be the normalization of the closure of S, let

$$p : P \to \widetilde{S}$$

be the G-torsor over $\widetilde{S}$ determined by the monodromy group G acting on the set B_S, and let

$$j = \epsilon \circ p : P \to X$$

denote the composition. Since the map ϵ is unramified, we can define the normal bundle N_ϵ which splits when pulled back to P:

$$p^* N_\epsilon = \oplus_{b \in B_S} N_b \,.$$

[27] Warning to the reader: the piecewise polynomials of [52] are the strict piecewise polynomials here.

The assignment $b \mapsto c_1(N_b)$ extends to a $\mathbb{Q}$-algebra homomorphism

$$c_1(N)\colon \mathbb{Q}[B_S] \to \mathsf{CH}^*(P)\,.$$

Given $F \in \mathbb{Q}[B_S]$, let

$$F(c_1(N)) \in \mathsf{CH}^*(P)$$

be the image under $c_1(N)$.

Definition 37. A *normally decorated strata* class of (X, D) is a class of the form

$$j_*F(c_1(N)) \in \mathsf{CH}^*(X)$$

where $F \in \mathbb{Q}[B_S]$ is a polynomial in the branches of D through a stratum S. $\Diamond$

6.2.2 Piecewise polynomials

Let (X, D) be a nonsingular algebraic stack equipped with a normal crossings divisor, viewed as a log stack. The combinatorial definition of (strict) piecewise polynomials given in Section 1.7.1 for $\overline{\mathcal{M}}_{g,n}$ carries over unchanged to (X, D). See [39, 52, 53] for detailed foundations. We will describe the piecewise polynomial of (X, D) corresponding to a normally decorated strata class.

Let Σ_X denote the fan of (X, D), and let σ be the stacky cone corresponding to a stratum $S \subset X$. The stacky cone σ has a strict cover by the cone $\tau = \mathbb{R}_{\geq 0}^{B_S}$, in the sense of [18, Definition 2.5]. In particular, the interior of σ is a quotient of the interior of τ by the monodromy group G. Let

$$\mathfrak{p}\colon \tau \to \sigma \quad \text{ and } \quad \mathsf{j} : \tau \to \Sigma_X$$

denote the quotient and the composition of the quotient with the inclusion. The set $\tau(1)$ of rays in τ is naturally identified with the set B_S of branches, so there is an identification of $\mathbb{Q}$-algebras

$$\mathsf{sPP}(\tau) = \mathbb{Q}[B_S]\,.$$

The monodromy group G acts naturally and compatibly on B_S, hence on τ, on $\mathbb{Q}[B_S]$, and on $\mathsf{sPP}(\tau)$.

Pullback of strict piecewise polynomials yields an injection

$$\mathfrak{i} : \mathsf{sPP}(\sigma) \to \mathsf{sPP}(\tau)\,. \tag{33}$$

The image of $\mathfrak{i}$ is contained in the G-invariant polynomials and contains those G-invariant polynomials which vanish on the boundary of τ; equivalently, the G-invariant polynomials which are divisible by

$$\delta_\tau = \prod_{b \in B_S} b\,.$$

We define a map

$$\mathfrak{p}_*\colon \mathsf{sPP}(\tau) \to \mathsf{sPP}(\sigma)\,, \quad F \mapsto \delta_\tau \sum_{g\in G} g^* F \tag{34}$$

which is *not* a retract of (33).

6.2.3 Extension

Let σ be the stacky cone, with strict cover τ, corresponding to a codimension k stratum $S \subset X$. We describe how to canonically extend the polynomial function $\mathfrak{p}_*F$ defined by (34) from σ to the whole fan Σ_X. We will construct a non-trivial extension of $\mathfrak{p}_*F$ over cones which contain σ. For cones not containing σ, the extension will be 0.

Let $\sigma' \supset \sigma$ be any stacky cone containing σ. As before, σ' is a quotient of a cone $\tau' = \mathbb{R}^{k'}_{\geq 0}$ for $k' \geq k$ with perhaps additional faces identified,

$$\mathfrak{p} : \tau' \to \sigma'\,.$$

The map $\mathfrak{p}$ sends some of the k-dimensional faces of τ' onto σ and the rest onto faces of σ' different from σ.

Let $\mathcal{F}_k$ be the set of the k-dimensional faces of the fiber product $\tau \times_{\sigma'} \tau'$. Explicitly, $\mathcal{F}_k$ is the set of pairs $(\widetilde{\tau}, \gamma)$ where $\widetilde{\tau}$ is a face of τ' and γ is an isomorphism fitting into the diagram

$$\begin{array}{ccc} \widetilde{\tau} & \xrightarrow{\ \subset\ } & \tau' \\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\mathfrak{p}} \\ \tau & \longrightarrow & \sigma'. \end{array}$$

Because the image of τ in σ' is generically the quotient $\sigma \cong \tau/G$, the group G acts freely on the isomorphisms

$$\gamma\colon \widetilde{\tau} \to \tau$$

commuting with the map $\tau \to \sigma'$, so $\mathcal{F}_k$ is a disjoint union of G-torsors.

The standard variables of the polynomials on τ' are in bijective correspondence to the set of rays $\tau'(1)$. Let

$$D^n(\tau') \subset \mathsf{sPP}(\tau')$$

be the $\mathbb{Q}$-linear subspace spanned by all monomials in at most n different variables. For example, if $|\tau'(1)| = 3$ with variables x_1, x_2, x_3, then

$$2x_1^9x_2^3 + 7x_2x_3 \in D^2(\tau') \quad \text{and} \quad x_1x_2x_3 \notin D^2(\tau')\,.$$

Elements in $D^n(\tau')$ are determined by their values on all of the n-dimensional faces of τ'.

The subspaces $D^n(\tau')$ define an increasing filtration of $\mathsf{sPP}(\tau')$ as n increases. By restriction, a filtration, denoted $D^n(\sigma')$, is defined on $\mathsf{sPP}(\sigma')$

Proposition 38. *Let $F \in \mathsf{sPP}(\tau)$. There is a unique polynomial*

$$\mathfrak{j}_*^{\sigma'} F \in D^k(\sigma')$$

which agrees with $\mathfrak{p}_* F$ on σ and is 0 on all other k-dimensional faces of σ'.

Proof. The uniqueness claim is immediate since the function is in D^k and its values on all of the k-dimensional faces of σ' are specified.

We will construct the desired function on τ' and then argue that it descends to σ'. For $(\widetilde{\tau}, \gamma) \in \mathcal{F}_k$, the polynomial $\delta_{\widetilde{\tau}} \gamma^* F \in \mathsf{sPP}(\widetilde{\tau})$ vanishes on the boundary of $\widetilde{\tau}$, and we write $\overline{\delta_{\widetilde{\tau}} \gamma^* F} \in \mathsf{sPP}(\tau')$ for the basic[28] extension. Define

$$\mathfrak{j}_*^{\sigma'} F = \sum_{(\widetilde{\tau}, \gamma) \in \mathcal{F}_k} \overline{\delta_{\widetilde{\tau}} \gamma^* F} \in \mathsf{sPP}(\tau').$$

Certainly, $\mathfrak{j}_*^{\sigma'} F \in D^k(\tau')$.

We claim that $\mathfrak{j}_*^{\sigma'} F$ lies in the image of $\mathsf{sPP}(\sigma') \hookrightarrow \mathsf{sPP}(\tau')$. Let G' be the monodromy group of σ' which acts on τ'. The group G' also acts on the set of k-dimensional faces of τ'. Since $\tau'/G' = \sigma'$ in the interior of τ', if a k-dimensional face $\kappa \subset \tau'$ is taken to another face κ' by $g' \in G'$, and the image of κ in σ' is σ, then the image of κ' is also σ. And G' permutes all k-dimensional faces that do not surject to σ. Therefore, $\mathfrak{j}_*^{\sigma'} F$ is invariant with respect to the G'-action. Furthermore, since $\mathfrak{j}_*^{\sigma'} F$ vanishes on the boundary of every k-dimensional cone and is symmetric with respect to any additional identification of faces, $\mathfrak{j}_*^{\sigma'} F$ lies in the image of $\mathsf{sPP}(\sigma') \hookrightarrow \mathsf{sPP}(\tau')$.

Finally, we must check that $\mathfrak{j}_*^{\sigma'} F$ agrees with $\mathfrak{p}_* F$ on σ and is 0 on all other k-dimensional faces of σ'. The restriction of the function $\mathfrak{j}_*^{\sigma'} F$ to σ is determined by the k-dimensional faces that surject onto σ. Choose such a face τ_0 of τ'. For any $(\widetilde{\tau}, \gamma) \in \mathcal{F}_k$ with $\widetilde{\tau} \neq \tau_0$, the function $\overline{\delta_{\widetilde{\tau}} \gamma^* F}$ vanishes on τ_0 by construction. Thus, since the isomorphisms $\gamma : \tau_0 \to \tau$ in $\mathcal{F}_k$ can be identified with elements of $g \in G$, we have

$$\mathfrak{j}_*^{\sigma'} F|_{\tau_0} = \sum_{g \in G} \overline{\delta_\tau g^* F} = \mathfrak{p}_* F\,,$$

which is the required agreement. ♦

Definition 39. Given $F \in \mathsf{sPP}(\tau)$, we define $\mathfrak{j}_* F \in \mathsf{sPP}(\Sigma_X)$ to be $\mathfrak{j}_*^{\sigma'} F$ on those stacky cones σ' of Σ_X which contain σ as a face and 0 on the other stacky cones of Σ_X. ♢

[28]The *basic extension* of a polynomial on a face of $\mathbb{R}^n_{\geq 0}$ is by pullback via the canonical projection $\mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ to the face.

Example 40. For $(X, D) = (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$, consider the cone $\sigma_{\Gamma_0} \subseteq \Sigma_X$ associated to the stable graph Γ_0 with one vertex of genus 0 and precisely g loops. Then $\tau = \mathbb{R}^g_{\geq 0}$ carries the strict piecewise polynomial $F = 1$. Then, the function[29]

$$\mathsf{j}_* F \in \mathsf{sPP}(\Sigma_X)$$

is given as follows:

- on cones σ_Γ where Γ has a vertex of positive genus, the function $\mathsf{j}_* F$ vanishes (since these do not contain σ_{Γ_0}),
- on the other cones σ_Γ, where Γ has first Betti number equal to g, we have

$$\mathsf{j}_* F = 2^g \cdot g! \cdot \sum_{\substack{T \subseteq \Gamma \\ \text{spanning tree}}} \prod_{e \in E(\Gamma) \setminus E(T)} \ell_e \,.$$

  Indeed, in this case the set $\mathcal{F}_g$ corresponds to the set of graph morphisms $\Gamma \to \Gamma_0$. Every such morphism precisely contracts a spanning tree $T \subseteq \Gamma$, and for a given tree T there are $2^g \cdot g!$ such morphisms. $\lozenge$

6.2.4 Commutation

The explicit correspondence of Section 6.2.5 requires the following basic commutation result.

Lemma 41. *Let S be a stratum of X of codimension k with branch set B_S and monodromy torsor*

$$p : P \to \widetilde{S}\,.$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Q}[B_S] & \xrightarrow{\mathsf{j}_*} & \mathsf{sPP}(\Sigma_X) \\ \big\downarrow{\scriptstyle c_1(N)} & & \big\downarrow{\scriptstyle \Phi} \\ \mathsf{CH}^*(P) & \xrightarrow{j_*} & \mathsf{CH}^*(X). \end{array} \tag{35}$$

Proof. The argument is carried out in three steps.

Step I: The simple normal crossings case.

We first treat the case where (X, D) is a scheme with simple normal crossings and $\bar{S}$ is the intersection of the irreducible components D_b corresponding to the

[29] The function was explained to us by D. Ranganathan.

branches $b \in B_S$. The closure $\bar{S}$ is then normal, the monodromy group G is trivial, and $N_b = \mathcal{O}(D_b)|_{\bar{S}}$. The commutation of (35) is then easily verified:

$$\begin{aligned} j_* F(c_1(N)) &= j_* j^* F(D_b : b \in B_S) = [\bar{S}] \cdot F(D_b : b \in B_S) \\ &= \left(\prod_{b \in B_S} D_b \right) \cdot F(D_b : b \in B_S) = \Phi(\mathsf{j}_* F)\,. \end{aligned}$$

Step II: Reduction of the general case to $X = \mathcal{A}_{\sigma'}$ a stacky Artin cone.

We can move the issue to the Artin fan $\mathcal{A}_X$ of X. There is a Cartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{j} & X \\ \downarrow{\scriptstyle t_P} & & \downarrow{\scriptstyle t_X} \\ \mathcal{P} & \xrightarrow{\mathcal{A}_j} & \mathcal{A}_X \end{array}$$

where t_X is smooth, and $\mathcal{P}$ is the monodromy torsor over the normalization of the closure of the stratum $t_X(S) \subset \mathcal{A}_X$. Furthermore,

$$N_{\mathcal{P}/\mathcal{A}_X} = \oplus_{b \in B_S} \mathcal{N}_b$$

and $N_{P/X} = t_P^* N_{\mathcal{P}/\mathcal{A}_X}$ compatibly with the splittings of $N_{P/X}$ and $N_{\mathcal{P}/\mathcal{A}_X}$. Then,

$$j_*(F(c_1(N)) = j_*(t_P^* F(c_1(\mathcal{N})) = t_X^* \mathcal{A}_{j*} F(c_1(\mathcal{N})).$$

So, instead of proving the commutation of (35), we may instead prove the commutation of:

$$\begin{array}{ccc} \mathbb{Q}[B_S] & \xrightarrow{\mathsf{j}_*} & \mathsf{sPP}(\Sigma_X) \\ \downarrow{\scriptstyle c_1(N)} & & \downarrow{\scriptstyle \Phi} \\ \mathsf{CH}^*(\mathcal{P}) & \xrightarrow{\mathcal{A}_{j*}} & \mathsf{CH}^*(\mathcal{A}_X). \end{array}$$

Furthermore, for the Artin fan,

$$\Phi \colon \mathsf{sPP}(\Sigma_X) \to \mathsf{CH}^*(\mathcal{A}_X)$$

is an isomorphism [52], so we drop Φ from the notation.

Because the Chow ring of an Artin fan satisfies étale descent (see Lemma 42 below), we may prove the desired equalities étale locally on $X = \mathcal{A}_X$. Since passing to a Zariski open does not affect the monodromy or the normal bundle, we may further assume that

$$X = \mathcal{A}_{\sigma'}\,,$$

where $\Sigma_X = \sigma'$ with $\sigma \subset \sigma'$.

Step III: Reduction to simple normal crossings.

Choose an étale cover $\mathcal{A}_{\tau'}$ of $\mathcal{A}_{\sigma'}$ with $\tau' = \mathbb{R}^{k'}_{\geq 0}$. The diagram commutes for $X = \mathcal{A}_{\tau'}$ by Step I. To prove commutativity for $X = \mathcal{A}_{\sigma'}$, we must keep track of how passing from $\mathcal{A}_{\sigma'}$ to $\mathcal{A}_{\tau'}$ affects the monodromy.

Consider the fiber diagram of stacks

$$\begin{array}{ccc} Q & \xrightarrow{h} & \mathcal{A}_{\tau'} \\ \downarrow p & & \downarrow q \\ P & \xrightarrow{j} & \mathcal{A}_{\sigma'} \end{array}$$

where $Q \stackrel{\sim}{=} P \times_{\mathcal{A}_{\sigma'}} \mathcal{A}_{\tau'}$. Let $\widetilde{\tau}$ be a cone in τ' that maps isomorphically to $\sigma \subset \sigma'$. Geometrically, such a cone corresponds to a stratum $T_{\widetilde{\tau}} \subset \mathcal{A}_{\tau'}$ that maps to S, or, equivalently, to a connected component of the preimage of S in $\mathcal{A}_{\tau'}$. By definition, P is a G-torsor over the normalization $\widetilde{S}$ of the closure $\overline{S}$ of S. Since q is étale, the pullback of $\widetilde{S}$ to $\mathcal{A}_{\tau'}$ is the normalization of the union of the closures $\overline{T}_{\widetilde{\tau}}$ of the $T_{\widetilde{\tau}}$. Since $\mathcal{A}_{\tau'}$ is simple normal crossings, the latter is simply the disjoint union of the closures $\overline{T}_{\widetilde{\tau}}$. Furthermore, as $\mathcal{A}_{\tau'}$ has no monodromy, the monodromy torsor of each $\overline{T}_{\widetilde{\tau}}$ is $\overline{T}_{\widetilde{\tau}}$ itself, and the pullback of P to $\mathcal{A}_{\tau'}$ is a trivial G-torsor. We conclude that Q is a disjoint union

$$Q \cong \coprod_{(\widetilde{\tau},\gamma)\in\mathcal{F}_k} Q_{(\widetilde{\tau},\gamma)}$$

with isomorphic connected components $Q_{(\widetilde{\tau},\gamma)} \cong \overline{T}_{\widetilde{\tau}}$.

Let $B_{\widetilde{\tau}}$ be the set of branches of the divisor in $\mathcal{A}_{\tau'}$ cutting out $T_{\widetilde{\tau}}$. By construction, the group G acts on $B_{\widetilde{\tau}}$, and the set of G-equivariant bijections between B_S and $B_{\widetilde{\tau}}$ is a G-torsor, naturally identified with the torsor of pairs $(\widetilde{\tau},\gamma) \in \mathcal{F}_k$. Since q is étale, we find that

$$p^* N_{P/X} = N_{Q/\mathcal{A}_{\tau'}}\,.$$

Writing $p_{(\widetilde{\tau},\gamma)}$ for the restriction of p to $Q_{(\widetilde{\tau},\gamma)}$, we have that for each $b \in B_S$,

$$p^*_{(\widetilde{\tau},\gamma)} N_b = N_{(\widetilde{\tau},\gamma)(b)}$$

where we have written $(\widetilde{\tau},\gamma)(b)$ for the image of b under the bijection

$$B_S \to B_{\widetilde{\tau}}$$

corresponding to $(\widetilde{\tau}, \gamma)$. Therefore, for any polynomial in $F \in \mathbb{Q}[B_S]$, an element $(\widetilde{\tau}, \gamma) \in \mathcal{F}_k$ determines a polynomial $\gamma^* F \in \mathbb{Q}[B_{\widetilde{\tau}}]$, and the pullback of $j_* F(c_1(N_{P/X}))$ to $\mathcal{A}_{\tau'}$ is the sum

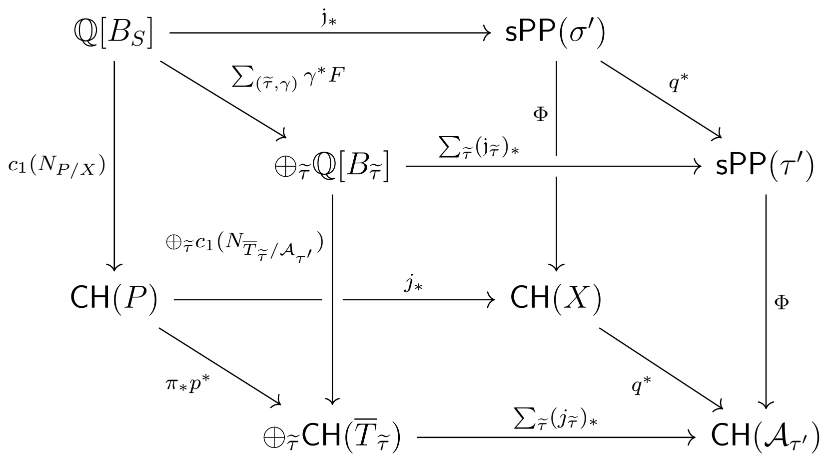
$$\sum_{(\widetilde{\tau},\gamma)\in\mathcal{F}_k} \gamma^* F(c_1(N_{\overline{T}_{\widetilde{\tau}}/\mathcal{A}_{\tau'}}))\,.$$

By Step I, the corresponding polynomial on $\mathcal{A}_{\tau'}$ is

$$\sum_{(\widetilde{\tau},\gamma)\in\mathcal{F}_k} \overline{\delta_{\widetilde{\tau}}\gamma^* F}\,,$$

which is precisely the polynomial on $\mathcal{A}_{\tau'}$ that descends to $\mathsf{j}_* F$ on X.

If we denote the projection $Q \to \coprod_{\widetilde{\tau}} \overline{T}_{\widetilde{\tau}}$ by π, the inclusion $\overline{T}_{\widetilde{\tau}} \to \mathcal{A}_{\tau'}$ by $j_{\widetilde{\tau}}$, and the homomorphism $\mathbb{Q}[B_{\widetilde{\tau}}] \to \mathsf{sPP}(\tau')$ by $(\mathsf{j}_{\widetilde{\tau}})_*$, we have proven that the following diagram commutes:



Here, the (direct) sums are taken over the set of cones $\widetilde{\tau}$ in τ' that map isomorphically to $\sigma \subset \sigma'$. Since we know the commutation for the front face of the cube, and all arrows from the back face to the front face are injective, the commutation for the back face follows. ♦

Lemma 42. *Let X be a smooth, quasi-separated*[30]*, log smooth, log algebraic stack with Artin fan $\mathcal{A}_X$. Let $(\mathcal{A}_X)_{et}$ be the small strict étale site of $\mathcal{A}_X$. Then, the functor*

$$\mathsf{CH}^* \colon (\mathcal{A}_X)_{et}^{\mathsf{op}} \to \mathbb{Q}\mathbf{Alg}$$

is a sheaf.

[30]Quasi-separated means an intersection of affine patches can be covered by finitely many affine patches.

Proof. The result is a rephrasing of [52, Theorem 14] which shows that the Chow ring of the Artin fan coincides with the algebra of piecewise polynomials.

The idea is to identify the presheaf CH with the sheaf sPP of strict piecewise polynomial functions. More precisely, we have a fully faithful functor

$$(\mathcal{A}_X)_{et} \to X_{et}\,, \quad U \mapsto U \times_{\mathcal{A}_X} X$$

to the small strict étale site of X, and sPP is a sheaf on X_{et}. It thus suffices to show that

$$\mathsf{CH}^*(U) = \mathsf{sPP}(U \times_{\mathcal{A}_X} X)$$

for every $U \in (\mathcal{A}_X)_{et}$. Formation of the Artin fan is functorial for strict morphisms, hence

$$U = \mathcal{A}_{U \times_{\mathcal{A}_X} X}$$

and by [52, Theorem 14] we know

$$\mathsf{CH}^*(U) = \mathsf{sPP}(U \times_{\mathcal{A}_X} X)\,.$$

The result is proven in [52] for the case where X is a variety, but all that is actually used is that the cone stack $\mathcal{A}_X$ can be written as a colimit of a *finite* diagram of cones. Now because we work with operational Chow rings with finite-type test objects we may immediately assume X to be quasi-compact, so we can choose a smooth cover of X by finitely many atomic log schemes. By quasi-separatedness we can then cover their overlaps with finitely many atomic log schemes, and the result is proven. ♦

6.2.5 Explicit correspondence

We return to our main case of interest: strict piecewise polynomials on the stack of log curves $\mathfrak{M}_g$.

• Let Γ be a graph corresponding to a cone σ_Γ. The rays of σ_Γ are identified with the edges $E(\Gamma)$ of Γ. The corresponding stratum is given by the immersion

$$\prod_{v \in V(\Gamma)} \mathfrak{M}^{sm}_{g(v),H(v)} / \operatorname{Aut}(\Gamma) \to \mathfrak{M}_g\,, \tag{36}$$

where $H(v)$ is the set of half-edges attached to the vertex v, see [6, Proposition 2.5]. Define

$$\mathfrak{M}_\Gamma = \prod_{v \in \mathrm{V}(\Gamma)} \mathfrak{M}_{g(v),H(v)}\,.$$

The normalisation of the closure of the stratum is given by

$$\mathfrak{M}_\Gamma / \operatorname{Aut}_\Gamma \to \mathfrak{M}_g\,,$$

and the universal monodromy torsor is given by

$$\mathfrak{M}_\Gamma \to \mathfrak{M}_\Gamma / \operatorname{Aut}_\Gamma ,$$

with the composite being the familiar map

$$j_\Gamma \colon \mathfrak{M}_\Gamma \to \mathfrak{M}_g .$$

• Let $d\colon E(\Gamma) \to \mathbb{Z}_{\geq 0}$. Define a polynomial on the set of edges of Γ:

$$F = \prod_{e \in E(\Gamma)} e^{d_e} \in \mathbb{Q}[E(\Gamma)],$$

Then, $\mathsf{j}_* F$ is the sPP function on $\mathfrak{M}$ defined as follows: for a stable graph Γ', the value on the cone $\sigma_{\Gamma'}$ is given by the formula

$$\sum_{f:\Gamma' \to \Gamma} \prod_{e \in E(\Gamma)} \ell_{f(e)}^{d_e+1} .$$

As in Example 40, the graph morphisms $f : \Gamma' \to \Gamma$ precisely correspond to the elements $(\widetilde{\tau}, \gamma)$ that we sum over in the definition of $\mathsf{j}_*^{\sigma_{\Gamma'}}$. The above formula automatically vanishes outside the star of the cone of Γ, since there we have no graph morphisms $\Gamma' \to \Gamma$. Define the class

$$\mathsf{m} = \prod_{e=(h,h') \in E(\Gamma)} (-\psi_h - \psi_{h'})^{d_e} \in \mathsf{CH}^*(\overline{\mathcal{M}}_\Gamma) .$$

Proposition 43. *The following explicit correspondence holds:*

$$j_{\Gamma,*}(\mathsf{m}) = \Phi(\mathsf{j}_* F) .$$

Proof. We apply Lemma 41 where S is the stratum of $X = \mathfrak{M}_g$ defined by (36). Then, B_S is naturally in bijection with $E(\Gamma)$. We view F as an element of the top left corner of (35). We must show that

$$j_{\Gamma,*}(\mathsf{m}) = j_*(F(c_1(N))) .$$

Since $j\colon P \to X$ is given exactly by $j_\Gamma : \mathfrak{M}_\Gamma \to \mathfrak{M}$, we need only show

$$\mathsf{m} = F(c_1(N)) \in \mathsf{CH}(\mathfrak{M}_\Gamma) ,$$

which is clear. ♦

6.3 Pixton's formula in terms of piecewise polynomials

6.3.1 Contributions

We translate here the formula of [5] for the universal double ramification cycle into the language of piecewise polynomials. The formula is written in the Chow cohomology of the universal Picard stack $\mathfrak{Pic}$ of degree 0 line bundles over the stack of genus g curves $\mathfrak{M}_g$. The result will be used in Section 7 to prove our formula for the logarithmic double ramification cycle.

Let $C \to S$ be a log curve of genus g, and let $\mathcal{L}$ be a line bundle on C of degree 0. The double ramification cycle

$$\mathsf{DR} \in \mathsf{CH}^g(\mathfrak{Pic})$$

can be naturally expressed in terms of strict piecewise polynomials on S. We require the following two constructions:

- Let $\eta^{\mathfrak{Pic}} = \pi_*(c_1(\mathcal{L})^2) \in \mathsf{CH}^1(\mathfrak{Pic})$.
- For every positive integer r, let Cont^r be the strict piecewise power series on S defined on the cone associated to a graph Γ of genus g by:

$$\mathrm{Cont}^r_\Gamma = \sum_w r^{-h^1(\Gamma)} \prod_{e \in E(\Gamma)} \exp\left(\frac{\overline{w}(\vec{e})\overline{w}(\overleftarrow{e})}{2}\ell_e\right) \in \mathbb{Q}[[\ell_e : e \in E(\Gamma)]]\,. \quad (37)$$

  The sum runs over *admissible weightings* mod r: flows w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

  $$\mathrm{div}(w) = \underline{\deg}(\mathcal{L}) \in (\mathbb{Z}/r\mathbb{Z})^{V(\Gamma)}\,.$$

  Inside the exponential, $\overline{w}(\vec{e})$ and $\overline{w}(\overleftarrow{e})$ denote the unique representative of $w(\vec{e}) \in \mathbb{Z}/r\mathbb{Z}$ and $w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ in $\{0, \ldots, r-1\}$.

Remark 44. To make sense of the contribution formula (37), we must show that the collection of power series Cont^r_Γ yields a strict piecewise power series on the cone stack of S. The issue is about the compatibility of the formula on different cones Γ.

An étale specialisation of geometric points of $\mathfrak{Pic}$,

$$\varphi \colon t \mapsto s\,,$$

yields a natural contraction map $\Gamma_s \to \Gamma_t$, an inclusion $E(\Gamma_t) \hookrightarrow E(\Gamma_s)$ of edges, and an injective map of rings

$$P_\varphi \colon \mathbb{Q}[[\ell_e \colon e \in E(\Gamma_t)]] \to \mathbb{Q}[[\ell_e \colon e \in E(\Gamma_s)]].$$

Suppose for every geometric point s in $\mathfrak{Pic}$, we are given a power series

$$f^s \in \mathbb{Q}[\ell_e \colon e \in E(\Gamma_s)]\,.$$

If, for every étale specialisation $\varphi\colon t \mapsto s$, we have

$$P_\varphi(f^t) = f^s\big|_{\{\ell_e = 0\,|\, e \notin E(\Gamma_t)\}}\,, \tag{38}$$

then the collection of power series $\{f_s\}$ glue to a global strict piecewise power series on the cone stack of S. Every truncation of the collection of power series $\{f^s\}$ can be viewed as a strictly piecewise polynomial on $\mathfrak{Pic}$.

If the f^s are compatible under graph contractions and graph automorphisms, then the compatibility (38) is satisfied over $\mathfrak{Pic}$. For the the power series Cont^r of (37), these two compatibilities hold. ♢

Lemma 45. *Formula* (37) *for* Cont^r *is compatible with respect to graph automorphisms and contractions.*

Proof. The claim for automorphisms is clear by definition (since only the structure of the graph Γ is used).

Since a general contraction is a composition of contractions of single edges, we need only study the contraction of a single edge e,

$$\Gamma \to \Gamma'\,.$$

There are two cases:

(i) The edge e is not a loop. Weightings on the contracted graph Γ' are then naturally in bijection with weightings on Γ, and $h^1(\Gamma') = h^1(\Gamma)$.

(ii) The edge e is a loop. Then, $h^1(\Gamma') = h^1(\Gamma) - 1$. For a given weighting on Γ' there are exactly r lifts to weightings on Γ (obtained by assigning weights i and $r - i$ to half edges of e for $0 \le i < r$).

Therefore, in both cases, compatibility (38) holds. ♦

Lemma 46. *For truncations of a fixed degree on quasi-compact opens of S,* Cont^r *is eventually polynomial in r.*

Proof. For every graph Γ, the eventual polynomiality in r of Cont^r_Γ is obtained from the theory of Ehrhart polynomials, see [40, Appendix A]. Quasi-compactness is requires to ensure that only finitely many graphs are considered. ♦

We define Cont to be the $r = 0$ value of the polynomial determined by Cont^r for large r. Every fixed degree truncation of Cont is a piecewise polynomial on the cone stack of $\mathfrak{Pic}$.

6.3.2 The universal DR cycle

With these preparations in place, we define

$$\mathbf{P}_g = \exp\Big(-\frac{1}{2}\eta^{\mathfrak{Pic}}\Big)\cdot \Phi(\mathrm{Cont}) \in \mathsf{CH}^*(\mathfrak{Pic})\,.$$

On the other hand, a mixed degree operational Chow class

$$\mathsf{P}_{g,\emptyset,0} \in \mathsf{CH}^*(\mathfrak{Pic})$$

is defined in [5] whose codimension g part $\mathsf{P}^g_{g,\emptyset,0}$ computes DR.

Theorem 47. *We have the equality*

$$\mathbf{P}_g = \mathsf{P}_{g,\emptyset,0} \in \mathsf{CH}^*(\mathfrak{Pic})\,.$$

Hence, we have $\mathsf{DR} = \mathbf{P}^g_g$*, where* $\mathbf{P}^g_g$ *denotes the codimension* g *part of* $\mathbf{P}_g$*.*

Proof. By the definition of [5], $\mathsf{P}_{g,\emptyset,0}$ is given by the coefficient of r^0 of[31]

$$\exp\left(-\frac{1}{2}\eta^{\mathfrak{Pic}}\right) \sum_{\substack{\Gamma\in\mathsf{G}_{g,0,0}\\ w\in\mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\, j_{\Gamma*}\Bigg[\prod_{e\in\mathrm{E}(\Gamma)} \frac{1-\exp\left(-\frac{w(\vec{e})w(\overleftarrow{e})}{2}(\psi_{\vec{e}}+\psi_{\overleftarrow{e}})\right)}{\psi_{\vec{e}}+\psi_{\overleftarrow{e}}}\Bigg]\,.$$

Following the notation of [5], the above sum is over the set of all possible graphs $\mathsf{G}_{g,0,0}$ for the universal Picard stack and over the set $\mathsf{W}_{\Gamma,r}$ of all possible admissible weightings mod r on Γ.

It suffices to show, for fixed r, that the class

$$\Phi(\mathrm{Cont}^r) = \Phi\left(\left\{\sum_w r^{-h^1(\Gamma)}\prod_e \exp\left(\frac{w(\vec{e})w(\overleftarrow{e})}{2}\ell_e\right)\right\}_\Gamma\right)$$

equals the class

$$\sum_{\substack{\Gamma\in\mathsf{G}_{g,0,0}\\ w\in\mathsf{W}_{\Gamma,r}}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\, j_{\Gamma*}\Bigg[\prod_{e\in\mathrm{E}(\Gamma)} \frac{1-\exp\left(-\frac{w(\vec{e})w(\overleftarrow{e})}{2}(\psi_{\vec{e}}+\psi_{\overleftarrow{e}})\right)}{\psi_{\vec{e}}+\psi_{\overleftarrow{e}}}\Bigg]\,.$$

[31] The half-edges h, h' appearing in the formula from [5] naturally correspond to the directed edges $\vec{e}, \overleftarrow{e}$, and under this correspondence, the notions of admissible weightings mod r likewise agree.

Consider a graph $\Gamma \in \mathsf{G}_{g,0,0}$. Let $\mathrm{Cont}^r(\Gamma)$ be the part of Cont^r containing those monomials in which *all* the ℓ_e for $e \in E(\Gamma)$ appear with *positive* exponent.[32] We easily see that

$$\mathrm{Cont}^r = \sum_{\Gamma \in \mathsf{G}_{g,0,0}} \mathrm{Cont}^r(\Gamma)\,.$$

We claim that the class $\Phi(\mathrm{Cont}^r(\Gamma))$ is exactly equal to

$$\sum_{w \in \mathsf{W}_{\Gamma,r}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\, j_{\Gamma*}\left[\prod_{e \in \mathrm{E}(\Gamma)} \frac{1-\exp\left(-\frac{w(\vec{e})w(\overleftarrow{e})}{2}(\psi_{\vec{e}}+\psi_{\overleftarrow{e}})\right)}{\psi_{\vec{e}}+\psi_{\overleftarrow{e}}}\right].$$

The equality follows from Lemma 43 together with the observation that replacing ℓ_e^d by $(-\psi_{\vec{e}}-\psi_{\overleftarrow{e}})^{d-1}$ everywhere in the expression $\exp(\frac{w(\vec{e})w(\overleftarrow{e})}{2}\ell_e)$ yields exactly

$$\frac{1-\exp\left(-\frac{w(\vec{e})w(\overleftarrow{e})}{2}(\psi_{\vec{e}}+\psi_{\overleftarrow{e}})\right)}{\psi_{\vec{e}}+\psi_{\overleftarrow{e}}}\,.$$

We see that the language of piecewise polynomials expresses Pixton's formula more efficiently. ♦

7 Theorems B and C

7.1 Families of curves

Let S be a log smooth log algebraic stack, let C/S be a log curve of genus g, and let $\mathcal{L}$ be a line bundle on C of degree 0. We assume, in addition, that we have:

(i) a log modification $\widehat{S} \to S$,

(ii) a quasi-stable model $\widehat{C} \to C \times_S \widehat{S}$,

(iii) $\alpha \in \overline{\mathsf{M}}^{\mathsf{gp}}_{\widehat{C}}(\widehat{C})$ a piecewise linear function (in the sense of Section 3.4).

For the proofs of Theorems B and C, the fundamental geometry is the following. Recall that $A = (a_1, \ldots, a_n)$ is a vector of integers summing to $k(2g-2+n)$. Let

[32] More precisely, for a graph $\Gamma \in \mathsf{G}_{g,0,0}$, we can describe the piecewise polynomial $\mathrm{Cont}^r(\Gamma)$ as follows. On the cone associated to a stable graph Γ', $\mathrm{Cont}^r(\Gamma)$ is given by those monomial terms of $\mathrm{Cont}^r_{\Gamma'}$ of (37) for which the contraction of all edges of Γ' associated to variables ℓ_e which *do not* appear in the monomial is a graph isomorphic to Γ.

C/S be the universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$ over the moduli space of stable curves with markings $x_1, \dots, x_n$, and let

$$\mathcal{L} = (\omega_C^{\mathsf{log}})^{\otimes k} \left(-\sum_{i=1}^{n} a_i x_i \right) .$$

The additional data is given by:

(i) $\widehat{S} = \overline{\mathcal{M}}_{g,A}^{\theta} \xrightarrow{\rho} \overline{\mathcal{M}}_{g,n}$ for a small nondegenerate stability condition θ,

(ii) $\widehat{C} = \mathcal{C}^{\theta} \to \mathcal{C} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,A}^{\theta}$ is the universal quasi-stable curve,

(iii) $\alpha = \alpha^{\theta}$ is the universal PL function on $\mathcal{C}^{\theta}$ and satisfies $\mathcal{L}^{\theta} = \mathcal{L}(\alpha^{\theta})$.

7.2 Geometric construction of a logarithmic class on S

The line bundle $\widehat{\mathcal{L}} = \mathcal{L}(\alpha)$ on $\widehat{C}$ defines a map

$$\varphi_{\widehat{\mathcal{L}}} \colon \widehat{S} \to \mathfrak{Pic} ,$$

and, in turn, a class

$$\mathsf{DR}(\widehat{\mathcal{L}}) = \varphi_{\widehat{\mathcal{L}}}^{*}(\mathsf{DR}) \in \mathsf{CH}^{g}(\widehat{S}) \subset \mathsf{logCH}^{g}(S) .$$

To emphasise that we are considering $\mathsf{DR}(\widehat{\mathcal{L}})$ as a logarithmic class on S, we use the notation

$$\mathsf{DR}_{S}^{\approx}(\widehat{S}, \widehat{\mathcal{L}}) = \mathsf{DR}(\widehat{\mathcal{L}}) \in \mathsf{logCH}^{g}(S) ,$$

an approximation to the true logarithmic double ramification cycle

$$\mathsf{logDR}(\mathcal{L}) \in \mathsf{logCH}^{g}(S)$$

of [39].

By Proposition 33, a sufficient condition for the equality

$$\mathsf{DR}_{S}^{\approx}(\widehat{S}, \widehat{\mathcal{L}}) = \mathsf{logDR}(\mathcal{L})$$

is that $(\widehat{C}/\widehat{S}, \widehat{\mathcal{L}})$ is almost twistable.

In Section 7.4, we give a formula for $\mathsf{DR}_{S}^{\approx}(\widehat{S}, \widehat{\mathcal{L}})$ which applies regardless of whether the family is almost twistable. Applied with Theorem A, the formula yields Theorem B. Theorem C is obtained from Pixton's relations in the universal setting [5].

7.3 The functions $\mathfrak{P}$ and $\mathfrak{L}$

We take a (strict) geometric point $\widehat{s}$ of $\widehat{S}$ mapping to s, a strict geometric point of S. Over these points, we have a graph $\widehat{\Gamma}$ of $\widehat{C}$ contracting to a graph Γ of C, a map of cones

$$\mathrm{Hom}(\overline{\mathsf{M}}_{\widehat{S},\widehat{s}},\mathbb{R}_{\geq 0}) = \widehat{\sigma} \to \sigma_\Gamma = \mathrm{Hom}(\overline{\mathsf{M}}_{S,s},\mathbb{R}_{\geq 0})\,,$$

and a PL function α on $\widehat{\Gamma}$.

Remark 48. In the geometric setting of Theorems B and C (as specified at the end of Section 7.1),

$$\sigma_\Gamma = \mathbb{R}_{\geq 0}^{E(\Gamma)}\,,$$

and the choice of the point $\widehat{s}$ determines a flow I. The cone $\widehat{\sigma}$ is then given by

$$\widehat{\sigma} = \sigma_{\widehat{\Gamma},I}$$

defined in (29). The PL function α is uniquely determined by the conditions:

- $\alpha(x_1) = 0$,
- the slopes of α are equal to I,

see (30). In particular, α here exactly matches (16). ◊

We resume the general case, continuing to view the PL function $\alpha \in \overline{\mathsf{M}}_{\widehat{C}}(\widehat{C})$ as a function

$$V(\widehat{\Gamma}) \to \overline{\mathsf{M}}^{\mathsf{gp}}_{\widehat{S},\widehat{s}}\,,$$

following Section 3.4. Let

$$\underline{\deg}(\widehat{\mathcal{L}}) : V(\widehat{\Gamma}) \to \mathbb{Z}$$

be the multidegree. We have a length function

$$\ell\colon E(\widehat{\Gamma}) \to \overline{\mathsf{M}}_{\widehat{S},\widehat{s}}\,,$$

and seeing elements of $\overline{\mathsf{M}}_{\widehat{S},\widehat{s}}$ as piecewise linear functions on $\widehat{\sigma}$, we can define a power series on $\widehat{\sigma}$ by

$$\mathrm{Cont}^r_{\widehat{\sigma}} = \sum_w r^{-h^1(\widehat{\Gamma})} \prod_{e\in E(\widehat{\Gamma})} \exp\left(\frac{\overline{w}(\vec{e})\overline{w}(\overleftarrow{e})}{2}\ell_e\right) \;\in\; \mathbb{Q}[[\ell_e : e \in E(\widehat{\Gamma})]]\,. \qquad (39)$$

The sum runs over *admissible weightings* mod r: flows w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

$$\mathrm{div}(w) = \underline{\deg}(\widehat{\mathcal{L}}) \in (\mathbb{Z}/r\mathbb{Z})^{V(\widehat{\Gamma})}\,.$$

Inside the exponential, $\overline{w}(\vec{e}), \overline{w}(\overleftarrow{e}) \in \{0,\ldots,r-1\}$ denote the unique representative of $w(\vec{e}), w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ respectively.

Definition 49 (The function $\mathfrak{P}$)**.** As explained in Section 6.3, the formula (39) is polynomial in r for sufficiently large r. The function $\mathfrak{P}$, a strict piecewise power series on the cone complex of $\widehat{S}$, is defined by the the $r = 0$ specialization of the formula (39). ♢

Definition 50 (The function $\mathfrak{L}$)**.** Following (17), we define a PL function $\mathfrak{L}$ on the cone stack of $\widehat{S}$. On $\widehat{\sigma}$, the definition is

$$\mathfrak{L} = \sum_{v \in V(\widehat{\Gamma})} \underline{\deg}(\mathcal{L} \otimes \widehat{\mathcal{L}})(v) \cdot \alpha(v)\,, \tag{40}$$

where[33] $\deg(\mathcal{L} \otimes \widehat{\mathcal{L}})(v)$ is an integer and $\alpha(v) \in \overline{\mathsf{M}}^{\mathsf{gp}}_{\widehat{S},\widehat{s}}$ is viewed as a linear function on $\widehat{\sigma}$. We extend the expression (40) to a strict piecewise linear function on the cone stack of $\widehat{S}$ using the method of Remark 44. ♢

7.4 The formula

Application of the map Φ of (11) to $\mathfrak{P}$ and $\mathfrak{L}$ yields classes

$$\Phi(\mathfrak{P}), \Phi(\mathfrak{L}) \in \mathsf{logCH}^*(S).$$

Let the logarithmic class $\eta \in \mathsf{logCH}^1(S)$ be the image of

$$\eta = \pi_*(c_1(\mathcal{L})^2) \in \mathsf{CH}^1(\widehat{S})$$

under the inclusion $\mathsf{CH}^*(\widehat{S}) \subset \mathsf{logCH}^*(S)$.

Definition 51 (The class $\mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}}$)**.** The logarithmic class associated to $(\widehat{C}/\widehat{S}, \widehat{L})$ by Pixton's formula is defined by:

$$\mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}} = \exp\Big(-\frac{1}{2}(\eta + \Phi(\mathfrak{L}))\Big) \cdot \Phi(\mathfrak{P}) \in \mathsf{logCH}^*(S)\,.$$

Let $\mathbf{P}^{g,\widehat{S}}_{g,\widehat{\mathcal{L}}}$ be the codimension g part of $\mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}}$,

$$\mathbf{P}^{g,\widehat{S}}_{g,\widehat{\mathcal{L}}} \in \mathsf{logCH}^*(S)\,.$$

♢

Following Section 6.3.2, the mixed degree operational Chow class

$$\mathsf{P}_{g,\emptyset,0} \in \mathsf{CH}^*(\mathfrak{Pic})$$

from [5] has codimension g part $\mathsf{P}^g_{g,\emptyset,0}$ which computes the universal DR class.

[33]Over $\widehat{S}$, we follow the convention that $\mathcal{L}$ denotes the pullback of $\mathcal{L}$ from C to $\widehat{C}$.

Theorem 52. *We have*

$$\varphi^*_{\widehat{\mathcal{L}}}(\mathsf{P}_{g,\emptyset,0}) = \mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}} \in \mathsf{logCH}^*(S)\,. \tag{41}$$

In particular, we have

$$\mathsf{DR}^{\approx}_{S}(\widehat{S},\widehat{\mathcal{L}}) = \mathbf{P}^{g,\widehat{S}}_{g,\widehat{\mathcal{L}}} \in \mathsf{logCH}^*(S)\,. \tag{42}$$

Proof. On $\widehat{S}$, we have the classes

$$\eta = \pi_*(c_1(\mathcal{L})^2) \quad \text{and} \quad \widehat{\eta} = \pi_*(c_1(\widehat{\mathcal{L}})^2)\,,$$

which are related by

$$\begin{aligned}\widehat{\eta} &= \pi_*(c_1(\widehat{\mathcal{L}})^2)\\ &= \pi_*(c_1(f^*\mathcal{L})^2) + 2\pi_*(c_1(f^*\mathcal{L})\cdot c_1(\mathcal{O}(\alpha))) + \pi_*(c_1(\mathcal{O}(\alpha))^2)\\ &= \eta + \pi_*(c_1(\mathcal{L}\otimes\widehat{\mathcal{L}})\cdot c_1(\mathcal{O}(\alpha)))\,.\end{aligned}$$

Next, for every line bundle $\mathcal{F}$ on $\widehat{C}$ we claim that

$$\pi_*(c_1(\mathcal{F})\cdot c_1(\mathcal{O}(\alpha))) = \Phi\Big(\sum_{v\in\widehat{V}}(\underline{\deg}\mathcal{F})(v)\cdot\alpha(v)\Big)\,, \tag{43}$$

To prove formula (43), we first reduce to the case where $c_1(\mathcal{O}_C(\alpha))$ is a vertical prime divisor D on $\widehat{C}$ lying over a prime divisor $\pi(D)$ on $\widehat{S}$. The left side yields $\pi(D)$ times the degree of F on the generic fibre of $D\to\pi(D)$, which is exactly what is computed by the right side.

Applying (43) yields $\widehat{\eta} = \eta + \Phi(\mathfrak{L})$. By Theorem 47, we have[34]

$$\begin{aligned}\varphi^*_{\widehat{\mathcal{L}}}(\mathsf{P}_{g,\emptyset,0}) &= \varphi^*_{\widehat{\mathcal{L}}}\left(\exp\Big(-\frac{1}{2}\eta^{\mathfrak{Pic}}\Big)\cdot\Phi(\mathrm{Cont})\right)\\ &= \exp\Big(-\frac{1}{2}\widehat{\eta}\Big)\cdot\varphi^*_{\widehat{\mathcal{L}}}\Phi(\mathrm{Cont})\\ &= \exp\Big(-\frac{1}{2}(\eta+\Phi(\mathfrak{L}))\Big)\cdot\varphi^*_{\widehat{\mathcal{L}}}\Phi(\mathrm{Cont})\end{aligned}$$

By definition, $\mathfrak{P}$ exactly matches the piecewise power series $\varphi^*_{\widehat{\mathcal{L}}}\Phi(\mathrm{Cont})$. Hence,

$$\varphi^*_{\widehat{\mathcal{L}}}(\mathsf{P}_{g,\emptyset,0}) = \exp\Big(-\frac{1}{2}(\eta+\Phi(\mathfrak{L}))\Big)\cdot\Phi(\mathfrak{P}) \in \mathsf{logCH}^*(S)\,,$$

which proves (41). Equation (42) follows from the codimension g part of (41) together with [5]. ♦

[34]Here, $\eta^{\mathfrak{Pic}}$ denotes the universal class on $\mathfrak{Pic}$ from Section 6.3.

7.5 Proofs of Theorems B and C

Proof of Theorem B. The result follows immediately by specializing Theorem 52 to the fundamental geometry specified in Section 7.1. By Theorem A, the left side of equation (42) specializes to $\mathsf{logDR}_{g,A}$, and the right side of equation (42) specializes specializes to $\mathbf{P}^{g,\theta}_{g,A}$. We exactly obtain the formula of Theorem B. ♦

Proof of Theorem C. Part (i) is a consequence of Theorem B, since both sides of the equality compute the cycle $\mathsf{logDR}_{g,A}$. Part (ii) follows from the vanishing

$$\mathsf{P}^h_{g,\emptyset,0} = 0 \in \mathsf{CH}^h(\mathfrak{Pic}) \quad \text{for } h > g \tag{44}$$

combined with equality (41) of Theorem 52. The vanishing (44) in the universal context is [5, Theorem 8] and is a form of Pixton's double ramification cycle relations proven earlier in [22]. ♦

8 Genus 1 calculations

8.1 Form of the answer

We apply Theorem B here to explicitly compute $\mathsf{logDR}_{g,A}$ in the case $g = 1$. The result, presented in Theorem 57, is of the form

$$\mathsf{logDR}_{1,A} = \mathsf{DR}_{1,A} + (\text{piecewise linear correction term}) \ \in \mathsf{logCH}^1(\overline{\mathcal{M}}_{1,n}),$$

where the term $\mathsf{DR}_{1,A}$ is the (pullback of the) standard double ramification cycle in $\mathsf{CH}^1(\overline{\mathcal{M}}_{1,n})$. The formula for $\mathsf{DR}_{1,A}$ is written in sPP language using Theorem 47. We have

$$\mathsf{DR}_{1,A} = -\frac{1}{2}\eta + \Phi(\mathrm{Cont}),$$

where η is the class from (18) and Cont is the piecewise linear function given by

$$\mathrm{Cont}_\Gamma = -\frac{1}{12}\left(\sum_{e \text{ a non-separating edge}} \ell_e\right) - \frac{1}{2}\left(\sum_{e \text{ a separating edge}} w(e)^2\ell_e\right), \tag{45}$$

where w is any flow on Γ satisfying $\mathrm{div}(w) = \underline{\deg}_{k,A}$ (which determines $|w(e)|$ for separating edges e).

8.2 θ-stability

Let θ be a small nondegenerate stability condition of type $(1, n)$ and degree 0. We can completely describe the θ-stable multidegrees D. A graph Γ of genus 1 has cycle number either 0 or 1. Nothing interesting happens when the cycle number is 0 (in which case the graph is a tree).

Lemma 53. *Let Γ be a tree. Let θ be a small nondegenerate stability condition of degree* 0*. Then, the only θ-stable multidegree D on Γ is $D = 0$.*

Proof. Let e be any edge of Γ which splits Γ into subgraphs Γ_1 and Γ_2. Because there is only one edge connecting Γ_1 with Γ_2, there is only one integer satisfying the θ-stability inequality for $D(\Gamma_1)$. Because θ is small, this integer must be 0. Therefore, D has total degree 0 on each of the subgraphs Γ_1 and Γ_2. We easily conclude that D has degree 0 on every vertex. ♦

The situation is slightly more complicated for a graph Γ with cycle number 1. Such a graph has a unique cycle C_Γ along with trees glued to the vertices of C_Γ.

Lemma 54. *Let Γ be a connected graph with cycle number* 1 *and cycle C_Γ. Let θ be a small nondegenerate stability condition of degree* 0*. Let D be a θ-stable multidegree D on Γ. Then, $D(v) = 0$ for all v not on the cycle C_Γ. Moreover, as one goes around the cycle, the nonzero values of $D(v)$ alternate between* 1 *and* -1*.*

Proof. The first claim follows by the same argument as for Lemma 53. For the second claim, it suffices to show that, for any subset of consecutive vertices around the cycle C_Γ, the sum of their degrees is in the set $\{-1, 0, 1\}$.

Consider the subgraph Γ_1 consisting of consecutive vertices around C_Γ together with the trees glued at those vertices. Because there are only two edges connecting Γ_1 with its complement, there are only two integers satisfying the θ-stability inequality for $D(\Gamma_1)$. Because θ is small, one of these integers must be 0, and then the other is ± 1. Therefore, $D(\Gamma_1) \in \{-1, 0, 1\}$, and since D has degree 0 outside the cycle, we obtain the desired property. ♦

The Lemmas 53 and 54 apply not only to a genus 1 stable graph Γ, but to any quasi-stable modification $\widehat{\Gamma}$.

8.3 The formula

The computation of logDR is particularly simple in genus 1 because the function $\mathfrak{P}$ of (19) in degree 1 is just the first term in (45): the piecewise linear function

$$-\frac{1}{12}\left(\sum_{e \text{ a non-separating edge}} \ell_e\right).$$

The interesting part of formula (19) in genus 1 is $\mathfrak{L}$, which we will compute explicitly.

We begin with a general result (which holds in any genus) rewriting $\mathfrak{L}$ in a more convenient form.

Lemma 55. *Let* $(\widehat{\Gamma}, D, I)$ *be the data giving a cone* $\sigma_{\widehat{\Gamma},I} \in \widetilde{\Sigma}^{\theta}_{\Gamma}$. *Let* J *be any flow on* $\widehat{\Gamma}$ *satisfying* $\operatorname{div}(J) = D$, *and let* $I_0 = I + J$ *(so* $\operatorname{div}(I_0) = \underline{\deg}_{k,A}$*). Then on* $\sigma_{\widehat{\Gamma},I}$, *we have*

$$\mathfrak{L} = \sum_{e \in E(\widehat{\Gamma})} (I_0(\vec{e})^2 - J(\vec{e})^2) \cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)},$$

where the sum runs over the unoriented edges of $\widehat{\Gamma}$*: any orientation of* e *can be chosen (the summand is invariant).*

Proof. We use the same notation as in the definition of $\mathfrak{L}$ in (16) and (17), but we omit the final change of variables $|_{\widehat{\ell}=\widehat{\ell}(\ell)}$ for brevity of notation. We have

$$\begin{aligned}
\mathfrak{L} &= \sum_{v \in V(\widehat{\Gamma})} (D + \underline{\deg}_{k,A})(v)\alpha(v) \\
&= \sum_{v, \vec{e} \to v} (I_0(\vec{e}) + J(\vec{e}))\alpha(v) \\
&= \sum_{e=(v,v') \in E(\widehat{\Gamma})} \Big((I_0(v \to v') + J(v \to v'))\alpha(v') + (I_0(v' \to v) + J(v' \to v))\alpha(v) \Big) \\
&= \sum_{e=(v,v') \in E(\widehat{\Gamma})} (I_0(v \to v') + J(v \to v'))(\alpha(v') - \alpha(v)) \\
&= \sum_{e \in E(\widehat{\Gamma})} (I_0(\vec{e}) + J(\vec{e}))I(\vec{e})\widehat{\ell}_e \\
&= \sum_{e \in E(\widehat{\Gamma})} (I_0(\vec{e}) + J(\vec{e}))(I_0(\vec{e}) - J(\vec{e}))\widehat{\ell}_e
\end{aligned}$$

as desired. In the last two lines, any orientation of e can be chosen (the summand is invariant). ♦

Let Γ be a stable graph of genus 1, and let $(\widehat{\Gamma}, D, I)$ be the data giving a cone

$$\sigma_{\widehat{\Gamma},I} \in \widetilde{\Sigma}^{\theta}_{\Gamma}.$$

We will compute $\mathfrak{L}$ on $\sigma_{\widehat{\Gamma},I}$. If Γ is a tree, then $D = 0$ and $\widehat{\Gamma} = \Gamma$ by Lemma 53. Nothing interesting is happening: there is a unique flow I with $\operatorname{div}(I) = \underline{\deg}_{k,A}$, and the resulting function $\mathfrak{L}$ simply contributes part of the regular double ramification cycle formula.

Assume now that Γ has cycle number 1 with cycle C_Γ. We fix an orientation on C_Γ, and let C be the flow on Γ (or on $\widehat{\Gamma}$) with value 1 around that cycle and 0 elsewhere. In order to apply Lemma 55, we must pick a flow J with $\operatorname{div}(J) = D$.

By Lemma 54, D must have a very specific form, and we can find such [35] a J supported on C_Γ that only takes on values 0 and 1 (when read in the chosen orientation around the cycle). We then have

$$\begin{aligned}\mathfrak{L} &= \sum_{e\in E(\widehat{\Gamma})} (I_0(e)^2 - J(e)^2)\cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \\ &= \sum_{e\in E(\widehat{\Gamma})} (I_0(e)^2 - J(e)C(e))\cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \\ &= \sum_{e\in E(\widehat{\Gamma})} (I_0(e)^2 - I_0(e)C(e))\cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \\ &= \sum_{e\in E(\Gamma)} (I_0(e)^2 - I_0(e)C(e))\cdot \ell_e\,,\end{aligned}$$

where the first equality is Lemma 55, the second equality follows from J only taking on values 0 and 1 (when oriented consistently with C around the cycle), the third equality follows from the relation between $I = I_0 - J$ and the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$, and the fourth equality follows because I_0 and C are constant along the edge subdivided by $\widehat{\Gamma}$.

This final expression only depends on I_0. Since $I_0 = I + J$ and

$$C(e)^2 \geq C(e)J(e) \geq 0$$

for all e, we have the inequalities

$$\sum_{e\in E(\widehat{\Gamma})} I_0(e)C(e)\widehat{\ell}_e \geq \sum_{e\in E(\widehat{\Gamma})} I(e)C(e)\widehat{\ell}_e \geq \sum_{e\in E(\widehat{\Gamma})} (I_0(e) - C(e))C(e)\widehat{\ell}_e\,.$$

The middle term here vanishes after the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$, while the first and third terms can be simplified because I_0 and C are constant along the edge subdivided by $\widehat{\Gamma}$. If we let $F = I_0$ denote the flow on Γ, we obtain the inequalities

$$\sum_{e\in E(\Gamma)} F(e)C(e)\ell_e \geq 0 \geq \sum_{e\in E(\Gamma)} (F(e) - C(e))C(e)\ell_e\,. \tag{46}$$

But for general ℓ_e, exactly one flow F with $\mathrm{div}(F) = \underline{\deg}_{k,A}$ satisfies these inequalities (since the set of such flows is an arithmetic progression with difference C). Therefore the inequalities (46) describe a (coarser) subdivision where $\mathfrak{L}$ can be defined. We summarize what we have computed in the following result.

[35] If $D \neq 0$, it is easy to see that such a J is actually unique.

Proposition 56. *Let Γ be a graph with cycle number 1. Let C be a flow on Γ with value 1 around the unique cycle and 0 elsewhere. Then*

$$\mathfrak{L} = \sum_{e \in E(\Gamma)} (F(e)^2 - F(e)C(e)) \cdot \ell_e,$$

where F is any flow on Γ satisfying $\mathrm{div}(F) = \underline{\deg}_{k,A}$ and the inequalities (46).

By putting the pieces of the formula for logDR together, we obtain a simple description of the piecewise linear function giving the difference between logDR and standard DR in genus 1.

Theorem 57. *In genus 1, we have*

$$\mathsf{logDR}_{1,A} = \mathsf{DR}_{1,A} - \frac{1}{2}\Phi(\mathfrak{L}'),$$

where $\mathfrak{L}'$ is the piecewise linear function defined as follows: $\mathfrak{L}'$ is zero on cones coming from a stable tree Γ, and otherwise $\mathfrak{L}'$ is given by

$$\mathfrak{L}' = \sum_{e \in E(C_\Gamma)} (F(e)^2 - F(e)C(e)) \cdot \ell_e$$

(for C and F as in Proposition 56).

Proof. In genus 1 and degree 1, the logDR formula (19) yields

$$\mathsf{logDR}_{1,A} = -\frac{1}{2}\eta + \Phi([\mathfrak{P}]_{\deg 1}) - \frac{1}{2}\Phi(\mathfrak{L}).$$

We compare the above with the standard DR formula discussed in Section 8.1. The η terms are identical, and we have already observed that the degree 1 part of $\mathfrak{P}$ agrees with the non-separating edge term in (45). Meanwhile, the part of $-\frac{1}{2}\Phi(\mathfrak{L})$ coming from separating edges in Proposition 56 agrees with the separating edge term in (45). An edge in Γ is non-separating if and only if it is in the cycle C_Γ, so removing those terms from $\mathfrak{L}$ leaves precisely $\mathfrak{L}'$. ♦

Remark 58. After a little combinatorial manipulation, we obtain an alternative description of the piecewise linear function $\mathfrak{L}'$. Suppose F is a flow on Γ satisfying $\mathrm{div}(F) = \underline{\deg}_{k,A}$. Define a piecewise linear function on σ_Γ by

$$m_F = \min\left(\sum_{\substack{e \in E(\Gamma) \\ F(e)C(e)>0}} F(e)C(e)\ell_e, \sum_{\substack{e \in E(\Gamma) \\ F(e)C(e)<0}} -F(e)C(e)\ell_e \right).$$

The function m_F is zero if F is not acyclic. It is then easily checked that on σ_Γ we have

$$\mathfrak{L}' = 2\sum_F m_F\,,$$

where the sum runs over all acyclic F. $\Diamond$

Remark 59. The logarithmic genus 1 computation can be used to compute an interesting class in $\mathsf{CH}^2(\overline{\mathcal{M}}_{1,n})$ as follows. Take two genus 1 cycles with the same n,

$$\mathsf{logDR}_{1,A}\,,\ \mathsf{logDR}_{1,B} \in \mathsf{logCH}^1(\overline{\mathcal{M}}_{1,\mathsf{n}})\,.$$

Individually, the cycles push forward to the standard cycles $\mathsf{DR}_{1,A}, \mathsf{DR}_{1,B} \in \mathsf{CH}^1(\overline{\mathcal{M}}_{1,n})$, but the pushforward of their product,

$$\pi_*(\mathsf{logDR}_{1,A}\cdot \mathsf{logDR}_{1,B}) \in \mathsf{CH}^2(\overline{\mathcal{M}}_{1,n})\,,$$

is *not* equal to the product of the standard DR cycles. These *double-double ramification cycles* are discussed further in Section 9.1. Here, we simply sketch how to compute them in genus 1 using the above formulas for logDR.

By Theorem 57 and a straightforward argument for the vanishing of the cross terms, we have

$$\pi_*(\mathsf{logDR}_{1,A}\cdot \mathsf{logDR}_{1,B}) = \mathsf{DR}_{1,A}\cdot \mathsf{DR}_{1,B} + \frac{1}{4}\pi_*(\Phi(\mathfrak{L}'_A\cdot \mathfrak{L}'_B))\,.$$

By Remark 58, this second term is defined on a cone σ_Γ by the polynomial

$$\pi_*\left(\sum_{F_A,F_B} m_{F_A} m_{F_B}\right),$$

where F_A, F_B run over acyclic flows balancing A and B respectively and π_* is the pushforward from piecewise polynomials on a subdivision of a cone to polynomials on that cone. After working out the algebra of the pushforward and translating into standard tautological class notation, we obtain

$$\begin{aligned}\pi_*(\mathsf{logDR}_{1,A}\cdot \mathsf{logDR}_{1,B}) &= \mathsf{DR}_{1,A}\cdot \mathsf{DR}_{1,B}\\ &\quad - \sum_{\Gamma\text{ double edge graph}} \frac{1}{2} j_{\Gamma *}\left(\frac{A_\Gamma^2 B_\Gamma^2 - A_\Gamma^2 - B_\Gamma^2 + \mathsf{gcd}(A_\Gamma, B_\Gamma)^2}{12}\right),\end{aligned}$$

where the sum runs over the isomorphism classes of stable graphs Γ with two vertices and a double edge connecting them and A_Γ, B_Γ are the absolute values of the degrees of $\underline{\deg}_{k,A}$, $\underline{\deg}_{k,B}$ respectively on one of the vertices (it does not matter which one since the total degree is 0). $\Diamond$

9 Computational tools and higher double ramification cycles

9.1 A Sage implementation of logDR

A Sage package `logtaut` for computations with logarithmic double ramification cycles is being developed as part of the `admcycles` package [24]. The package is able to compute/manipulate the following mathematical structures discussed in our paper:

- the vector space of stability conditions θ of type (g, n) and degree d (with a description of a natural basis),
- cone stacks and their (strict) piecewise polynomial functions (together with natural operations such as pushforwards under subdivision maps),
- the particular cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$, the subdivision induced by a vector $A \in \mathbb{Z}^n$ together with a nondegenerate stability condition θ of degree $|A|$, and the natural map
$$\mathsf{sPP}(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \to \mathsf{R}^*(\overline{\mathcal{M}}_{g,n})$$
discussed in Section 6.2,
- the piecewise polynomial functions $\mathfrak{L}$ and $\mathfrak{P}$ of equation (19) determining the logarithmic double ramification cycle via Theorem B.

As an example of a calculation in `logtaut`, we can write an explicit relation in $\mathsf{logR}^2(\overline{\mathcal{M}}_{1,3})$ obtained by Theorem C (ii). Concretely, we can combine the vanishing from Theorem C with the classical vanishing of the formula $\mathsf{P}^h_{g,A}$ for the double ramification cycle in degree h higher than g.[36] Taking the difference, we obtain

$$\Delta = \mathbf{P}^{2,\theta}_{1,(2,-1,-1)} - \mathsf{P}^2_{1,(2,-1,-1)} = 0 \in \mathsf{logR}^2(\overline{\mathcal{M}}_{1,3}),$$

for a small and non-degenerate stability condition θ.[37] After expressing $\mathsf{P}^2_{1,(2,-1,-1)}$ in terms of piecewise polynomials via Theorem 47, the advantage of the difference Δ is that the piecewise polynomial parts of $\mathbf{P}^{2,\theta}_{1,(2,-1,-1)}$ and $\mathsf{P}^2_{1,(2,-1,-1)}$ cancel on all but two maximal cones in $\Sigma_{\overline{\mathcal{M}}_{1,3}}$. With `logtaut`, we then compute that Δ is the codimension 2 part of the mixed-degree class

$$\left(1 + 2\psi_1 + \frac{1}{2}(\psi_2 + \psi_3)\right) \cdot \Phi(\mathfrak{P}) \tag{47}$$

for the piecewise polynomial function $\mathfrak{P}$ on $\Sigma_{\overline{\mathcal{M}}_{1,3}}$ illustrated in Figure 5.

[36] The formula $\mathsf{P}^h_{g,A}$ was first presented in [40] and the claimed vanishing was proven in [22].

[37] The choice of the stability condition θ does not affect the relation Δ in this case.

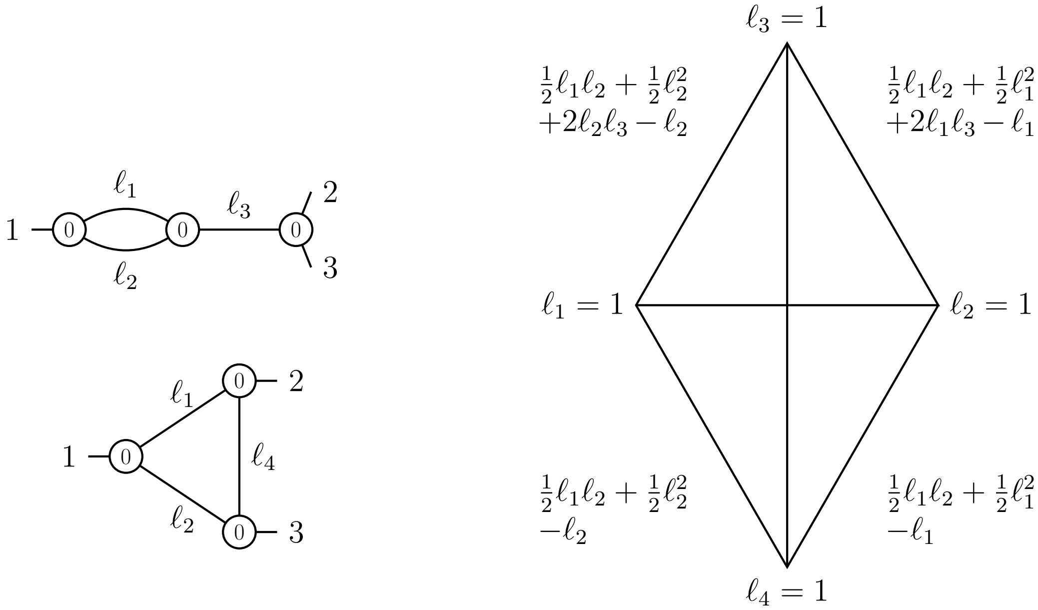
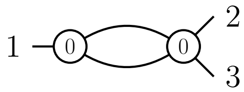
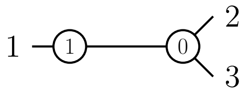


Figure 5: The piecewise polynomial $\mathfrak{P}$, which is non-vanishing on precisely two (neighboring) maximal cones of $\Sigma_{\overline{\mathcal{M}}_{1,3}}$. We illustrate the stable graphs associated to these cones and the slice through them corresponding to all edge lengths summing to 1. The cones are subdivided at the ray $\ell_1 = \ell_2 = 1, \ell_3 = \ell_4 = 0$, and we give the piecewise polynomial $\mathfrak{P}$ on the resulting four chambers.

Using the methods of Section 6.2, we can translate the relation $\Delta = 0$ into the language of normally decorated strata classes: consider the graphs

$\Gamma =$  , $\delta_{2,3} =$  .

Let $\pi : \widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{1,3}$ be the blow-up of the stratum associated to Γ, and denote by E_Γ the exceptional divisor of π. After expressing $\Phi(\mathfrak{P})$ in terms of normally decorated strata classes, we obtain

$$\Delta = \frac{1}{2} \cdot \pi^* [\Gamma] - (2\psi_1 + \frac{1}{2}(\psi_2 + \psi_3) - 2\delta_{23}) \cdot E_\Gamma + \frac{1}{2} E_\Gamma^2 \in \mathsf{CH}^2(\widehat{\mathcal{M}}) \,. \qquad (48)$$

It is interesting to note that the coefficient $(2\psi_1 + \frac{1}{2}(\psi_2 + \psi_3) - 2\delta_{23})$ appearing here is precisely the compact type part of the formula $\mathsf{P}^1_{1,(2,-1,-1)}$, giving the pullback of the theta divisor under the Abel-Jacobi map associated to $(2, -1, -1)$.

9.2 Double-double ramification cycles

One of the initial motivations for the study of logDR was that of understanding the *double-double ramification cycle* which arises in the virtual localization formula for the Gromov-Witten theory of log toric surfaces [30] as a vertex term (replacing the quadratic Hodge integrals in the localization formula for the Gromov-Witten theory of toric surfaces [31]). The geometry of these higher double ramification cycles plays a central role in the study of stable maps to log toric targets [63]. Using the formula of Theorem B and the Sage package `logtaut`, we can now compute these cycles in moderate genus (limited by computing capacity).

To motivate the higher double ramification cycles, we begin by recalling that the natural formula

$$\mathsf{DR}_{g,A} \cdot \mathsf{DR}_{g,B} \stackrel{?}{=} \mathsf{DR}_{g,A+B} \cdot \mathsf{DR}_{g,B}$$

holds on the locus of curves of compact type, but fails on $\overline{\mathcal{M}}_{g,n}$. However, by a result of [38], the analogue for logDR is always true:

$$\mathsf{logDR}_{g,A} \cdot \mathsf{logDR}_{g,B} = \mathsf{logDR}_{g,A+B} \cdot \mathsf{logDR}_{g,B} \in \mathsf{logCH}^{2g}(\overline{\mathcal{M}}_{g,n}) \,, \tag{49}$$

which is a special case of a $\mathrm{GL}_2(\mathbb{Z})$-invariance property for logDR. The *double-double ramification cycle* is defined by

$$\mathsf{DDR}_{g,A,B} = \pi_* \left(\mathsf{logDR}_{g,A} \cdot \mathsf{logDR}_{g,B}\right) \in \mathsf{CH}^{2g}(\overline{\mathcal{M}}_{g,n}),$$

and may be viewed as a "corrected" version of the product $\mathsf{DR}_{g,A} \cdot \mathsf{DR}_{g,B}$. Higher double ramification cycles are defined by pushforwards of products of more logarithmic double ramification cycles.

The double-double ramification cycle is trivial in genus 0:

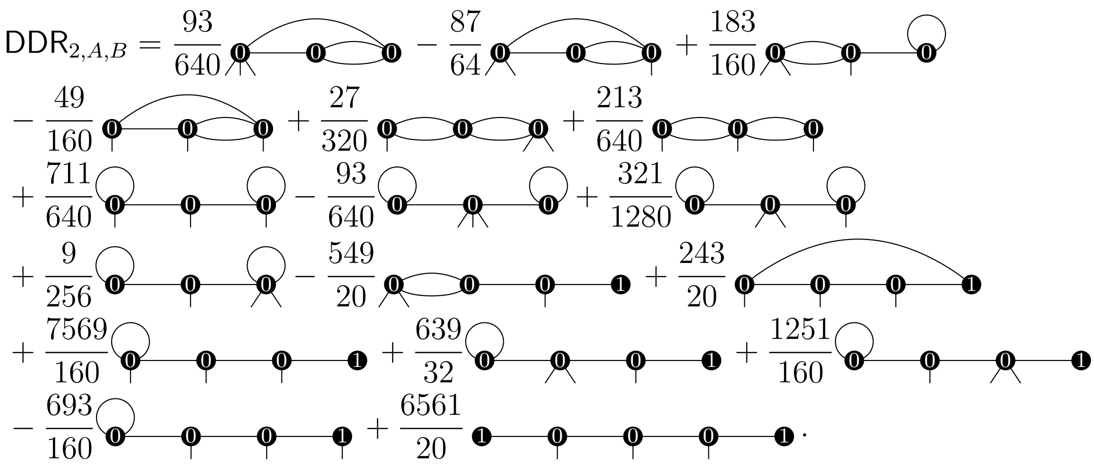
$$\mathsf{DDR}_{0,A,B} = 1 \in \mathsf{logCH}^0(\overline{\mathcal{M}}_{0,n}) \,.$$

A full calculation in genus 1 of $\mathsf{DDR}_{1,A,B}$ was presented in Remark 59 using the formula for logDR of Theorem B. That $\mathsf{DDR}_{g,A,B}$ and the higher analogues are tautological classes on the moduli spaces of curves for all g follows also from results of [39, 53], but the formula of Theorem B provides the only known effective method of calculation. The package `logtaut` produces explicit formulas for $\mathsf{DDR}_{g,A,B}$ in the tautological ring.

We illustrate below the double-double cycle in case $g = 2$, $n = 3$, $A = [3, -3, 0]$, and $B = [0, 3, -3]$. Since the class

$$\mathsf{DDR}_{2,A,B} \in \mathsf{CH}^4(\overline{\mathcal{M}}_{2,3})$$

is invariant under permuting the markings (as can be seen by applying (49)), we can express the answer via `logtaut` as a sum over graphs[38] without specifying the ordering of the legs:



Two constructions of tautological relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$ were given in Theorem C. The $\mathrm{GL}_2(\mathbb{Z})$-invariance property for the double-double ramification cycle yields a third construction: apply Theorem B to all four terms of (49).

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