# DESCENDENTS ON LOCAL CURVES: RATIONALITY

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Abstract. We study the stable pairs theory of local curves in 3-folds with descendent insertions. The rationality of the partition function of descendent invariants is established for the full local curve geometry (equivariant with respect to the scaling 2-torus) including relative conditions and odd degree insertions for higher genus curves. The capped 1-leg descendent vertex (equivariant with respect to the 3-torus) is also proven to be rational. The results are obtained by combining geometric constraints with a detailed analysis of the poles of the descendent vertex.

Contents



## 0. Introduction

0.1. **Descendents.** Let $X$ be a nonsingular 3-fold, and let

$$\beta \in H_2(X, \mathbb{Z})$$

be a nonzero class. We will study here the moduli space of stable pairs

$$[\mathcal{O}_X \xrightarrow{s} F] \in P_n(X, \beta)$$

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where $F$ is a pure sheaf supported on a Cohen-Macaulay subcurve of $X$, $s$ is a morphism with 0-dimensional cokernel, and

$$\chi(F) = n, \quad [F] = \beta.$$

The space $P_n(X,\beta)$ carries a virtual fundamental class obtained from the deformation theory of complexes in the derived category [24]. A review can be found in Section 1.

Since $P_n(X,\beta)$ is a fine moduli space, there exists a universal sheaf

$$\mathbb{F} \to X \times P_n(X,\beta),$$

see Section 2.3 of [24]. For a stable pair $[\mathcal{O}_X \to F] \in P_n(X,\beta)$, the restriction of $\mathbb{F}$ to the fiber

$$X \times [\mathcal{O}_X \to F] \subset X \times P_n(X,\beta)$$

is canonically isomorphic to $F$. Let

$$\pi_X \colon X \times P_n(X,\beta) \to X,$$
$$\pi_P \colon X \times P_n(X,\beta) \to P_n(X,\beta)$$

be the projections onto the first and second factors. Since $X$ is nonsingular and $\mathbb{F}$ is $\pi_P$-flat, $\mathbb{F}$ has a finite resolution by locally free sheaves. Hence, the Chern character of the universal sheaf $\mathbb{F}$ on $X \times P_n(X,\beta)$ is well-defined. By definition, the operation

$$\pi_{P*}\big(\pi_X^*(\gamma)\cdot \mathrm{ch}_{2+i}(\mathbb{F}) \cap (\pi_P^*(\ \cdot\ )\big) \colon H_*(P_n(X,\beta)) \to H_*(P_n(X,\beta))$$

is the action of the descendent $\tau_i(\gamma)$, where $\gamma \in H^*(X,\mathbb{Z})$.

For nonzero $\beta \in H_2(X,\mathbb{Z})$ and arbitrary $\gamma_i \in H^*(X,\mathbb{Z})$, define the stable pairs invariant with descendent insertions by

$$\begin{aligned}\left\langle \prod_{j=1}^k \tau_{i_j}(\gamma_j)\right\rangle_{n,\beta}^X &= \int_{[P_n(X,\beta)]^{vir}} \prod_{j=1}^k \tau_{i_j}(\gamma_j) \\ &= \int_{P_n(X,\beta)} \prod_{j=1}^k \tau_{i_j}(\gamma_j)\Big([P_n(X,\beta)]^{vir}\Big).\end{aligned}$$

The partition function is

$$\mathsf{Z}_\beta^X\left(\prod_{j=1}^k \tau_{i_j}(\gamma_j)\right) = \sum_n \left\langle \prod_{j=1}^k \tau_{i_j}(\gamma_j)\right\rangle_{n,\beta}^X q^n.$$

Since $P_n(X,\beta)$ is empty for sufficiently negative $n$, $\mathsf{Z}_\beta^X\big(\prod_{j=1}^k \tau_{i_j}(\gamma_j)\big)$ is a Laurent series in $q$. The following conjecture was made in [25].

**Conjecture 1.** *The partition function $\mathsf{Z}_\beta^X\big(\prod_{j=1}^k \tau_{i_j}(\gamma_j)\big)$ is the Laurent expansion of a rational function in $q$.*

If only primary field insertions $\tau_0(\gamma)$ appear, Conjecture 1 is known for toric $X$ by [13, 17] and for Calabi-Yau $X$ by [3, 28] together with [9]. In the presence of descendents $\tau_{i>0}(\gamma)$, very few results have been obtained.

The central result of the present paper is the proof of Conjecture 1 in case $X$ is the total space of an rank 2 bundle over a curve, a *local curve*. In fact, the rationality of the stable pairs descendent theory of relative local curves is proven.

0.2. **Local curves.** Let $N$ be a split rank 2 bundle on a nonsingular projective curve $C$ of genus $g$,

$$N = L_1 \oplus L_2. \tag{1}$$

The splitting determines a scaling action of a 2-dimensional torus

$$T = \mathbb{C}^* \times \mathbb{C}^*$$

on $N$. The *level* of the splitting is the pair of integers $(k_1, k_2)$ where,

$$k_i = \deg(L_i).$$

Of course, the scaling action and the level depend upon the choice of splitting (1).

Let $s_1, s_2 \in H^*_{\mathbf{T}}(\bullet)$ be the first Chern classes of the standard representations of the first and second $\mathbb{C}^*$-factors of $T$ respectively. We define

$$\left\langle \prod_{j=1}^{k} \tau_{i_j}(\gamma_j) \right\rangle^N_{n,d} = \int_{[P_n(N,d)]^{vir}} \prod_{j=1}^{k} \tau_{i_j}(\gamma_j) \quad \in \mathbb{Q}(s_1, s_2)\ . \tag{2}$$

Here, the curve class is $d$ times the zero section $C \subset N$ and

$$\gamma_j \in H^*(C, \mathbb{Z})\ .$$

The right side of (2) is defined by $T$-equivariant residues as in [4, 20]. Let

$$\mathsf{Z}^N_d \left( \prod_{j=1}^{k} \tau_{i_j}(\gamma_j) \right)^T = \sum_n \left\langle \prod_{j=1}^{k} \tau_{i_j}(\gamma_j) \right\rangle^N_{n,d} q^n.$$

**Theorem 1.** *$\mathsf{Z}^N_d \big( \prod_{j=1}^{k} \tau_{i_j}(\gamma_j) \big)^T$ is the Laurent expansion in $q$ of a rational function in $\mathbb{Q}(q, s_1, s_2)$.*

The rationality of Theorem 1 holds even when $\gamma_j \in H^1(C, \mathbb{Z})$. Theorem 1 is proven via the stable pairs theory of relative local curves and the 1-leg descendent vertex. The proof provides a method to compute $\mathsf{Z}^N_d \big( \prod_{j=1}^{k} \tau_{i_j}(\gamma_j) \big)^T$.

0.3. **Relative local curves.** The fiber of $N$ over a point $p \in C$ determines a $T$-invariant divisor

$$N_p \subset N$$

isomorphic to $\mathbb{C}^2$ with the standard $T$-action. For $r > 0$, we will consider the local theory of $N$ relative to the divisor

$$S = \bigcup_{i=1}^{r} N_{p_i} \subset N$$

determined by the fibers over $p_1, \ldots, p_r \in C$. Let $P_n(N/S, d)$ denote the relative moduli space of stable pairs, see [24] for a discussion.

For each $p_i$, let $\eta^i$ be a partition of $d$ weighted by the equivariant Chow ring,

$$A_T^*(N_{p_i}, \mathbb{Q}) \cong \mathbb{Q}[s_1, s_2],$$

of the fiber $N_{p_i}$. By Nakajima's construction, a weighted partition $\eta^i$ determines a $T$-equivariant class

$$\mathsf{C}_{\eta^i} \in A_T^*(\mathrm{Hilb}(N_{p_i}, d), \mathbb{Q})$$

in the Chow ring of the Hilbert scheme of points. In the theory of stable pairs, the weighted partition $\eta^i$ specifies relative conditions via the boundary map

$$\epsilon_i : P_n(N/S, d) \to \mathrm{Hilb}(N_{p_i}, d).$$

An element $\eta \in \mathcal{P}(d)$ of the set of partitions of $d$ may be viewed as a weighted partition with all weights set to the identity class

$$1 \in H_T^*(N_{p_i}, \mathbb{Q}) \ .$$

The Nakajima basis of $A_T^*(\mathrm{Hilb}(N_{p_i}, d), \mathbb{Q})$ consists of identity weighted partitions indexed by $\mathcal{P}(d)$. The $T$-equivariant intersection pairing in the Nakajima basis is

$$g_{\mu\nu} = \int_{\mathrm{Hilb}(N_{p_i}, d)} \mathsf{C}_\mu \cup \mathsf{C}_\nu = \frac{1}{(s_1 s_2)^{\ell(\mu)}} \frac{(-1)^{d-\ell(\mu)}}{\mathfrak{z}(\mu)} \ \delta_{\mu,\nu},$$

where

$$\mathfrak{z}(\mu) = \prod_{i=1}^{\ell(\mu)} \mu_i \cdot |\mathrm{Aut}(\mu)|.$$

Let $g^{\mu\nu}$ be the inverse matrix.

The notation $\eta([0])$ will be used to set all weights to $[0] \in A_T^*(N_{p_i}, \mathbb{Q})$. Since

$$[0] = s_1 s_2 \in A_T^*(N_{p_i}, \mathbb{Q}),$$

the weight choice has only a mild effect.

Following the notation of [4, 20], the relative stable pairs partition function with descendents,

$$\mathsf{Z}^{N/S}_{d,\eta^1,\dots,\eta^r}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^T = \sum_{n\in\mathbb{Z}} q^n \int_{[P_n(N/S,d)]^{vir}} \prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\ \prod_{i=1}^{r}\epsilon_i^*(\mathsf{C}_{\eta^i}),$$

is well-defined for local curves.

**Theorem 2.** $\mathsf{Z}^{N/S}_{d,\eta^1,\dots,\eta^r}\big(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\big)^T$ *is the Laurent expansion in $q$ of a rational function in $\mathbb{Q}(q,s_1,s_2)$.*

Theorem 2 implies Theorem 1 by the degeneration formula. The proof of Theorem 2 uses the TQFT formalism exploited in [4, 20] together with an analysis of the capped 1-leg descendent vertex.

0.4. **Capped 1-leg descendent vertex.** The capped 1-leg geometry concerns the trivial bundle,

$$N = \mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1} \to \mathbb{P}^1\ ,$$

relative to the fiber

$$N_\infty \subset N$$

over $\infty \in \mathbb{P}^1$. Capped geometries have been studied (without descendents) in [13].

The total space $N$ naturally carries an action of a 3-dimensional torus

$$\mathbf{T} = T\times\mathbb{C}^*\ .$$

Here, $T$ acts as before by scaling the factors of $N$ and preserving the relative divisor $N_\infty$. The $\mathbb{C}^*$-action on the base $\mathbb{P}^1$ which fixes the points $0,\infty\in\mathbb{P}^1$ lifts to an additional $\mathbb{C}^*$-action on $N$ fixing $N_\infty$.

The equivariant cohomology ring $H^*_{\mathbf{T}}(\bullet)$ is generated by the Chern classes $s_1$, $s_2$, and $s_3$ of the standard representation of the three $\mathbb{C}^*$-factors. We define

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^{\mathbf{T}} = \sum_{n\in\mathbb{Z}} q^n \int_{[P_n(N/N_\infty,d)]^{vir}} \prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\ \cup\epsilon_\infty^*(\mathsf{C}_\eta), \tag{3}$$

by $\mathbf{T}$-equivariant residues.[1] Here, $\gamma_j \in H^*_{\mathbf{T}}(\mathbb{P}^1,\mathbb{Z})$. By definition, the partition function (3) is a Laurent series in $q$ with coefficients in the field $\mathbb{Q}(s_1,s_2,s_3)$.

[1]The $T$-equivariant series associated to the cap will be denoted

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^{T}\ ,$$

for $\gamma_j \in H^*(\mathbb{P}^1,\mathbb{Z})$.

**Theorem 3.** $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^{\mathbf{T}}$ *is the Laurent expansion in $q$ of a rational function in $\mathbb{Q}(q,s_1,s_2,s_3)$.*

Theorem 3 is the main contribution of the paper. The result relies upon a delicate cancellation of poles in the vertex formula of [25] for stable pairs invariants. Theorem 2 is derived as a consequence.

0.5. **Stationary theory.** In [22], we prove reduction rules for stationary descendents in the $T$-equivariant local theory of curves. Let $\mathsf{p} \in H^2(C,\mathbb{Z})$ be the class of a point on a nonsingular curve $C$. The stationary descendents are $\tau_i(\mathsf{p})$. For the degree $d$ local theory of $C$, we find universal formulas expressing the descendents $\tau_{i>d}(\mathsf{p})$ in terms of the descendents $\tau_{i\leq d}(\mathsf{p})$. The reduction rules provide an alternative (and more effective) approach to the rationality of Theorem 2 in the stationary case.

The exact calculation in [22] of the basic stationary descendent series

$$\mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))^T = \frac{q^d}{d!}\left(\frac{s_1+s_2}{s_1s_2}\right)\frac{1}{2}\sum_{i=1}^{d}\frac{1+(-q)^i}{1-(-q)^i}$$

plays a special role. The coefficient of $q^d$,

$$\langle \tau_d, (d)\rangle_{\mathrm{Hilb}(\mathbb{C}^2,d)} = \frac{1}{2\cdot(d-1)!}\left(\frac{s_1+s_2}{s_1s_2}\right),$$

is the classical $T$-equivariant pairing on the Hilbert scheme of $d$ points in $\mathbb{C}^2$.

The $T$-equivariant stationary descendent theory is simpler than the full descendent theories studied here. We do not know an alternative approach to the rationality of the full $T$-equivariant descendent theory of local curves. Even the rationality of the $\mathbf{T}$-equivariant stationary theory of the cap does not appear to be accessible via [22].

The methods of [22] also prove a functional equation for the partition function for stationary descendents which is a special case of the following conjecture we make here.

**Conjecture 2.** *Let $\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^T$ be the Laurent expansion in $q$ of $F(q,s_1,s_2)\in\mathbb{Q}(q,s_1,s_2)$. Then, $F$ satisfies the functional equation*

$$F(q^{-1},s_2,s_2) = (-1)^{\Delta+|\eta|-\ell(\eta)+\sum_{j=1}^{k} i_j} q^{-\Delta}F(q,s_1,s_2),$$

*where the constants are defined by*

$$\Delta = \int_\beta c_1(T_N), \quad |\eta| = \sum_{i=1}^{r}|\eta^i|, \quad \textit{and} \quad \ell(\eta)=\sum_{i=1}^{r}\ell(\eta^i)\ .$$

Here, $T_N$ is the tangent bundle of the 3-fold $N$, and $\beta$ is the curve class given by $d$ times the 0-section. We believe the straightforward generalization of Conjecture 2 to all descendent partition functions for the stable pairs theories of relative 3-folds (equivariant and non-equivariant) holds. If there are no descendents, the functional equation is known to hold in the toric case [13]. The strongest evidence with descendents is the stationary result of Theorem 2 of [22].

0.6. **Denominators.** The descendent partition functions for the stable pairs theory of local curves have very restricted denominators when considered as rational functions in $q$ with coefficients in $\mathbb{Q}(s_1, s_2)$ for Theorems 1-2 and rational functions in $q$ with coefficients in $\mathbb{Q}(s_1, s_2, s_3)$ for Theorem 3.

**Conjecture 3.** *The denominators of the degree $d$ descendent partition functions $\mathsf{Z}$ of Theorems 1, 2, and 3 are products of factors of the form $q^k$ and*

$$1 - (-q)^r$$

*for $1 \leq r \leq d$.*

In other words, the poles in $-q$ are conjectured to occur only at 0 and $r^{th}$ roots for $r$ at most $d$ (and have no dependence on the variables $s_i$). Conjecture 3 is proven in Theorem 5 of Section 9 for descendents of even cohomology. The denominator restriction yields new results about the 3-point functions of the Hilbert scheme of points of $\mathbb{C}^2$ stated as a Corollary to Theorem 5.

0.7. **Descendent theory of toric 3-folds.** Calculation of the descendent theory of stable pairs on nonsingular toric 3-folds requires knowledge of the capped 3-leg descendent vertex.[2] The rationality of the capped 3-leg descendent vertex is proven in [23] via a geometric reduction to the 1-leg case of Theorem 3. As a result, Conjecture 1 is established for all nonsingular toric 3-folds. The rationality of the descendent theory of several log Calabi-Yau geometries is also proven in [23].

0.8. **Plan of the paper.** After a brief review of the theory of stable pairs in Section 1, the vertex formalism of [25] is summarized in Section 2. The proof of Theorem 3 is presented in Section 3 for descendents of the nonrelative $\mathbf{T}$-fixed point $0 \in \mathbb{P}^1$ modulo the pole cancellation property established in Section 4. Depth and the rubber calculus for stable pairs of local curves are discussed in Sections 5 and 6. The

[2]The capped 2-leg descendent vertex is, of course, a specialization of the 3-leg vertex.

full statement of Theorem 3 is obtained in Section 7. In fact, the rationality of the $\mathbf{T}$-equivariant descendent theories of all twisted caps and tubes is established in Section 7. Theorems 1 and 2 are proven as a consequence of Theorem 3 in Section 8 using the methods of [4, 18, 20]. Denominators are studied in Section 9.

0.9. **Other directions.** Whether parallel results can be obtained for the local Gromov-Witten theory of curves [4] is an interesting question. Although conjectured to be equivalent, the descendent theory of stable pairs on 3-folds appears more accessible than descendents in Gromov-Witten theory. The direct vertex analysis undertaken here for Theorem 3 must be replaced in Gromov-Witten theory with a deeper understanding of Hodge integrals [6].

Another advantage of stable pairs, at least for Calabi-Yau geometries, is the possibility of using motivic integrals with respect to Behrend's $\chi$-function [1], see [26] for an early use. Recently, D. Maulik and R. P. Thomas have been pursuing $\chi$-functions in the log Calabi-Yau setting. Applications to the rationality of descendent series in Fano geometries might be possible.

A principal motivation of studying descendents for stable pairs is the perspective of [16]. Descendents constrain relative invariants. With the degeneration formula, the possibility emerges of studying stable pairs on arbitrary (non-toric) 3-folds.

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## 1. Stable pairs on 3-folds

1.1. **Definitions.** Let $X$ be a nonsingular quasi-projective 3-fold over $\mathbb{C}$ with polarization $L$. Let $\beta \in H_2(X, \mathbb{Z})$ be a nonzero class. The moduli space $P_n(X, \beta)$ parameterizes *stable pairs*

$$\mathcal{O}_X \xrightarrow{s} F \tag{4}$$

where $F$ is a sheaf with Hilbert polynomial

$$\chi(F \otimes L^k) = k \int_\beta c_1(L) + n$$

and $s \in H^0(X, F)$ is a section. The two stability conditions are:

(i) the sheaf $F$ is pure with proper support,
(ii) the section $\mathcal{O}_X \xrightarrow{s} F$ has 0-dimensional cokernel.

By definition, *purity* (i) means every nonzero subsheaf of $F$ has support of dimension 1 [8]. In particular, purity implies the (scheme-theoretic) support $C_F$ of $F$ is a Cohen-Macaulay curve. A quasi-projective moduli space of stable pairs can be constructed by a standard GIT analysis of Quot scheme quotients [10].

For convenience, we will often refer to the stable pair (4) on $X$ simply by $(F, s)$.

1.2. **Virtual class.** A central result of [24] is the construction of a virtual class on $P_n(X, \beta)$. The standard approach to the deformation theory of pairs fails to yield an appropriate 2-term deformation theory for $P_n(X, \beta)$. Instead, $P_n(X, \beta)$ is viewed in [24] as a moduli space of complexes in the derived category.

Let $D^b(X)$ be the bounded derived category of coherent sheaves on $X$. Let

$$I^\bullet = \{\mathcal{O}_X \to F\} \in D^b(X)$$

be the complex determined by a stable pair. The tangent-obstruction theory obtained by deforming $I^\bullet$ in $D^b(X)$ while fixing its determinant is 2-term and governed by the groups[3]

$$\mathrm{Ext}^1(I^\bullet, I^\bullet)_0, \quad \mathrm{Ext}^2(I^\bullet, I^\bullet)_0.$$

The virtual class

$$[P_n(X, \beta)]^{vir} \in A_{\dim^{vir}}(P_n(X, \beta), \mathbb{Z})$$

is then obtained by standard methods [2, 11]. The virtual dimension is

$$\dim^{vir} = \int_\beta c_1(T_X).$$

Apart from the derived category deformation theory, the construction of the virtual class of $P_n(X, \beta)$ is parallel to virtual class construction in DT theory [27].

[3]The subscript 0 denotes traceless Ext.

1.3. **Characterization.** Consider the kernel/cokernel exact sequence associated to a stable pair $(F, s)$,

$$0 \to \mathscr{I}_{C_F} \to \mathcal{O}_X \xrightarrow{s} F \to Q \to 0. \tag{5}$$

The kernel is the ideal sheaf of the Cohen-Macaulay support curve $C_F$ by Lemma 1.6 of [24]. The cokernel $Q$ has dimension 0 support by stability. The *reduced* support scheme, $\mathrm{Support}^{red}(Q)$, is called the *zero locus* of the pair. The zero locus lies on $C_F$.

Let $C \subset X$ be a fixed Cohen-Macaulay curve. Stable pairs with support $C$ and bounded zero locus are characterized as follows. Let

$$\mathfrak{m} \subset \mathcal{O}_C$$

be the ideal in $\mathcal{O}_C$ of a 0-dimensional subscheme. Since

$$\mathscr{H}om(\mathfrak{m}^r/\mathfrak{m}^{r+1}, \mathcal{O}_C) = 0$$

by the purity of $\mathcal{O}_C$, we obtain an inclusion

$$\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C) \subset \mathscr{H}om(\mathfrak{m}^{r+1}, \mathcal{O}_C).$$

The inclusion $\mathfrak{m}^r \hookrightarrow \mathcal{O}_C$ induces a canonical section

$$\mathcal{O}_C \hookrightarrow \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C).$$

**Proposition 1.** *A stable pair $(F, s)$ with support $C$ satisfying*

$$\mathrm{Support}^{red}(Q) \subset \mathrm{Support}(\mathcal{O}_C/\mathfrak{m})$$

*is equivalent to a subsheaf of $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$, $r \gg 0$.*

Alternatively, we may work with coherent subsheaves of the quasi-coherent sheaf

$$\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C \tag{6}$$

Under the equivalence of Proposition 1, the subsheaf of (6) corresponds to $Q$, giving a subsheaf $F$ of $\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)$ containing the canonical subsheaf $\mathcal{O}_C$ and the sequence

$$0 \to \mathcal{O}_C \xrightarrow{s} F \to Q \to 0.$$

Proposition 1 is proven in [24].

## 2. $\mathbf{T}$-fixed points with one leg

2.1. **Affine chart.** Let $N$ be the 3-fold total space of

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathbb{P}^1$$

carrying the action of the 3-dimensional torus $\mathbf{T}$ as in Section 0.4. Let

$$[\mathcal{O}_N \xrightarrow{s} F] \in P_n(N,d)^{\mathbf{T}} \tag{7}$$

be a $\mathbf{T}$-fixed stable pair. The curve class is $d[\mathbb{P}^1]$.

Let $U \subset N$ be the $\mathbf{T}$-invariant affine chart associated to the $\mathbf{T}$-fixed point of $N$ lying over $0 \in \mathbb{P}^1$. The restriction of the stable pair (7) to the chart $U$,

$$\mathcal{O}_U \xrightarrow{s_U} F_U \ , \tag{8}$$

determines an invariant section $s_U$ of an equivariant sheaf $F_U$.

Let $x_1, x_2, x_3$ be coordinates on the affine chart $U$ in which the $\mathbf{T}$-action takes the diagonal form,

$$(t_1, t_2, t_3) \cdot x_i = t_i x_i.$$

By convention, $x_1$ and $x_2$ are coordinates on the fibers of $N$ and $x_3$ is a coordinate on the base $\mathbb{P}^1$.

We will characterize the restricted data $(F_U, s_U)$ in the coordinates $x_i$ closely following the presentation of [25].

2.2. **Monomial ideals and partitions.** Let $x_1, x_2$ be coordinates on the plane $\mathbb{C}^2$. A subscheme $S \subset \mathbb{C}^2$ invariant under the action of the diagonal torus,

$$(t_1, t_2) \cdot x_i = t_i x_i$$

must be defined by a monomial ideal $\mathscr{I}_S \subset \mathbb{C}[x_1, x_2]$. If

$$\dim_{\mathbb{C}} \mathbb{C}[x_1, x_2]/\mathscr{I}_S < \infty$$

then $\mathscr{I}_S$ determines a finite partition $\mu_S$ by considering lattice points corresponding to monomials of $\mathbb{C}[x_1, x_2]$ *not* contained in $\mathscr{I}_S$. Conversely, each partition $\mu$ determines a monomial ideal

$$\mu[x_1, x_2] \subset \mathbb{C}[x_1, x_2].$$

Similarly, the subschemes $S \subset \mathbb{C}^3$ invariant under the diagonal $\mathbf{T}$-action are in bijective correspondence with 3-dimensional partitions.

2.3. **Cohen-Macaulay support.** The first step in the characterization of the restricted data (8) is to determine the scheme-theoretic support $C_U$ of $F_U$. If nonempty, $C_U$ is a $\mathbf{T}$-invariant, Cohen-Macaulay subscheme of pure dimension 1.

The $\mathbf{T}$-fixed subscheme $C_U \subset \mathbb{C}^3$ is defined by a monomial ideal

$$\mathscr{I}_C \subset \mathbb{C}[x_1, x_2, x_3].$$

associated to the 3-dimensional partition $\pi$. The localisation

$$(\mathscr{I}_C)_{x_3} \subset \mathbb{C}[x_1, x_2, x_3]_{x_3},$$

is $T$-fixed and corresponds to a 2-dimensional partition $\mu$. Alternatively, the 2-dimensional partitions $\mu$ can be defined as the infinite limit of the $x_3$-constant cross-sections of $\pi$. In order for $C_U$ to have dimension 1, $\mu$ can not be empty.

There exists a unique *minimal* $\mathbf{T}$-fixed subscheme

$$C_\mu \subset \mathbb{C}^3$$

with outgoing partition $\mu$. The 3-dimensional partition corresponding to $C_\mu$ is the infinite cylinder on the $x_3$-axis determined by the 2-dimensional partitions $\mu$. Let

$$\mathscr{I}_\mu = \mu[x_1, x_2] \cdot \mathbb{C}[x_1, x_2, x_3], \qquad C_\mu = \mathcal{O}_{\mathbb{C}^3}/\mathscr{I}_\mu \ .$$

2.4. **Module $M_3$.** The kernel/cokernel sequence associated to the $\mathbf{T}$-fixed restricted data (8) takes the form

$$0 \to \mathscr{I}_{C_\mu} \to \mathcal{O}_U \xrightarrow{s} F_U \to Q_U \to 0 \tag{9}$$

for an outgoing partition $\mu$.

Since the support of the quotient $Q_U$ in (9) is 0-dimensional by stability and $\mathbf{T}$-fixed, $Q_U$ must be supported at the origin. By Proposition 1, the pair $(F_U, s_U)$ corresponds to a $\mathbf{T}$-invariant subsheaf of

$$\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{C_\mu})/\mathcal{O}_{C_\mu},$$

where $\mathfrak{m}$ is the ideal sheaf of the origin in $C_\mu \subset \mathbb{C}^3$. Let

$$M_3 = (\mathcal{O}_{C_\mu})_{x_3}$$

be the $\mathbb{C}[x_1, x_2, x_3]$-module obtained by localisation. Explicitly

$$M_3 = \mathbb{C}[x_3, x_3^{-1}] \otimes \frac{\mathbb{C}[x_1, x_2]}{\mu[x_1, x_2]} \ .$$

By elementary algebraic arguments,

$$\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{C_\mu}) \ \cong \ M_3 \ .$$

The $\mathbf{T}$-equivariant $\mathbb{C}[x_1,x_2,x_3]$-module $M_3$ has a canonical $\mathbf{T}$-invariant element 1. By Proposition 1, the $\mathbf{T}$-fixed pair $(F_U, s_U)$ corresponds to a finitely generated $\mathbf{T}$-invariant $\mathbb{C}[x_1,x_2,x_3]$-submodule

$$Q_U \subset M_3/\langle 1\rangle. \tag{10}$$

Conversely, *every* finitely generated[4] $\mathbf{T}$-invariant $\mathbb{C}[x_1,x_2,x_3]$-submodule

$$Q \subset M_3/\langle 1\rangle$$

occurs as the restriction to $U$ of a $\mathbf{T}$-fixed stable pair on $N$.

2.5. **The 1-leg stable pairs vertex.** Let $R$ be the coordinate ring,

$$R = \mathbb{C}[x_1,x_2,x_3] \cong \Gamma(U).$$

Following the conventions of Section 0.4, the $\mathbf{T}$-action on $R$ is

$$(t_1,t_2,t_3)\cdot x_i = t_i x_i\,.$$

Since the tangent spaces are dual to the coordinate functions, the tangent weight of $\mathbf{T}$ along the third axis is $-s_3$.

Let $Q_U \subset M/\langle 1\rangle$ be a $\mathbf{T}$-invariant submodule viewed as a stable pair on $U$. Let $\mathbb{I}^\bullet_U$ denote the universal complex on $[Q_U]\times U$. Consider a $\mathbf{T}$-equivariant free resolution[5] of $\mathbb{I}^\bullet_U$,

$$\{\mathscr{F}_s \to \cdots \to \mathscr{F}_{-1}\} \cong \mathbb{I}^\bullet_U \ \in D^b([Q_U]\times U). \tag{11}$$

Each term in (11) can be taken to have the form

$$\mathscr{F}_i = \bigoplus_j R(d_{ij})\,, \quad d_{ij}\in\mathbb{Z}^3.$$

The Poincaré polynomial

$$P_U = \sum_{i,j} (-1)^{i+1}\ t^{d_{ij}} \ \in \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm]$$

does not depend on the choice of the resolution (11).

We denote the $\mathbf{T}$-character of $F_U$ by $\mathsf{F}_U$. By the sequence

$$0 \to \mathcal{O}_{C_U} \to F_U \to Q_U \to 0,$$

we have a complete understanding of the representation $\mathsf{F}_U$. The $\mathbf{T}$-eigenspaces of $F_U$ correspond to the $\mathbf{T}$-eigenspaces of $\mathcal{O}_{C_U}$ and $Q_U$. The result determines

$$\mathsf{F}_U \in \mathbb{Z}(t_1,t_2,t_3).$$

The rational dependence on the $t_i$ is elementary.

[4]Here, finitely generated is equivalent to finite dimensional or Artinian.

[5]Here, $\mathbb{I}^\bullet_U$ is viewed to live in degrees 0 and -1.

From the resolution (11), we see that the Poincaré polynomial $P_U$ is related to the $\mathbf{T}$-character of $F_U$ as follows:

$$\mathsf{F}_U = \frac{1+P_U}{(1-t_1)(1-t_2)(1-t_3)}\,. \tag{12}$$

The virtual represention $\chi(\mathbb{I}_U^\bullet, \mathbb{I}_U^\bullet)$ is given by the following alternating sum

$$\begin{aligned}\chi(\mathbb{I}_U^\bullet, \mathbb{I}_U^\bullet) &= \sum_{i,j,k,l} (-1)^{i+k} \operatorname{Hom}_R(R(d_{ij}), R(d_{kl})) \\ &= \sum_{i,j,k,l} (-1)^{i+k} R(d_{kl}-d_{ij})\,.\end{aligned}$$

Therefore, the $\mathbf{T}$-character is

$$\mathrm{tr}_{\chi(\mathbb{I}_U,\mathbb{I}_U)} = \frac{P_U\,\overline{P}_U}{(1-t_1)(1-t_2)(1-t_3)}\,.$$

The bar operation

$$\gamma \in \mathbb{Z}(\!(t_1,t_2,t_3)\!) \mapsto \mathbb{Z}(\!(t_1^{-1},t_2^{-1},t_3^{-1})\!)$$

is $t_i \mapsto t_i^{-1}$ on the variables.

We find the $\mathbf{T}$-character of the $U$ summand of virtual tangent space $\mathcal{T}_{[I^\bullet]}$ of the moduli space of stable pairs of the 1-leg cap is

$$\mathrm{tr}_{R-\chi(\mathbb{I}_U^\bullet,\mathbb{I}_U^\bullet)} = \frac{1-P_U\,\overline{P}_U}{(1-t_1)(1-t_2)(1-t_3)}\,,$$

see [25]. Using (12), we may express the answer in terms of $\mathsf{F}_U$,

$$\mathrm{tr}_{R-\chi(\mathbb{I}_U^\bullet,\mathbb{I}_U^\bullet)} = \mathsf{F}_U - \frac{\overline{\mathsf{F}}_U}{t_1t_2t_3} + \mathsf{F}_U\overline{\mathsf{F}}_U\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}\,. \tag{13}$$

On the right side of (13), the rational functions should be expanded in ascending powers in the $t_i$.

The stable pairs vertex is obtained from (13) after a redistribution of edge terms following [25]. Let

$$\mathsf{F}_\mu = \sum_{(k_1,k_2)\in\mu} t_1^{k_1}t_2^{k_2}$$

correspond to the outgoing partition $\mu$. Define

$$\mathsf{G}_\mu = -\mathsf{F}_\mu - \frac{\overline{\mathsf{F}}_\mu}{t_1t_2} + \mathsf{F}_\mu\overline{\mathsf{F}}_\mu\frac{(1-t_1)(1-t_2)}{t_1t_2}\,.$$

Define the vertex character $\mathsf{V}_U$ by the following modification,

$$\mathsf{V}_U = \mathrm{tr}_{R-\chi(\mathbb{I}_U^\bullet,\mathbb{I}_U^\bullet)} + \frac{\mathsf{G}_\mu(t_1,t_2)}{1-t_3}\,. \tag{14}$$

The character $\mathsf{V}_U$ depends *only on the local data* $Q_U$. By the results of [25], $\mathsf{V}_U$ is a Laurent polynomial in $t_1$, $t_2$, and $t_3$.

2.6. **Descendents.** Let $[0] \in H^*_{\mathbf{T}}(\mathbb{P}^1, \mathbb{Z})$ be the class of the $\mathbf{T}$-fixed point $0 \in \mathbb{P}^1$. Consider the $\mathbf{T}$-equivariant descendent (with value in the $\mathbf{T}$-equivariant cohomology of a point),

$$(15)\quad \langle \tau_{i_1}([0]) \cdots \tau_{i_k}([0]) \rangle^N_{n,d} = \int_{P_n(N,d)} \prod_{j=1}^{k} \tau_{i_j}([0]) \Big( [P_n(N,d)]^{vir} \Big) \in \mathbb{Q}(s_1, s_2, s_3)\ ,$$

following the notation of Section 0.1.

In order to calculate (15) by $\mathbf{T}$-localization, we must determine the action of the operators $\tau_i([0])$ on the $\mathbf{T}$-equivariant cohomology of the $\mathbf{T}$-fixed loci. The calculation of [25] yields a formula for the descendent weight,

$$(16)\quad \mathsf{w}_{i_1,\cdots,i_m}(Q_U) = e(-\mathsf{V}_U) \cdot \prod_{j=1}^{m} \mathrm{ch}_{2+i_j} \big( \mathsf{F}_U \cdot (1-t_1)(1-t_2)(1-t_3) \big)\ .$$

The *descendent vertex* $\mathsf{W}^{\mathsf{Vert}}_\mu(\tau_{i_1}([0]) \cdots \tau_{i_m}([0]))$ is obtained from the descendent weight,

$$(17)\quad \mathsf{W}^{\mathsf{Vert}}_\mu(\tau_{i_1}([0]) \cdots \tau_{i_k}([0])) = \left( \frac{1}{s_1 s_2} \right)^k \sum_{Q_U} \mathsf{w}_{i_1,\cdots,i_k}(Q_U)\ q^{\ell(Q_U)+|\mu|}\ \in \mathbb{Q}(s_1, s_2, s_3)((q))\ .$$

Here, $\ell(Q_U)$ is the length of $Q_U$.

2.7. **Edge weights.** The edge weight in the cap geometry is

$$\mathsf{W}^{(0,0)}_\mu = e(\mathsf{G}_\mu)\ \in \mathbb{Q}(s_1, s_2).$$

In fact, $\mathsf{W}^{(0,0)}_\mu$ is simply the inverse product of the tangent weights of the Hilbert scheme of points of $\mathbb{C}^2$ at the $T$-fixed point corresponding to the partition $\mu$.

## 3. Capped 1-leg descendents: stationary

3.1. **Overview.** Consider the capped geometry of Section 0.4. As before, let $0 \in \mathbb{P}^1$ be the $\mathbf{T}$-fixed point away from the relative divisor over $\infty \in \mathbb{P}^1$, and let

$$[0] \in H^*_{\mathbf{T}}(\mathbb{P}^1, \mathbb{Z})$$

be the associated class. The $\mathbf{T}$-weight on the tangent space to $\mathbb{P}^1$ at 0 is $-s_3$. We study here the stationary[6] series

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}} . \tag{18}$$

Our main result is a special case of Theorem 3.

**Proposition 2.** $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}}$ *is the Laurent expansion in $q$ of a rational function in $\mathbb{Q}(q,s_1,s_2,s_3)$.*

3.2. **Dependence on $s_3$.** The function (18) is the generating series of the integrals

$$\left\langle\prod_{j=1}^{k}\tau_{i_j}([0])\right\rangle^{\mathsf{cap},\mathbf{T}}_{n,\eta} = \int_{[P_n(N/N_\infty,d)]^{vir}}\prod_{j=1}^{k}\tau_{i_j}([0])\ \cup\epsilon^*_\infty(C_\eta)\ , \tag{19}$$

following the notation of Section 0.4.

Let $\ell(\eta)$ denote the length of the partition $\eta$ of $d$, and let

$$\delta=\sum_{j=1}^{k} i_j + d - \ell(\eta)\ . \tag{20}$$

The dimension of $[P_n(N/N_\infty,d)]^{vir}$ after applying the integrand of (19) is $2d-\delta$.

**Lemma 1.** *The integral* $\left\langle\prod_{j=1}^{k}\tau_{i_j}([0])\right\rangle^{\mathsf{cap},\mathbf{T}}_{n,\eta}$ *is a* polynomial *in $s_3$ of degree $\delta$ with coefficients in the subring*

$$\mathbb{Q}[s_1,s_2]_{(s_1s_2)}\subset\mathbb{Q}(s_1,s_2).$$

*Proof.* Let $N=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}$. Let $\mathbb{F}\to\mathcal{N}$ denote the universal sheaf over the universal total space

$$\mathcal{N}\to P_n(N/N_\infty,d).$$

Since $N=\mathbb{P}^1\times\mathbb{C}^2$, there is a proper morphism

$$\mathcal{N}\to P_n(N/N_\infty,d)\times\mathbb{C}^2.$$

The locations and multiplicities of the supports of the universal sheaf determine a morphism of Hilbert-Chow type,

$$\iota: P_n(N/N_\infty,d)\to \mathrm{Sym}^d(\mathbb{C}^2).$$

A $\mathbf{T}$-equivariant, proper morphism,

$$\widehat{\iota}: \mathrm{Sym}^d(\mathbb{C}^2)\to\oplus_1^d(\mathbb{C}^2),$$

[6]Stationary refers to descendents of point classes.

is obtained via the higher moments,

$$\widehat{\iota}\Big(\ \{(x_i,y_i)\}\ \Big) = \Big(\sum_i x_i, \sum_i y_i\Big) \oplus \Big(\sum_i x_i^2, \sum_i y_i^2\Big) \oplus \cdots \oplus \Big(\sum_i x_i^d, \sum_i y_i^d\Big).$$

Let $\rho = \widehat{\iota} \circ \iota$.

Since $\rho$ is a $\mathbf{T}$-equivariant, proper morphism, there is a $\mathbf{T}$-equivariant push-forward

$$\rho_* : A_*^{\mathbf{T}}(P_n(N/N_\infty, d), \mathbb{Q}) \to A_*^{\mathbf{T}}(\oplus_1^d(\mathbb{C}^2), \mathbb{Q}).$$

Descendent invariants are defined via the $\mathbf{T}$-equivariant residue of

$$\left(\prod_{j=1}^k \tau_{i_j}([0]) \cup \epsilon_\infty^*(C_\eta)\right) \ \cap [P_n(N/S,d)]^{vir} \ \in A_*^{\mathbf{T}}(P_n(N/N_\infty, d), \mathbb{Q}).$$

We may instead calculate the $\mathbf{T}$-equivariant residue of

$$\rho_*\left(\left(\prod_{j=1}^k \tau_{i_j}([0]) \cup \epsilon_\infty^*(C_\eta)\right) \ \cap [P_n(N/N_\infty,d)]^{vir}\right) \tag{21}$$

in $A_*^{\mathbf{T}}(\oplus_1^d(\mathbb{C}^2), \mathbb{Q})$.

The codimension of the class (21) in $\oplus_1^d(\mathbb{C}^2)$ is $\delta$. Since the third factor of $\mathbf{T}$ acts trivially on $\oplus_1^d(\mathbb{C}^2)$, the class (21) may be written as

$$\gamma_0 s_3^0 + \gamma_1 s_3^1 + \ldots + \gamma_\delta s_3^\delta \tag{22}$$

where $\gamma_i \in A_{2d-\delta+i}^T(\oplus_1^d(\mathbb{C}^2), \mathbb{Q})$. Since the space $\oplus_1^d(\mathbb{C}^2)$ has a unique $T$-fixed point with tangent weights,

$$-s_1, -s_2, -2s_1, -2s_2, \ldots, -ds_1, -ds_2,$$

we conclude the localization of $\gamma_i$ has only monomial poles in the variables $t_1$ and $t_2$. □

As a consequence of Lemma 1, we may write

$$\mathsf{Z}_{d,\eta}^{\mathsf{cap}}\left(\prod_{j=1}^k \tau_{i_j}([0])\right)^{\mathbf{T}} = \sum_{r=0}^{\delta} s_3^r \cdot \Gamma_r(q, s_1, s_2) \tag{23}$$

where $\Gamma_r \in \mathbb{Q}(s_1, s_2)((q))$.

3.3. **Localization: rubber contribution.** The $\mathbf{T}$-equivariant localization formula for the series $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}}$ has three parts:

(i) vertex terms over $0 \in \mathbb{P}^1$,
(ii) edge terms,
(iii) rubber integrals over $\infty \in \mathbb{P}^1$.

The vertex and edge terms have been explained already in Section 2. We discuss the rubber integrals here.

The stable pairs theory of *rubber*[7] naturally arises at the boundary of $P_n(N/N_\infty, d)$. Let $R$ be a rank 2 bundle of level $(0,0)$ over $\mathbb{P}^1$. Let

$$R_0, R_\infty \subset R$$

denote the fibers over $0, \infty \in \mathbb{P}^1$. The 1-dimensional torus $\mathbb{C}^*$ acts on $R$ via the symmetries of $\mathbb{P}^1$. Let $P_n(R/R_0 \cup R_\infty, d)$ be the relative moduli space of stable pairs, and let

$$P_n(R/R_0 \cup R_\infty, d)^\circ \subset P_n(R/R_0 \cup R_\infty, d)$$

denote the open set with finite stabilizers for the $\mathbb{C}^*$-action and *no* destabilization over $\infty \in \mathbb{P}^1$. The rubber moduli space,

$$P_n(R/R_0 \cup R_\infty, d)^\sim = P_n(R/R_0 \cup R_\infty, d)^\circ / \mathbb{C}^*,$$

denoted by a superscripted tilde, is determined by the (stack) quotient. The moduli space is empty unless $n > d$. The rubber theory of $R$ is defined by integration against the rubber virtual class,

$$[P_n(R/R_0 \cup R_\infty, d)^\sim]^{vir}.$$

All of the above rubber constructions are $T$-equivariant for the scaling action on the fibers of $R$ with weights $s_1$ and $s_2$.

The rubber moduli space $P_n(R/R_0 \cup R_\infty, d)^\sim$ carries a cotangent line at the dynamical point $0 \in \mathbb{P}^1$. Let

$$\psi_0 \in A^1_T(P_n(R/R_0 \cup R_\infty, d)^\sim, \mathbb{Q})$$

denote the associated cotangent line class. Let

$$\mathsf{P}_\mu \in A^{2d}_T(\mathrm{Hilb}(\mathbb{C}^2, d), \mathbb{Z})$$

be the class corresponding to the $T$-fixed point determined by the monomial ideal $\mu[x_1, x_2] \subset \mathbb{C}[x_1, x_2]$.

In the localization formula for the cap, special rubber integrals with relative conditions $\mathsf{P}_\mu$ over 0 and $\mathsf{C}_\eta$ (in the Nakajima basis) over $\infty$

[7]We follow the terminology and conventions of the parallel rubber discussion for the local Donaldson-Thomas theory of curves treated in [20].

arise. Let

$$\mathsf{S}^\mu_\eta = \sum_{n\geq d} q^n \left\langle \mathsf{P}_\mu \ \middle| \ \frac{1}{s_3-\psi_0} \ \middle| \ \mathsf{C}_\eta \right\rangle^{\sim}_{n,d} \in \mathbb{Q}(s_1,s_2,s_3)((q)) \ .$$

The bracket on the right is the rubber integral defined by $T$-equivariant residues. If $n=d$, the rubber moduli space in undefined — the bracket is then taken to be the $T$-equivariant intersection pairing between the classes $\mathsf{P}_\mu$ and $\mathsf{C}_\eta$ in $\mathrm{Hilb}(\mathbb{C}^2,d)$.

The $s_3$ dependence of the rubber integral

$$\left\langle \mathsf{P}_\mu \ \middle| \ \frac{1}{s_3-\psi_0} \ \middle| \ \mathsf{C}_\eta \right\rangle^{\sim}_{n,d} \in \mathbb{Q}(s_1,s_2,s_3)$$

enter *only* through the term $s_3-\psi_0$. On the $T$-fixed loci of the moduli space $P_n(R/R_0\cup R_\infty,d)^\sim$, the cotangent line class $\psi_0$ is either equal to a weight of $\mathrm{Tan}_\mu$ (if 0 lies on a twistor component) or is nilpotent (if 0 lies on a non-twistor component). We conclude the following result.

**Lemma 2.** *The evaluation of* $\mathsf{S}^\mu_\eta$ *at*

$$s_3 = n_1s_1+n_2s_2, \qquad n_1,n_2\in\mathbb{Q}$$

*is well-defined if* $(n_1,n_2)\neq(0,0)$ *and* $n_1s_1+n_2s_2$ *is not a weight of* $\mathrm{Tan}_\mu$.

The weights of $\mathrm{Tan}_\mu$ are either proportional to $s_1$ or $s_2$ or of the form

$$n_1s_1+n_2s_2, \qquad n_1,n_2\neq 0$$

where $n_1$ is the *opposite* sign of $n_2$.

3.4. **Localization: full formula.** The localization formula [7] for the capped 1-leg descendent vertex is the following:

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^k \tau_{i_j}([0])\right)^{\mathbf{T}} = \sum_{|\mu|=d} \mathsf{W}^{\mathsf{Vert}}_\mu \left(\prod_{j=1}^k \tau_{i_j}([0])\right)\cdot \mathsf{W}^{(0,0)}_\mu \cdot \mathsf{S}^\mu_\eta \ . \tag{24}$$

The form is the same as the Donaldson-Thomas localization formulas used in [13, 20].

3.5. **Proof of Proposition 2.** We will consider the evaluations of $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}(\prod_{j=1}^k \tau_{i_j}([0]))^{\mathbf{T}}$ at the values

$$s_3 = \frac{1}{a}(s_1+s_2) \tag{25}$$

for all integers $a>0$. By Theorem 4, the main cancellation of poles result of Section 4, the evaluation (25) of $\mathsf{W}^{\mathsf{Vert}}_\mu\left(\prod_{j=1}^k \tau_{i_j}([0])\right)$ is well-defined and yields a Laurent *polynomial* in $q$ with coefficients in $\mathbb{Q}(s_1,s_2)$.

The edge term $\mathsf{W}_\mu^{(0,0)}$ has no $s_3$ dependence (and $q$ dependence given by $q^{-d}$). The evaluation (25) of $\mathsf{S}_\eta^\mu$ is well-defined by Lemma 2 and is the Laurent series associated to a rational function in $\mathbb{Q}(q,s_1,s_2)$ by Lemma 3 below.

We have proven the evalution of $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^k \tau_{i_j}([0])\right)^{\mathbf{T}}$ at (25) for all integers $a>0$ is well-defined and yields a rational function in $\mathbb{Q}(q,s_1,s_2)$. By (23) and the invertibility of the Vandermonde matrix, we see

$$\Gamma_r(q,s_1,s_2)\in\mathbb{Q}(q,s_1,s_2)$$

for all $0\le r\le\delta$. □

3.6. **Evaluation of $\mathsf{S}_\eta^\mu$.** The following result is well-known from the study of the quantum differential equation of the Hilbert scheme of points [19, 21]. We include the proof for the reader's convenience.

**Lemma 3.** *For all integers $a\neq 0$, the evaluation*

$$\mathsf{S}_\eta^\mu|_{s_3=\frac{1}{a}(s_1+s_2)}$$

*yields the Laurent series associated to a rational function in $\mathbb{Q}(q,s_1,s_2)$.*

*Proof.* Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ with tangent weights $-s_3$ and $s_3$ at $0,\infty\in\mathbb{P}^1$ respectively. Lift the $\mathbb{C}^*$-action to $\mathcal{O}_{\mathbb{P}^1}(-a)$ with fiber weights[8] $as_3$ and $0$ over $0,\infty\in\mathbb{P}^1$. Lift $\mathbb{C}^*$ to $\mathcal{O}_{\mathbb{P}^1}$ with fiber weights 0 and 0 over $0,\infty\in\mathbb{P}^1$. The $(-a,0)$-tube is the geometry of total space of

$$\mathcal{O}_{\mathbb{P}^1}(-a)\oplus\mathcal{O}_{\mathbb{P}^1}\to\mathbb{P}^1 \tag{26}$$

relative to the fibers over both $0,\infty\in\mathbb{P}^1$.

The 2-dimensional torus $T$ acts on the $(-a,0)$-tube as before by scaling the line summands. For

$$\mathbf{T}=T\times\mathbb{C}^*,$$

we obtain a $\mathbf{T}$-action on the $(-a,0)$-tube. Define the generating series of $\mathbf{T}$-equivariant integrals

$$\mathsf{Z}^{(-a,0),\mathbf{T}}_{d,\eta^0,\eta^\infty}=\sum_n q^n\left\langle \mathsf{C}_{\eta^0}\ \middle|\ 1\ \middle|\ \mathsf{C}_{\eta^\infty}\right\rangle^{(-a,0)}_{n,d}\in\mathbb{Q}(s_1,s_2,s_3)((q)) \tag{27}$$

where the superscript $(-a,0)$ refers to the geometry (26).

[8]Remember, weights on the coordinate functions are the opposite of the weights on the fibers.

The series $\mathsf{Z}^{(-a,0),\mathbf{T}}_{d,\eta^0,\eta^\infty}$ has no insertions. Hence, the results of [13, 17] show $\mathsf{Z}^{(-a,0),\mathbf{T}}_{d,\eta^0,\eta^\infty}$ is actually the Laurent series associated to a rational function in $\mathbb{Q}(q,s_1,s_2,s_3)$. The $\mathbf{T}$-equivariant localization formula yields

$$\mathsf{Z}^{(-a,0),\mathbf{T}}_{d,\eta^0,\eta^\infty} = \sum_{|\mu|=d} \mathsf{S}^{\mu}_{\eta^0}\Big|_{s_1=s_1-as_3,s_2,s_3=-s_3} \cdot \mathsf{W}^{(-a,0)}_{\mu} \cdot \mathsf{S}^{\mu}_{\eta^\infty} \ .$$

The formula for the edge term $\mathsf{W}^{(-a,0)}_{\mu}$ can be found in Section 4.6 of [25].

Next, we consider the evaluation of the three terms of the localization formula at

$$s_3 = \frac{1}{a}(s_1+s_2) \ . \tag{28}$$

After evaluation, the first term becomes

$$\mathsf{S}^{\mu}_{\eta^0}\Big|_{s_1=-s_2,s_2,s_3=-s_3} \tag{29}$$

which only has $q^d$ terms by holomorphic symplectic vanishing [17, 20]. The evaluation of $\mathsf{W}^{(-a,0)}_{\mu}$ at (28) is easily seen to be well-defined and nonzero by inspection of the formulas in Section 4.6 of [25]. The $q$ dependence of $\mathsf{W}^{(-a,0)}_{\mu}$ is monomial. The evaluation of the third term $\mathsf{S}^{\mu}_{\eta^\infty}$ at (28) is well-defined by Lemma 2. We conclude the evaluation of $\mathsf{Z}^{(-a,0),\mathbf{T}}_{d,\eta^0,\eta^\infty}$ at (28) is a well-defined rational function in $\mathbb{Q}(q,s_1,s_2)$.

By the invertibility of (29) and the edge terms, $\mathsf{S}^{\mu}_{\eta^\infty}$ must also be a rational function in $\mathbb{Q}(q,s_1,s_2)$ after the evaluation (28). □

3.7. **Twisted cap.** The twisted $(a_1,a_2)$-cap is the geometry of the total space of

$$\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \to \mathbb{P}^1 \tag{30}$$

relative to the fiber over $\infty \in \mathbb{P}^1$.

We lift the $\mathbb{C}^*$-action on $\mathbb{P}^1$ to $\mathcal{O}_{\mathbb{P}^1}(a_i)$ with fiber weights 0 and $-a_is_3$ over $0,\infty \in \mathbb{P}^1$. The 2-dimensional torus $T$ acts on the $(a_1,a_2)$-cap by scaling the line summands, so we obtain a $\mathbf{T}$-action on the $(a_1,a_2)$-cap.

Define the generating series of $\mathbf{T}$-equivariant integrals

$$\mathsf{Z}^{(a_1,a_2)}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}} = \sum_n q^n \left\langle \prod_{j=1}^{k}\tau_{i_j}([0]) \,\middle|\, \mathsf{C}_\eta \right\rangle^{(a_1,a_2)}_{n,d} \in \mathbb{Q}(s_1,s_2,s_3)((q))$$

where the superscript $(a_1,a_2)$ refers to the geometry (30).

**Proposition 3.** $\mathsf{Z}^{(a_1,a_2)}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}}$ *is the Laurent expansion in* $q$ *of a rational function in* $\mathbb{Q}(q,s_1,s_2,s_3)$.

*Proof.* The twisted $(a_1,a_2)$-cap admits a $\mathbf{T}$-equivariant degeneration to a standard $(0,0)$-cap and an $(a_1,a_2)$-tube by bubbling off $0\in\mathbb{P}^1$. The insertions $\tau_{i_j}([0])$ are sent $\mathbf{T}$-equivariantly to the non-relative point of the $(0,0)$-cap. The rationality of $\mathsf{Z}^{(a_1,a_2)}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}}$ then follows from Proposition 2, the $\mathbf{T}$-equivariant rationality results for the $(a_1,a_2)$-tube without insertions [17, 20], and the degeneration formula. $\square$

## 4. Cancellation of poles

4.1. **Overview.** Our goal here is to prove the following result.

**Theorem 4.** *For all integers* $a>0$, *the evaluation*

$$\mathsf{W}^{\mathrm{Vert}}_\mu\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)\Bigg|_{s_3=\frac{1}{a}(s_1+s_2)}$$

*is well-defined and yields a Laurent polynomial in* $q$ *with coefficients in* $\mathbb{Q}(s_1,s_2)$.

We regard the partition $\mu$, the descendent factor $\prod_{j=1}^{k}\tau_{i_j}([0])$, and the integer $a$ as fixed throughout Section 4.

Recall $\mathsf{W}^{\mathrm{Vert}}_\mu\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)$ is defined as an infinite sum over the fixed loci $Q_U$,

$$\mathsf{W}^{\mathrm{Vert}}_\mu\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right) = \left(\frac{1}{s_1s_2}\right)^k \sum_{Q_U} \mathsf{w}_{\tau_{i_1},\ldots,\tau_{i_k}}(Q_U)\, q^{l(Q_U)+|\mu|}. \tag{31}$$

The $Q_U$ are determined by $\mathsf{F}_U$, the weight of the corresponding box configuration. Although $\mathsf{F}_U$ is just a Laurent series in $t_1,t_2,t_3$, the product $(1-t_3)\mathsf{F}_U$ is a Laurent polynomial.

Our approach to proving Theorem 4 is to break (31) into finite sums based on the Laurent polynomial

$$(1-t_3)\mathsf{F}_U\big|_{t_3=(t_1t_2)^{\frac{1}{a}}}\ .$$

For any Laurent polynomial $f \in \mathbb{Z}[t_1, t_2, (t_1t_2)^{-\frac{1}{a}}]$, define

$$\mathcal{S}_f = \left\{ Q_U \;\middle|\; (1-t_3)\mathsf{F}_U\big|_{t_3=(t_1t_2)^{\frac{1}{a}}} = f \right\}\ .$$

Theorem 4 follows from the following result regarding the subsums of (31) corresponding to the sets $\mathcal{S}_f$.

**Proposition 4.** *Let $f \in \mathbb{Z}[t_1, t_2, (t_1t_2)^{-\frac{1}{a}}]$ be a Laurent polynomial. The evaluation*

$$\left( \sum_{Q_U \in \mathcal{S}_f} \mathsf{w}_{i_1,\ldots,i_k}(Q_U) \right)\Bigg|_{s_3=\frac{1}{a}(s_1+s_2)}$$

*is well-defined. Moreover, the evaluation vanishes for all but finitely many choices of $f$.*

4.2. **Notation and Preliminaries.** We introduce here the notation and conventions required to analyze the sums appearing in Proposition 4.

First, we view the partition $\mu$ as a subset of $\mathbb{Z}^2_{\geq 0}$. The lattice points, for which we use the coordinates $(i, j) \in \mu$, correspond to the lower left corners of the boxes of $\mu$. We also write

$$(\delta; j) = (i, j)$$

for $\delta = i - j$.

The points $(\delta; j) \in \mu$ for fixed $\delta$ lie on a single diagonal. The diagonals will play an important role. Let $\mu_\delta = \{j \mid (\delta; j) \in \mu\}$, and define

$$\mathrm{Sym}_\mu = \prod_{\delta \in \mathbb{Z}} \mathrm{Sym}(\mu_\delta),$$

where $\mathrm{Sym}(S)$ is the group of permutations of a set $S$. Thus, $\mathrm{Sym}_\mu$ may be viewed as the group of permutations of $\mu$ which move points only inside their diagonals. Let

$$\mathrm{sgn} : \mathrm{Sym}_\mu \to \{\pm 1\}$$

be the sign of the permutation of $\mu$.

Recall the Laurent polynomials $(1-t_3)\mathsf{F}_U$ are of the form

$$(1-t_3)\mathsf{F}_U = \sum_{(i,j)\in\mu} t_1^i t_2^j t_3^{-h_U(i,j)},$$

where $h_U(i,j)$ is the depth of the box arrangement below $(i,j)$.

Because of our reparametrization of the partition $\mu$ and the evaluation $t_3 = (t_1t_2)^{\frac{1}{a}}$, the following change of variables will be convenient:

$$v_1 = t_1, \quad v_2 = t_1t_2, \quad v_3 = t_1t_2t_3^{-a}$$

and $u_i = e(v_i)$, so

$$u_1 = s_1, \quad u_2 = s_1 + s_2, \quad u_3 = s_1 + s_2 - as_3.$$

The evaluations under consideration are then simply $v_3 = 1$ and $u_3 = 0$.

From now on we will assume $\mathcal{S}_f$ to be nonempty, so

$$f = (1-t_3)\mathsf{F}_U\big|_{t_3=(t_1t_2)^{\frac{1}{a}}}$$

for some $Q_U$ and thus $f$ can be written in the form

$$f = \sum_{(\delta;j)\in\mu} v_1^{\delta} v_2^{e_\delta(j)}$$

for some exponents $e_\delta(j)$. These exponents are made unique by requiring that $e_\delta(j)$ is a weakly decreasing function of $j$, for each $\delta$. We generally regard $f$ as fixed and thus do not indicate the $f$-dependence in $e_\delta(j)$.

We now classify all $Q_U \in \mathcal{S}_f$. Given any $\sigma = (\sigma_\delta) \in \mathrm{Sym}_\mu$, we define a function $h_\sigma : \mu \to \mathbb{Z}$ by

$$h_\sigma(\delta;j) = a\cdot(j - e_\delta(\sigma_\delta^{-1}(j))).$$

When $h_\sigma$ defines a valid box arrangement, we say $\sigma$ is *admissible*. Admissibility is equivalent to the following conditions on $\sigma$:

$$\begin{aligned}
\sigma_0(j) &\neq 0 \text{ if } e_0(j) > 0 \\
\sigma_{\delta+1}(j) &\neq \sigma_\delta(k) \text{ if } e_{\delta+1}(j) > e_\delta(k) \\
\sigma_\delta(j) &\neq \sigma_{\delta+1}(k) + 1 \text{ if } e_\delta(j) > e_{\delta+1}(k) + 1.
\end{aligned}$$

For admissible $\sigma$, let $Q_\sigma$ denote the corresponding $\mathbf{T}$-fixed locus.

Unraveling the definitions, we compute

$$\begin{aligned}
(1-t_3)\mathsf{F}_\sigma &= \sum_{(i,j)\in\mu} t_1^i t_2^j t_3^{-h_\sigma(i-j,j)} \\
&= \sum_{(\delta;j)\in\mu} v_1^\delta v_2^j v_2^{-\frac{1}{a}h_\sigma(\delta;j)} v_3^{\frac{1}{a}h_\sigma(\delta;j)} \\
&= \sum_{(\delta;j)\in\mu} v_1^\delta v_2^{e_\delta(\sigma_\delta^{-1}(j))} v_3^{j-e_\delta(\sigma_\delta^{-1}(j))} \\
&= \sum_{(\delta;j)\in\mu} v_1^\delta v_2^{e_\delta(j)} v_3^{\sigma_\delta(j)-e_\delta(j)}.
\end{aligned}$$

We conclude $(1-t_3)\mathsf{F}_\sigma|_{v_3=1} = f$ and $Q_\sigma \in \mathcal{S}_f$. In fact, a direct examination shows every $Q_{U'} \in \mathcal{S}_f$ can be obtained as $Q_\sigma$ for some admissible $\sigma \in \mathrm{Sym}_\mu$. If we let $\mathrm{Sym}_\mu^0$ be the subgroup of $\mathrm{Sym}_\mu$ consisting of elements $\tau$ such that $e_\delta(\tau_\delta(j)) = e_\delta(j)$, then $Q_\sigma = Q_{\sigma'}$ if and only if $\sigma^{-1}\sigma' \in \mathrm{Sym}_\mu^0$.

We thus can replace the sum over $Q_U \in \mathcal{S}_f$ with a sum over admissible $\sigma \in \mathrm{Sym}_\mu$:

$$\left(\sum_{Q_U \in \mathcal{S}_f} \mathsf{w}_{i_1,\ldots,i_k}(Q_U)\right)\Bigg|_{s_3=\frac{1}{a}(s_1+s_2)} = \frac{1}{|\,\mathrm{Sym}_\mu^0\,|}\left(\sum_{\sigma \in \mathrm{Sym}_\mu \text{ admissible}} \mathsf{w}_{i_1,\ldots,i_k}(Q_\sigma)\right)\Bigg|_{s_3=\frac{1}{a}(s_1+s_2)}. \tag{32}$$

We will show the evaluation is well-defined by choosing $\kappa_0$ such that each term $\mathsf{w}_{i_1,\ldots,i_k}(Q_\sigma)$ in the above sum has order of vanishing along $u_3 = 0$ at least $-\kappa_0$, and then showing

$$\sum_{\sigma \in \mathrm{Sym}_\mu \text{ admissible}} \left(\frac{\partial}{\partial u_3}\right)^\kappa (u_3^{\kappa_0}\mathsf{w}_{i_1,\ldots,i_k}(Q_\sigma))\Bigg|_{u_3=0} = 0 \tag{33}$$

for $0 \le \kappa < \kappa_0$.

The second part of Proposition 4, the vanishing of the evaluation (32) for all but finitely many $f$, is then equivalent to proving that (33) holds for $\kappa = \kappa_0$ (for all but finitely many $f$).

In order to prove these vanishing results, we will need to analyze the dependence of the terms $\left(\frac{\partial}{\partial u_3}\right)^\kappa (u_3^{\kappa_0}\mathsf{w}_{i_1,\ldots,i_k}(Q_\sigma))\Big|_{u_3=0}$ on the permutation $\sigma \in \mathrm{Sym}_\mu$. For each $\kappa$, we will find the corresponding term is equal to a polynomial in the values $\sigma_\delta(j)$ of relatively low degree which vanishes at all inadmissible permutations $\sigma$.

Let $\mathbb{Q}[\sigma]$ and $\mathbb{Q}(\sigma)$ denote the ring of polynomials and the field of rational functions respectively in the variables $\sigma_\delta(j)$. For a polynomial $P \in \mathbb{Q}[\sigma]$, let $\deg(P)$ be the (total) degree of $P$. For rational functions $\frac{P}{Q} \in \mathbb{Q}(\sigma)$, we set

$$\deg\left(\frac{P}{Q}\right) = \deg(P) - \deg(Q).$$

We observe that if $P \in \mathbb{Q}[\sigma]$ has degree $\deg(P) < \sum_\delta \frac{1}{2}|\mu_\delta|(|\mu_\delta|-1)$, then

$$\sum_{\sigma \in \mathrm{Sym}_\mu} \mathrm{sgn}(\sigma)P(\sigma) = 0,$$

since a nonzero alternating polynomial with respect to $\mathrm{Sym}_\mu$ would have to have greater degree.

4.3. **Proof of Proposition 4.** We need to study the $\sigma$-dependence of

$$\mathsf{w}_{i_1,\dots,i_k}(Q_\sigma) = e(-\mathsf{V}_\sigma)\prod_{j=1}^{k}\mathrm{ch}_{2+i_j}(\mathsf{F}_\sigma\cdot(1-t_1)(1-t_2)(1-t_3)).$$

We begin by explicitly writing $\mathsf{V}_\sigma$ in terms of $\sigma$ and the numbers $e_\delta(j)$. Recall

$$\mathsf{V}_\sigma = \frac{\mathsf{F}'_\sigma - \mathsf{F}'_0}{1-t_3} + \frac{\overline{\mathsf{F}'_\sigma} - \overline{\mathsf{F}'_0}}{t_1t_2(1-t_3)} - \frac{\mathsf{F}'_\sigma\overline{\mathsf{F}'_\sigma} - \mathsf{F}'_0\overline{\mathsf{F}'_0}}{1-t_3}(1-t_1^{-1})(1-t_2^{-1}),$$

where $\mathsf{F}'_\sigma = (1-t_3)\mathsf{F}_\sigma$ and

$$\mathsf{F}'_0 = \sum_{(i,j)\in\mu} t_1^i t_2^j.$$

In particular, $\mathsf{V}_\sigma|_{v_3=1}$ does not depend on $\sigma$. Hence, the order of vanishing of $e(-\mathsf{V}_\sigma)$ along $u_3 = 0$ is an integer $-\kappa_0$ independent of $\sigma$. Since the descendent factor is a polynomial in $u_1, u_2, u_3$, the order of vanishing of $\mathsf{w}_{i_1,\dots,i_k}(Q_\sigma)$ along $u_3 = 0$ is at least $-\kappa_0$. If $\kappa_0 \le 0$, then the evaluation is well-defined on each $\mathsf{w}_{i_1,\dots,i_k}(Q_\sigma)$ and thus on their sum. If $\kappa_0 < 0$, then the evaluation in fact yields zero. So we may assume $\kappa_0 \ge 0$.

We now rewrite $\mathsf{V}_\sigma$ in terms of $v_1, v_2, v_3$. We find $\mathsf{V}_\sigma$ equals

$$\begin{aligned}
&\sum_{(\delta;j)\in\mu} \frac{v_1^{\delta}v_2^{e_\delta(j)}v_3^{\sigma_\delta(j)-e_\delta(j)} - v_1^\delta v_2^j}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}}\\
&+\sum_{(\delta;j)\in\mu} \frac{v_1^{-\delta}v_2^{-e_\delta(j)-1}v_3^{-\sigma_\delta(j)+e_\delta(j)} - v_1^{-\delta}v_2^{-j-1}}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}}\\
&-\sum_{(\delta_1;j_1),(\delta_2;j_2)\in\mu} \frac{v_1^{\delta_1-\delta_2}v_2^{e_{\delta_1}(j_1)-e_{\delta_2}(j_2)}v_3^{\sigma_{\delta_1}(j_1)-\sigma_{\delta_2}(j_2)-e_{\delta_1}(j_1)+e_{\delta_2}(j_2)} - v_1^{\delta_1-\delta_2}v_2^{j_1-j_2}}{(1-(\frac{v_2}{v_3})^{\frac{1}{a}})\cdot(1-v_1^{-1})^{-1}(1-v_1v_2^{-1})^{-1}}.
\end{aligned}$$

Let $C > 2\max(e_\delta(j))$ be a large positive integer. We break up each of the three above sums above using $C$. Then, $\mathsf{V}_\sigma$ equals

$$\begin{aligned}
&\sum_{(\delta;j)\in\mu} \frac{v_1^\delta v_2^{e_\delta(j)} v_3^{\sigma_\delta(j)-e_\delta(j)} - v_1^\delta v_2^{-C} v_3^{\sigma_\delta(j)+C}}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}} \\
&+ \sum_{(\delta;j)\in\mu} \frac{v_1^\delta v_2^{-C} v_3^{j+C} - v_1^\delta v_2^j}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}} \\
&+ \sum_{(\delta;j)\in\mu} \frac{v_1^{-\delta} v_2^{-e_\delta(j)-1} v_3^{-\sigma_\delta(j)+e_\delta(j)} - v_1^{-\delta} v_2^{-C-1} v_3^{-\sigma_\delta(j)+C}}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}} \\
&+ \sum_{(\delta;j)\in\mu} \frac{v_1^{-\delta} v_2^{-C-1} v_3^{-j+C} - v_1^{-\delta} v_2^{-j-1}}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}} \\
&- \sum_{(\delta_1;j_1),(\delta_2;j_2)\in\mu} \left( \frac{v_1^{\delta_1-\delta_2} v_2^{e_{\delta_1}(j_1)-e_{\delta_2}(j_2)} v_3^{\sigma_{\delta_1}(j_1)-\sigma_{\delta_2}(j_2)-e_{\delta_1}(j_1)+e_{\delta_2}(j_2)}}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}} \right. \\
&\qquad\qquad \left. - \frac{v_1^{\delta_1-\delta_2} v_2^{-C} v_3^{\sigma_{\delta_1}(j_1)-\sigma_{\delta_2}(j_2)+C}}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}} \right) \cdot (1-v_1^{-1})(1-v_1 v_2^{-1}) \\
&- \sum_{(\delta_1;j_1),(\delta_2;j_2)\in\mu} \frac{v_1^{\delta_1-\delta_2} v_2^{-C} v_3^{j_1-j_2+C} - v_1^{\delta_1-\delta_2} v_2^{j_1-j_2}}{1-(\frac{v_2}{v_3})^{\frac{1}{a}}} \cdot (1-v_1^{-1})(1-v_1 v_2^{-1}).
\end{aligned}$$

We now expand out the above sums into monomials: all of the resulting terms will be of the form

$$\pm v_1^x v_2^y v_3^{z(\sigma)},$$

where $x$ and $y$ have no dependence on the permutation $\sigma = (\sigma_\delta)$ and $z \in \mathbb{Q}[\sigma]$ is a linear function of the values $\sigma_\delta(j)$. After separating out the monomials with $x = y = 0$, we write

$$\mathsf{V}_\sigma = \sum_{(c,0,0,z)\in S} c v_3^{z(\sigma)} + \sum_{\substack{(c,x,y,z)\in S \\ (x,y)\neq(0,0)}} c v_1^x v_2^y v_3^{z(\sigma)},$$

where $S$ is a finite set containing the data of the monomials which appear (with coefficients $c \in \mathbb{Z}$). Then

$$e\left( - \sum_{(c,0,0,z)\in S} c v_3^{z(\sigma)} \right) = \phi(\sigma) u_3^{-\kappa_0},$$

for a rational function $\phi = \phi_f \in \mathbb{Q}(\sigma)$ which will be explicitly described below.

We analyze first the descendent factors in $\mathsf{w}_{i_1,\ldots,i_k}(Q_\sigma)$. The descendent terms can be expressed in the form

$$\prod_{j=1}^{k} \mathrm{ch}_{2+i_j}\left(\mathsf{F}_\sigma \cdot (1-t_1)(1-t_2)(1-t_3)\right) = \prod_{j=1}^{k} \sum_{(c',x,y,z)\in S'_j} c'(xu_1+yu_2+z(\sigma)u_3)^{2+i_j},$$

where the $S'_j$ are more fixed finite sets containing the data of the terms which appear. As before, $z \in \mathbb{Q}[\sigma]$ is linear. We then find

$$u_3^{\kappa_0}\mathsf{w}_{i_1,\ldots,i_k}(Q_\sigma) = \phi(\sigma) \prod_{\substack{(c,x,y,z)\in S\\ (x,y)\neq(0,0)}} (xu_1+yu_2+z(\sigma)u_3)^{-c} \cdot \prod_{j=1}^{k} \sum_{(c',x,y,z)\in S'_j} c'(xu_1+yu_2+z(\sigma)u_3)^{2+i_j}.$$

Differentiating the above product $\kappa$ times with respect to $u_3$ and then setting $u_3$ equal to 0 is easily done. We obtain

$$\left(\frac{\partial}{\partial u_3}\right)^{\kappa} (u_3^{\kappa_0}\mathsf{w}_{i_1,\ldots,i_k}(Q_\sigma))|_{u_3=0} = \sum_{i\in\mathcal{I}} \phi(\sigma)Z_i(\sigma)R_i(u_1,u_2),$$

where $\mathcal{I}$ is an indexing set, $Z_i \in \mathbb{Q}[\sigma]$ has degree at most $\kappa$, and $R_i(u_1,u_2) \in \mathbb{Q}(u_1,u_2)$ does not depend on $\sigma$.

Proposition 4 will follow from the claim that

$$\sum_{\sigma\in\mathrm{Sym}_\mu \text{ admissible}} \phi(\sigma)Z(\sigma) = 0 \tag{34}$$

for any polynomial $Z$ of degree $\kappa < \kappa_0$ (or degree $\kappa = \kappa_0$ for all but finitely many $f$). The vanishing property (34) is purely a property of the rational function $\phi \in \mathbb{Q}(\sigma)$.

We will now study $\phi$ in more detail. The goal is to find a polynomial $\psi \in \mathbb{Q}[\sigma]$ of sufficiently low degree satisfying

$$\phi(\sigma) = \mathrm{sgn}(\sigma)\psi(\sigma)$$

for every admissible $\sigma \in \mathrm{Sym}_\mu$ and satisfying $\psi(\sigma) = 0$ for every inadmissible $\sigma \in \mathrm{Sym}_\mu$. From the formula for $\mathsf{V}_\sigma$, we can describe $\phi \in \mathbb{Q}(\sigma)$

explicitly as a product of linear factors:

$$\phi(\sigma) = \left(\prod_{\substack{(0;j)\in\mu \\ e_0(j)>0}} \sigma_0(j)\right) \left(\prod_{\substack{(0;j)\in\mu \\ j>0}} j\right)^{-1} \left(\prod_{\substack{(0;j)\in\mu \\ e_0(j)<-1}} (-\sigma_0(j)-1)\right)$$
$$\left(\prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ e_\delta(j_1)>e_\delta(j_2)}} (\sigma_\delta(j_1)-\sigma_\delta(j_2))\right)^{-1} \left(\prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ j_1>j_2}} (j_1-j_2)\right)$$
$$\left(\prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ e_\delta(j_1)>e_\delta(j_2)+1}} (\sigma_\delta(j_1)-\sigma_\delta(j_2)-1)\right)^{-1} \left(\prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ j_1>j_2+1}} (j_1-j_2-1)\right)$$
$$\left(\prod_{\substack{(\delta+1;j_1),(\delta;j_2)\in\mu \\ e_{\delta+1}(j_1)>e_\delta(j_2)}} (\sigma_{\delta+1}(j_1)-\sigma_\delta(j_2))\right) \left(\prod_{\substack{(\delta+1;j_1),(\delta;j_2)\in\mu \\ j_1>j_2}} (j_1-j_2)\right)^{-1}$$
$$\left(\prod_{\substack{(\delta;j_1),(\delta+1;j_2)\in\mu \\ e_\delta(j_1)>e_{\delta+1}(j_2)+1}} (\sigma_\delta(j_1)-\sigma_{\delta+1}(j_2)-1)\right) \left(\prod_{\substack{(\delta;j_1),(\delta+1;j_2)\in\mu \\ j_1>j_2+1}} (j_1-j_2-1)\right)^{-1}.$$

The degree of $\phi$ is easily computed to be $-\kappa_0$, since there are the same number of constant factors appearing on the numerator and denominator in the above expression.

**Lemma 4.** *We have*

$$\frac{\prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ j_1>j_2}} (j_1-j_2)}{\prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ e_\delta(j_1)>e_\delta(j_2)}} (\sigma_\delta(j_1)-\sigma_\delta(j_2))} = \pm\,\mathrm{sgn}(\sigma) \prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ e_\delta(j_1)=e_\delta(j_2) \\ j_1>j_2}} (\sigma_\delta(j_1)-\sigma_\delta(j_2))$$

*for every* $\sigma \in \mathrm{Sym}_\mu$.

*Proof.* The formula is obtained by cancelling equal terms on the left side. □

Suppose that $\{\delta \mid \mu_\delta \neq \emptyset\} = \{\delta \mid a \leq \delta \leq b\}$. By using the identity of Lemma 4 and grouping terms appropriately, we find

$$\phi(\sigma) = \mathrm{sgn}(\sigma)\phi_0(\sigma)$$

for $\phi_0 \in \mathbb{Q}(\sigma)$ given by

$$\phi_0 = XPQ \frac{\prod_{a\leq\delta\leq b-1} R_\delta}{\prod_{a+1\leq\delta\leq b-1} S_\delta}, \tag{35}$$

where

$$P = \prod_{\substack{j\in\mu_0 \\ e_0(j)<-1}} (-\sigma_0(j)-1), \quad Q = \prod_{\substack{j\in\mu_0 \\ e_0(j)>0}} \sigma_0(j),$$

$$R_\delta = \left( \prod_{\substack{j_1\in\mu_{\delta+1}, j_2\in\mu_\delta \\ e_{\delta+1}(j_1)>e_\delta(j_2)}} (\sigma_{\delta+1}(j_1)-\sigma_\delta(j_2)) \right) \left( \prod_{\substack{j_1\in\mu_\delta, j_2\in\mu_{\delta+1} \\ e_\delta(j_1)>e_{\delta+1}(j_2)+1}} (\sigma_\delta(j_1)-\sigma_{\delta+1}(j_2)-1) \right),$$

$$S_\delta = \prod_{\substack{j_1,j_2\in\mu_\delta \\ e_\delta(j_1)>e_\delta(j_2)+1}} (\sigma_\delta(j_1)-\sigma_\delta(j_2)-1),$$

and $X \in \mathbb{Q}[\sigma]$ is a polynomial. The total degree of the rational function $\phi_0$ is

$$\deg(\phi)+\deg\left( \prod_{\substack{(\delta;j_1),(\delta;j_2)\in\mu \\ j_1>j_2}} (\sigma_\delta(j_1)-\sigma_\delta(j_2)) \right) = -\kappa_0+\sum_\delta \frac{1}{2}|\mu_\delta|(|\mu_\delta|-1).$$

We now require an algebraic result in order to convert $\phi_0$ into a polynomial. Let $m, n \geq 0$ be integers, and let

$$A = \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_m].$$

Let $P$ be the collection of $n!m!$ points

$$(x_1,\ldots,x_n,y_1,\ldots,y_m) \in \mathbb{Q}^{n+m}$$

satisfying $\{x_1,\ldots,x_n\} = \{1,\ldots,n\}$ and $\{y_1,\ldots,y_m\} = \{1,\ldots,m\}$. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be integers with $0 \leq a_i < i$, and set

$$F = \prod_{1\leq j\leq a_i} (x_j - x_i + 1) \in A.$$

The following Proposition will be proven in Section 5.

**Proposition 5.** *If $G \in A$ vanishes when evaluated at every point of $P$ at which $F$ vanishes, then there exists $H \in A$ with*

$$\deg(H) \leq \deg(G) - \deg(F)$$

*satisying $G = FH$ for every point of $P$.*

If $S_{\delta+1}(\sigma) = 0$ for a given $\sigma \in \mathrm{Sym}_\mu$ (which is then necessarily inadmissible), then

$$R_\delta(\sigma) = R_{\delta+1}(\sigma) = 0. \tag{36}$$

By reindexing the permutation sets $\mu_\delta$ and $\mu_{\delta+1}$ as necessary, we can apply Proposition 5 with $G = R_\delta$ and $F = S_\delta$, since $S_\delta$ is of the appropriate form.[9] Thus for $a+1 \leq \delta \leq b-1$, there exist polynomials $T_\delta \in \mathbb{Q}[\sigma]$ with $\deg(T_\delta) \leq \deg(R_\delta) - \deg(S_\delta)$ satisfying

$$T_\delta(\sigma) = \frac{R_\delta(\sigma)}{S_\delta(\sigma)}$$

for all $\sigma$ for which which $S_\delta(\sigma) \neq 0$. Then

$$\psi = XPQR_a \prod_{a+1 \leq \delta \leq b-1} T_\delta \in \mathbb{Q}[\sigma]$$

has degree at most equal to that of $\phi_0$ and satisfies

$$\operatorname{sgn}(\sigma)\psi(\sigma) = \operatorname{sgn}(\sigma)\phi_0(\sigma) = \phi(\sigma)$$

for any admissible $\sigma$.

For a polynomial $\theta \in \mathbb{Q}[\sigma]$, let $V(\theta)$ denote the set of $\sigma \in \operatorname{Sym}_\mu$ such that $\theta(\sigma) = 0$. We see

$$\begin{aligned} V(\psi) &\supseteq V(Q) \cup V(R_a) \cup \left( \bigcup_{a+1 \leq \delta \leq b-1} V(T_\delta) \right) \\ &\supseteq V(Q) \cup V(R_a) \cup \left( \bigcup_{a+1 \leq \delta \leq b-1} (V(R_\delta) - V(S_\delta)) \right) \\ &\supseteq V(Q) \cup \left( \bigcup_{a \leq \delta \leq b-1} V(R_\delta) \right) \\ &= \{\sigma \in \operatorname{Sym}_\mu \mid \sigma \text{ is not admissible}\}. \end{aligned}$$

The third inclusion is by repeated application of (36). We conclude $\psi$ vanishes when evaluated at any inadmissible $\sigma$.

We are finally able to evaluate the sum (34). We have

$$\sum_{\sigma \in \operatorname{Sym}_\mu \text{ admissible}} \phi(\sigma) Z(\sigma) = \sum_{\sigma \in \operatorname{Sym}_\mu} \operatorname{sgn}(\sigma)\psi(\sigma)Z(\sigma).$$

If $\deg(Z) < \kappa_0$, then $\deg(\psi Z) < \sum_\delta \frac{1}{2}|\mu_\delta|(|\mu_\delta| - 1)$, and thus

$$\sum_{\sigma \in \operatorname{Sym}_\mu} \operatorname{sgn}(\sigma)\psi(\sigma)Z(\sigma) = 0.$$

We have proven the evaluation of Proposition 4 is well-defined.

[9]By definition, $e_\delta(j)$ is a weakly decreasing function of $j$. We use the *opposite* ordering on the variables $\sigma_\delta(j)$ to write $S_\delta$ in the desired form. Explicitly, if

$$\mu_\delta = \{A, A+1, \ldots, B\},$$

then we take $x_i = \sigma_\delta(B - i + 1) - A + 1$.

The second part of Proposition 4 asserts the vanishing of the evaluation for all but finitely many $f$. We will use a combination of two ideas to prove the assertion. First, if $\mathcal{S}(f) = \emptyset$, then the evaluation is trivially zero. Second, we replace the polynomial $\psi$ above with another polynomial $\psi'$ which assumes the same values but has lower degree. Then

$$\deg(\psi') < \deg(\phi_0) = -\kappa_0 + \sum_\delta \frac{1}{2}|\mu_\delta|(|\mu_\delta| - 1).$$

So for $\deg(Z) \leq \kappa_0$,

$$\sum_{\sigma \in \mathrm{Sym}_\mu} \mathrm{sgn}(\sigma)\psi'(\sigma)Z(\sigma) = 0.$$

As we have seen, a choice of $f$ such that $\mathcal{S}(f) \neq \emptyset$ uniquely determines constants $e_\delta(j)$ weakly decreasing in $j$. We use linear inequalities in the constants $e_\delta(j)$ to describe four cases in which either $\mathcal{S}(f) = \emptyset$ or $\psi$ can be replaced by $\psi'$ as above. In the end, we will check that only finitely many possibilities avoid all four cases. The finiteness will come from giving upper and lower bounds for the $e_\delta(j)$. For the lower bound, since $e_\delta(j)$ is weakly decreasing in $j$, we introduce the notation

$$m_\delta = \max(\mu_\delta)$$

and focus on the values $e_\delta(m_\delta)$.

**Case I.** Let $J = \max\{j \mid (\delta; j) \in \mu \text{ for some } \delta\}$ and suppose $e_\delta(j) > J$ for some $(\delta; j) \in \mu$. Then for any $\sigma \in \mathrm{Sym}_\mu$,

$$h_\sigma(\delta; \sigma_\delta(j)) = a \cdot (\sigma_\delta(j) - e_\delta(j)) < 0,$$

so $\sigma$ is not admissible. Thus $\mathcal{S}(f) = \emptyset$.

**Case II.** Consider the sequence

$$e_0(0) \geq e_0(1) \geq \cdots \geq e_0(m_0).$$

Suppose there exists $i \in \{0, \ldots, m_0\}$ for which the conditions

- $e_0(i) < -1$
- $i = 0$ or $e_0(i) < e_0(i-1) - 1$

hold. Then, for admissible $\sigma \in \mathrm{Sym}_\mu$, the factor $\sigma_0$ must map $\{i, \ldots, m_0\}$ to itself, as the box configuration function

$$h_\sigma(\delta; j) = a(j - e_\delta(\sigma_\delta^{-1}(j)))$$

must be weakly increasing in $j$. The factor $P$ of $\psi$ is a multiple of

$$\prod_{j=i}^{m_0}(-\sigma_0(j) - 1).$$

Since $\frac{\psi}{P}$ vanishes at all inadmissible $\sigma$, we can take

$$\psi' = \frac{\prod_{j=i}^{m_0}(-j-1)}{\prod_{j=i}^{m_0}(-\sigma_0(j)-1)}\psi,$$

and then $\psi'(\sigma) = \psi(\sigma)$ at all $\sigma \in \mathrm{Sym}_\mu$. We have $\deg(\psi') < \deg(\psi)$, as desired.

**Case III.** Suppose $\delta \geq 0$ and $e_{\delta+1}(m_{\delta+1}) + 1 < e_\delta(m_\delta)$.

Then, either $m_{\delta+1} = m_\delta - 1$ or $m_{\delta+1} = m_\delta$. We consider the two options separately.

(**i**) If $m_{\delta+1} = m_\delta - 1$, then for any $\sigma \in \mathrm{Sym}_\mu$, we can take

$$i = \sigma_\delta^{-1}(\sigma_{\delta+1}(m_{\delta+1}) + 1)\ .$$

Then, $\sigma_\delta(i) = \sigma_{\delta+1}(m_{\delta+1}) + 1$ and $e_\delta(i) \geq e_\delta(m_\delta) > e_{\delta+1}(m_{\delta+1}) + 1$, so $\sigma$ is not admissible. Thus $\mathcal{S}(f) = \emptyset$.

(**ii**) If $m_{\delta+1} = m_\delta$, then we have $e_{\delta+1}(m_\delta) + 1 < e_\delta(m_\delta) \leq e_\delta(j)$ for $0 \leq j \leq m_\delta$, so $R_\delta$ is a multiple of

$$\prod_{j=0}^{m_\delta}(\sigma_\delta(j) - \sigma_{\delta+1}(m_\delta) - 1). \tag{37}$$

The product (37) vanishes unless $\sigma_{\delta+1}(m_\delta) = m_\delta$. Hence

$$(-m_\delta - 1)\prod_{j=1}^{m_\delta}(j - \sigma_{\delta+1}(m_\delta) - 1)$$

equals (37) for all $\sigma \in \mathrm{Sym}_\mu$ and is of lower degree, so we may replace $\psi$ with $\psi'$ of lower degree.

**Case IV.** Suppose $\delta < 0$ and $e_\delta(m_\delta) < e_{\delta+1}(m_{\delta+1})$.

The situation is parallel to Case III. As before, either $\mathcal{S}(f) = \emptyset$ or we can replace a divisor of $R_\delta$ with a polynomial of lower degree.

To complete the proof of Proposition 4, we must check there are only finitely many $f$ which avoid Cases I-IV. If $f$ does not fall into Case I, then $e_\delta(j) \leq J$ for all $(\delta; j) \in \mu$. If $f$ does not fall into Case II, then $e_0(j) \geq -j - 1$ for each $j$, and in particular $e_0(m_0) \geq -m_0 - 1$. If $f$ also does not fall into either of the other two cases, we can extend the inequality to obtain

$$e_\delta(m_\delta) \geq -m_0 - 1 - \max\{\delta \mid \mu_\delta \neq \emptyset\}$$

for all $\delta$. Since $e_\delta(j)$ is a weakly decreasing function of $j$, the bounds imply bounds for all of the $e_\delta(j)$. Since the $e_\delta(j)$ belong to $\frac{1}{a}\mathbb{Z}$, we conclude there are only a finite number of possibilities for each if $f$ does not fall into any of the Cases I-IV. $\square$

4.4. **Proof of Proposition 5.** Let $R = \mathbb{Q}[x_1, \ldots, x_n]$, and let

$$e_1, e_2, \ldots, e_n \in R$$

be the elementary symmetric polynomials with $c_1, c_2, \ldots, c_n \in \mathbb{Z}$ their evaluations at $x_i = i$. Let

$$I = (e_1 - c_1, \ldots, e_n - c_n) \subset R$$

denote the ideal of polynomials vanishing on every permutation of $(1, \ldots, n)$. For a polynomial $f \in R$, let $f_0$ denote the homogeneous part of $f$ of highest degree. For an ideal $J \subset R$, let $J_0$ denote the homogeneous ideal generated by the top-degree parts,

$$J_0 = \langle\, f_0 \mid f \in J \,\rangle\ .$$

Using the regularity of $e_1, \ldots, e_n$, we easily see $I_0 = (e_1, \ldots, e_n)$.

We define $R' = \mathbb{Q}[y_1, \ldots, y_m]$ and ideals $I', I'_0 \subset R'$ as above with respect to the permutations of $(1, \ldots, m)$. We have

$$A = R \otimes_{\mathbb{Q}} R' = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_m]\ .$$

For notational convenience, we let

$$I, I_0, I', I'_0 \subset A$$

denote the extensions of the respective ideals of $R$ and $R'$ in $A$. The ideal of $A$ vanishing on the set $P \subset \mathbb{Q}^{n+m}$ of Proposition 5 is precisely $I + I'$. The basic equality

$$(I + I')_0 = I_0 + I'_0$$

holds.

Let $\widehat{P} = \{p \in P \mid F(p) \neq 0\}$. Let $H \in A$ be a polynomial with the prescribed values

$$H(p) = \frac{G(p)}{F(p)}$$

for $p \in \widehat{P}$, of minimum possible degree $d = \deg(H)$. We must show $d \leq \deg(G) - \deg(F)$. For contradiction, assume $d > \deg(G) - \deg(F)$. Then, the polynomial $G - FH$ vanishes at every $p \in P$ and has top degree part $F_0 H_0$.

Since $F_0 \in R$, we verify the following equality

$$H_0 \in \{f \in A \mid F_0 f \in (I + I')_0\} = \{r \in R \mid F_0 r \in I_0\} + I'_0 \quad \subset A.$$

We claim the above ideal is equal to

$$\{f \in A \mid Ff \in I + I'\}_0 = \{r \in R \mid Fr \in I\}_0 + I'_0 \quad \subset A,$$

Assuming the equality, there exists $H' \in A$ with top degree part $H_0$ and $FH' \in I + I'$ vanishing at every $p \in P$. But then $H_0 - H'$ has degree less than that of $H_0$ and still interpolates the desired values, so we have a contradiction.

To complete the proof of Proposition 5, we must show

$$\{r \in R \mid F_0 r \in I_0\} + I'_0 = \{r \in R \mid Fr \in I\}_0 + I'_0,$$

or equivalently

$$\{r \in R \mid F_0 r \in I_0\} = \{r \in R \mid Fr \in I\}_0. \tag{38}$$

The left hand side contains the right hand side. The equality (38) is thus a consequence of the following Lemma which implies the two sides have equal (and finite) codimension in $R$.

**Lemma 5.** *Let $n \geq 0$ be an integer, and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be integers satisfying $0 \leq a_i < i$. Let*

$$F = \prod_{1 \leq j \leq a_i} (x_j - x_i + 1) \qquad \text{and} \qquad F_0 = \prod_{1 \leq j \leq a_i} (x_j - x_i).$$

*Then, we have*

$$\begin{aligned} \operatorname{rank}_{\mathbb{Q}}(m_F : R/I \to R/I) &= \operatorname{rank}_{\mathbb{Q}}(m_{F_0} : R/I_0 \to R/I_0) \\ &= \prod_{i=1}^{n} (i - a_i), \end{aligned}$$

*where $m_F$ and $m_{F_0}$ denote multiplication operators by $F$ and $F_0$ respectively.*

*Proof.* We first show $\mathrm{rank}_{\mathbb{Q}}(m_F) = \prod_{i=1}^n (i - a_i)$. Since $R/I$ is the coordinate ring of the set of $n!$ permutations of $(1, \ldots, n)$, the rank is simply the number of permutations at which $F$ does not vanish. We must count the number of permutations $\sigma \in \mathrm{Sym}_n$ satisfying

$$\sigma(i) - 1 \neq \sigma(j)$$

for $1 \leq j \leq a_i$.

We view the permutation $\sigma$ (extended by $\sigma(0) = 0$) as a directed path on vertices labeled $0, 1, \ldots, n$ with an edge from $i$ to $j$ if $\sigma(i) - 1 = \sigma(j)$. We are then counting permutations which do not have an edge from $i$ to $j$ if $1 \leq j \leq a_i$.

We count the number of ways of building such a path by first choosing an edge leading out of $n$, then an edge leading out of $n - 1$, and so on. The edge leading out of $n$ can go to 0 or to any $j$ with $a_n < j < n$; there are $n - a_n$ choices. After placing the edges leading out of $n, n - 1, \ldots, k + 1$, the digraph will be a disjoint union of $k + 1$ paths. One of these paths will end at $k$ and $a_k$ of the other paths will end at $1, \ldots, a_k$, so the choices for the edge leading out of $k$ are to go to the start of one of the $k - a_k$ other paths. Thus, the number of such permutations is indeed the product $(n - a_n) \cdots (1 - a_1)$.

Proving $\mathrm{rank}_{\mathbb{Q}}(m_{F_0}) = \prod_{i=1}^n (i - a_i)$ will require more work. Let

$$J = \{f \in R \mid F_0 f \in I_0\},$$

so multiplication by $F_0$ induces an isomorphism between $R/J$ and $\mathrm{Image}(m_{F_0}) \subset R/I_0$. We will show

$$\mathrm{rank}_{\mathbb{Q}}(R/J) = \prod_{i=1}^{n} (i - a_i) \ . \tag{39}$$

In fact, we claim $R/J$ is a 0-dimensional complete intersection of multidegree $(1 - a_1, \ldots, n - a_n)$. The dimension (39) will then follow from Bezout's Theorem. For $1 \leq k \leq n$, let

$$f_k = \sum_{i=k}^{n} x_i \prod_{j=a_k+1}^{k-1} (x_j - x_i).$$

We claim $J = (f_1, \ldots, f_n)$. Note $f_k$ has degree $k - a_k$ as desired.

We will prove this claim by induction on the sequence $(a_i)_{i=1}^n$. The base case is $a_i = 0$ for all $i$ where

$$F = 1, \quad J = I_0, \quad \text{and} \quad f_k = \sum_{i=k}^{n} x_i \prod_{j=1}^{k-1} (x_j - x_i) \ .$$

We must show $(f_1, \ldots, f_n) = (e_1, \ldots, e_n)$.

First, suppose $f_1 = f_2 = \cdots = f_n = 0$ at some point

$$(t_1, \ldots, t_n) \in \overline{\mathbb{Q}}^n.$$

From $f_n = 0$, we find either $t_n = 0$ or $t_n = t_i$ for some $i < n$. Since $f_{n-1} = 0$, either $t_{n-1} = 0$ or $t_{n-1} = t_i$ for some $i < n-1$. Continuing, we conclude for every $k$, either $t_k = 0$ or $t_k = t_i$ for some $i < k$. Thus, $t_k = 0$ for all $k$. Therefore $R/(f_1, \ldots, f_n)$ is a complete intersection and has $\mathbb{Q}$-rank

$$(\deg f_1) \cdots (\deg f_n) = n! = \mathrm{rank}_{\mathbb{Q}}(R/(e_1, \ldots, e_n)) \ .$$

By the rank computation, we need only show

$$(f_1, \ldots, f_n) \subseteq (e_1, \ldots, e_n) \tag{40}$$

to complete the base case of the induction. But the inclusion (40) is easily seen. For every $k$, we have

$$\begin{aligned} f_k &= \sum_{i=1}^{n} x_i \prod_{j=1}^{k-1} (x_j - x_i) \\ &= \sum_{i=1}^{n} \sum_{e=1}^{n} c_e x_i^e \\ &= \sum_{e=1}^{n} c_e \left( \sum_{i=1}^{n} x_i^e \right), \end{aligned}$$

where $c_e \in R$. The power sum $\sum_{i=1}^n x_i^e$ is symmetric and can be written as a polynomial in the elementary symmetric functions $e_1, \ldots, e_n$. The base case is now established.

We now consider two sets of indices $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ for which such that $a'_i = a_i$ except when $i = l$ and

$$a'_l = a_l + 1. \tag{41}$$

We moreover require either $l = n$ or $a_{l+1} = a_l + 1$. We assume inductively our claim holds for $a_1, \ldots, a_n$ and show the claim for $a'_1, \ldots, a'_n$. Every $(a'_i)_{1 \leq i \leq n}$ which is not identically zero can be reached by taking $l = \min\{l \mid a'_l = a'_n\}$, so the inductive step will imply the Lemma. Let $J$,$J'$ be the corresponding ideals and let $f_1, \ldots, f_n$ and $f'_1 \ldots, f'_n$ be the claimed generators. We are assuming $J = (f_1, \ldots, f_n)$ and want to prove $J' = (f'_1 \ldots, f'_n)$.

From the definition of $J$ and $J'$, we easily see

$$J' = \{g \in R \mid (x_{a_l+1} - x_l) g \in J\}.$$

Also note $f'_k = f_k$ for $k \neq l$. If $l = n$, then

$$f'_l = \frac{f_l}{x_{a_l+1} - x_l}$$

and otherwise

$$f'_l = \frac{f_l - f_{l+1}}{x_{a_l+1} - x_l}$$

by condition (41).

Let $\overline{R} = R/(x_{a_l+1} - x_l)$. For an element $r \in R$, let $\overline{r}$ denote the projection in $\overline{R}$. Consider the $\overline{R}$-module homomorphism

$$\psi : \overline{R}^n \to \overline{R}$$

defined by $\psi(\overline{r}_1, \ldots, \overline{r}_n) = \overline{f}_1\overline{r}_1 + \cdots + \overline{f}_n\overline{r}_n$. Let $s_i^{(j)}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ be such that the $m$ elements $(\overline{s}_1^{(j)}, \ldots, \overline{s}_n^{(j)}) \in \overline{R}^n$ generate the kernel of $\psi$. Clearly, $J'$ is the ideal generated by $J$ and the $m$ elements

$$\frac{1}{x_{a_l+1} - x_l} \sum_{i=1}^{n} f_i s_i^{(j)} \ .$$

In other words, we must find all the relations between the elements

$$\overline{f}_1, \ldots, \overline{f}_n.$$

Now $\overline{f}_l = \overline{f}_{l+1}$ if $l \neq n$, or $\overline{f}_l = 0$ if $l = n$, so we need only consider relations between the $n-1$ elements with $\overline{f}_l$ removed. These $n-1$ elements in $\overline{R}$ form a complete intersection, so the relations are generated by the trivial ones $\overline{f}_i\overline{f}_j - \overline{f}_j\overline{f}_i = 0$.

We have proven that $J'$ is the ideal generated by $J = (f_1, \ldots, f_n)$, either $\frac{f_l}{x_{a_l+1}-x_l}$ if $l = n$ or $\frac{f_l - f_{l+1}}{x_{a_l+1}-x_l}$ otherwise, and the elements

$$\frac{f_i f_j - f_j f_i}{x_{a_l+1} - x_l} = 0.$$

Thus $J' = (f'_1 \ldots, f'_n)$, as desired. □

## 5. Descendent depth

5.1. ***T*-Depth.** Let $N$ be a split rank 2 bundle on a nonsingular projective curve $C$ of genus $g$. Let $S \subset N$ be the relative divisor associated to the points $p_1, \ldots, p_r \in C$. We consider the $T$-equivariant stable pairs theory of $N/S$ with respect to the scaling action.

The $T$-*depth* $m$ theory of $N/S$ consists of all $T$-equivariant series

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j'=1}^{k'} \tau_{i'_{j'}}(\mathsf{1}) \ \prod_{j=1}^{k} \tau_{i_j}(\mathsf{p})\right)^T \tag{42}$$

where $k' \leq m$. As before, $\mathsf{p} \in H^2(C, \mathbb{Z})$ is the class of a point. The $T$-depth $m$ theory has at most $m$ descendents of 1 and arbitrarily many descendents of $\mathsf{p}$ in the integrand. The $T$-depth $m$ theory of $N/S$ is *rational* if all $T$-depth $m$ series (42) are Laurent expansions in $q$ of rational functions in $\mathbb{Q}(q, s_1, s_2)$.

The $T$-depth 0 theory concerns only descendents of $\mathsf{p}$. By taking the specialization $s_3 = 0$ of Proposition 2,

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k} \tau_{i_j}(\mathsf{p})\right)^{T} = \mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k} \tau_{i_j}([0])\right)^{\mathbf{T}}\Bigg|_{s_3=0},$$

we see the depth 0 theory of the cap is rational.

**Lemma 6.** *The $T$-depth 0 theory of $N/S$ over a curve $C$ is rational.*

*Proof.* By the degeneration formula, all the descendents $\tau_{i_j}(\mathsf{p})$ can be degenerated on to a $(0,0)$-cap. The $T$-depth 0 theory of the cap is rational. The pairs theory of local curves without any insertions is rational by [17, 20]. Hence, the result follows by the degeneration formula. $\square$

5.2. **Degeneration.** We have already used the degeneration formula in simple cases in Proposition 3 and Lemma 6 above. We review here the full $T$-equivariant formula for descendents of $1, \mathsf{p} \in H^*(C, \mathbb{Z})$.

Let $C$ degenerate to a union $C_1 \cup C_2$ of nonsingular projective curves $C_i$ meeting at a node $p'$. Let $N$ degenerate to split bundles

$$N_1 \to C_1, \qquad N_2 \to C_2 \ .$$

The levels of $N_i$ must sum to the level of $N$. The relative points $p_i$, distributed to nonsingular points of $C_1 \cup C_2$, specify relative points $S_i \subset C_i$ away from $p'$. Let $S_i^+ = S_i \cup \{p'\}$.

In order to apply the degeneration formula to the series (42), we must also specify the distribution of the point classes occuring in the descendents $\tau_{i_j}(\mathsf{p})$. The disjoint union

$$J_1 \cup J_2 = \{1, \ldots, k\}$$

specifies the descendents $\tau_{i_j}(\mathsf{p})$ distribute to $C_i$ for $j \in J_i$. The degeneration formula for (42) is

$$\sum_{J'_1 \cup J'_2 = \{1,\ldots,k'\}} \mathsf{Z}^{N_1/S_1^+}_{d,\eta^1,\ldots,\eta^{|S_1|},\mu}\left(\prod_{j'\in J'_1} \tau_{i'_{j'}}(1) \prod_{j\in J_1} \tau_{i_j}(\mathsf{p})\right)^T \frac{g^{\mu\widehat{\mu}}}{q^d}$$
$$\cdot \mathsf{Z}^{N_2/S_2^+}_{d,\eta^{|S_1|+1},\ldots,\eta^{|S_2|},\widehat{\mu}}\left(\prod_{j'\in J'_2} \tau_{i'_{j'}}(1) \prod_{j\in J_2} \tau_{i_j}(\mathsf{p})\right)^T$$

A crucial point in the derivation of the degeneration formula is the pre-deformability condition (ii) of Section 3.7 of [24]. The condition insures the existence of finite resolutions of the universal sheaf $\mathbb{F}$ in the relative geometry (needed for the definition of the descendents) and guarantees the splitting of the descendents under pull-back via the gluing maps of the relative geometry. The foundational treatment for stable pairs is essentially the same as for ideal sheaves [12].

5.3. **Induction I.** To obtain the rationality of the $T$-depth $m$ theory of $N/S$ over a curve $C$, further knowledge of the descendent theory of twisted caps is required.

**Lemma 7.** *The rationality of the $T$-depth $m$ theories of all twisted caps implies the rationality of the $T$-depth $m$ theory of $N/S$ over a curve $C$.*

*Proof.* We start by proving rationality for the $T$-depth $m$ theories of all $(0,0)$ geometries,

$$\mathcal{O}_{\mathbb{C}} \oplus \mathcal{O}_{\mathbb{C}} \to \mathbb{P}^1 \ , \tag{43}$$

relative to $p_1,\ldots,p_r \in \mathbb{P}^1$. If $r = 1$, the geometry is the cap and rationality of the $T$-depth $m$ theory is given. Assume rationality holds for $r$. We will show rationality holds for $r+1$.

Let $p(d)$ be the number of partitions of size $d > 0$. Consider the $\infty \times p(d)$ matrix $M_d$, indexed by monomials

$$L = \prod_{i\geq 0} \tau_i(\mathsf{p})^{n_i}$$

in the descendents of $\mathsf{p}$ and partitions $\mu$ of $d$, with coefficient $\mathsf{Z}^{\mathsf{cap}}_{d,\mu}(L)^T$ in position $(L,\mu)$. The lowest Euler characteristic for a degree $d$ stable pair on the cap is $d$. The leading $q^d$ coefficients of $M_d$ are well-known to be of maximal rank.[10] Hence, the full matrix $M_d$ is also of maximal rank.

Consider the level $(0,0)$ geometry over $\mathbb{P}^1$ relative to $r+1$ points in $T$-depth $m$,

$$\mathsf{Z}^{(0,0)}_{d,\eta^1,\dots,\eta^r,\mu}\left(\prod_{j'=1}^{k'}\tau_{i'_{j'}}(1)\ \prod_{j=1}^{k}\tau_{i_j}(\mathsf{p})\right)^T . \tag{44}$$

We will determine the series (44) from the $T$-depth $m$ series relative to $r$ points,

$$\mathsf{Z}^{(0,0)}_{d,\eta^1,\dots,\eta^r}\left(L\ \prod_{j'=1}^{k'}\tau_{i'_{j'}}(1)\ \prod_{j=1}^{k}\tau_{i_j}(\mathsf{p})\right)^T \tag{45}$$

defined by all monomials $L$ in the descendents of $\mathsf{p}$.

Consider the $T$-equivariant degeneration of the $(0,0)$ geometry relative to $r$ points obtained by bubbling off a single $(0,0)$-cap. All the descendents of $\mathsf{p}$ remain on the original $(0,0)$ geometry in the degeneration except for those in $L$ which distribute to the cap. By induction on $m$, we need only analyze the terms of the degeneration formula in which the descendents of the identity distribute away from the cap. Then, since $M_d$ has full rank, the invariants (44) are determined by the invariants (45).

We have proven the rationality of the $T$-depth $m$ theory of the $(0,0)$-cap implies the rationality of the $T$-depth $m$ theories of all $(0,0)$ relative geometries over $\mathbb{P}^1$. By degenerations of higher genus curves $C$ to rational curves with relative points, the rationality of the $(0,0)$ relative geometries over curves $C$ of arbitrary genus is established.

Finally, consider a relative geometry $N/S$ over $C$ of level $(a_1,a_2)$. We can degenerate $N/S$ to the union of a $(0,0)$ relative geometry over $C$ and a twisted $(a_1,a_2)$-cap. Since the rationality of the $T$-depth $m$ theory of the twisted cap is given, we conclude the rationality of $N/S$ over $C$. $\square$

The proof of Lemma 7 yields a slightly refined result which will be half of our induction argument relating the descendent theory of the $(0,0)$-cap and the $(0,0)$-tube.

**Lemma 8.** *The rationality of the $T$-depth $m$ theory of the $(0,0)$-cap implies the rationality of the $T$-depth $m$ theory of the $(0,0)$-tube.*

[10] The leading $q^d$ coefficients are obtained from the Chern characters of the tautological rank $d$ bundle on $\mathrm{Hilb}(N_\infty,d)$. The Chern characters generate the ring $H_T^*(\mathrm{Hilb}(N_\infty,d),\mathbb{Q})$ after localization as can easily be seen in the $T$-fixed point basis. A more refined result is discussed in Section 9.

5.4. **T-depth.** The **T**-*depth* $m$ theory of the $(a_1, a_2)$-cap consists of all the **T**-equivariant series

$$\mathsf{Z}^{(a_1,a_2)}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\ \prod_{j'=1}^{k'}\tau_{i'_{j'}}([\infty])\right)^{\mathbf{T}} \tag{46}$$

where $k' \leq m$. Here, $0 \in \mathbb{P}^1$ is the non-relative **T**-fixed point and $\infty \in \mathbb{P}^1$ is the relative point. The **T**-depth $m$ theory of the $(a_1, a_2)$-cap is *rational* if all **T**-equivariant depth $m$ series (42) are Laurent expansions in $q$ of rational functions in $\mathbb{Q}(q, s_1, s_2, s_3)$.

**Lemma 9.** *The rationality of the* **T**-*depth $m$ theory of the $(a_1, a_2)$-cap implies the rationality of the $T$-depth $m$ theory of the $(a_1, a_2)$-cap.*

*Proof.* The identity class $1 \in H^*_T(\mathbb{P}^1, \mathbb{Z})$ has a well-known expression in terms of the **T**-fixed point classes

$$1 = -\frac{[0]}{s_3} + \frac{[\infty]}{s_3}\ .$$

We can calculate at most $m$ descendents of 1 in the $T$-equivariant theory via at most $m$ descendents of $[\infty]$ in the **T**-equivariant theory (followed the specialization $s_3 = 0$). ☐

## 6. Rubber calculus

6.1. **Overview.** We collect here results concerning the rubber calculus which will be needed to complete the proof of Theorem 3. Our discussion of the rubber calculus follows the treatment given in Section 4.8-4.9 of [20].

6.2. **Universal 3-fold $\mathcal{R}$.** Consider the moduli space of stable pairs on rubber $P_n(R/R_0 \cup R_\infty)^\sim$ discussed in Section 3.3. Let

$$\pi : \mathcal{R} \to P_n(R/R_0 \cup R_\infty, d)^\sim$$

denote the universal 3-fold. The space $\mathcal{R}$ can be viewed as a moduli space of stable pairs on rubber *together* with a point $r$ of the 3-fold rubber. The point $r$ is *not* permitted to lie on the relative divisors $R_0$ and $R_\infty$. The stability condition is given by finiteness of the associated automorphism group. The virtual class of $\mathcal{R}$ is obtained via $\pi$-flat pull-back,

$$[\mathcal{R}]^{vir} = \pi^*\Big([P_n(R/R_0 \cup R_\infty, d)^\sim]^{vir}\Big).$$

As before, let

$$\mathbb{F} \to \mathcal{R}$$

denote the universal sheaf on $\mathcal{R}$.

The target point $r$ together with $R_0$ and $R_\infty$ specifies 3 distinct points of the destabilized $\mathbb{P}^1$ over which the rubber is fibered. By viewing the target point as $1 \in \mathbb{P}^1$, we obtain a rigidification map to the tube,

$$\phi : \mathcal{R} \to P_n(N/N_0 \cup N_\infty, d),$$

where $N = \mathcal{O}^1_{\mathbb{P}} \oplus \mathcal{O}^1_{\mathbb{P}}$ is the trivial bundle over $\mathbb{P}^1$. By a comparison of deformation theories,

$$[\mathcal{R}]^{vir} = \phi^*\Big([P_n(N/N_0 \cup N_\infty, d)]^{vir}\Big). \tag{47}$$

6.3. **Rubber descendents.** Rubber calculus transfers $T$-equivariant rubber descendent integrals to $T$-equivariant descendent integrals for the $(0,0)$-tube geometry via the maps $\pi$ and $\phi$. Consider the rubber descendent

$$\Big\langle \mu \ \Big| \ \psi_0^\ell \ \tau_c \cdot \prod_{j=1}^{k} \tau_{i_j} \ \Big| \ \nu \Big\rangle^{\sim}_{n,d} . \tag{48}$$

As before, $\psi_0$ is the cotangent line at the dynamical point $0 \in \mathbb{P}^1$. The action of the rubber descendent $\tau_i$ is defined via the universal sheaf $\mathbb{F}$ by the operation

$$\pi_*\big(\mathrm{ch}_{2+i}(\mathbb{F}) \cap (\pi^*(\ \cdot\ )\big) \colon H_*(P_n(N/N_0 \cup N_\infty, d)) \to H_*(P_n(N/N_0 \cup N_\infty, d)) .$$

By the push-pull formula, the integral (48) equals

$$\Big\langle \mu \ \Big| \ \mathrm{ch}_{2+c}(\mathbb{F}) \ \pi^* \left( \psi_0^\ell \cdot \prod_{j=1}^{k} \tau_{i_j} \right) \ \Big| \ \nu \Big\rangle^{\mathcal{R}\sim}_{n,d} . \tag{49}$$

Next, we compare the cotangent lines $\pi^*(\psi_0)$ and $\phi^*(\psi_0)$ on $\mathcal{R}$. A standard argument yields

$$\pi^*(\psi_0) = \phi^*(\psi_0) - \phi^*(D_0),$$

where

$$D_0 \subset I_n(N/N_0 \cup N_\infty, d)$$

is the virtual boundary divisor for which the rubber over $\infty$ carries Euler characteristic $n$. We will apply the cotangent line comparisons to (49). The basic vanishing

$$\psi_0|_{D_0} = 0 \tag{50}$$

holds.

Consider the Hilbert scheme of points $\mathrm{Hilb}(R_0, d)$ of the relative divisor. The boundary condition $\mu$ corresponds to a Nakajima basis element of $A^*_T(\mathrm{Hilb}(R_0, d))$. Let $\mathbb{F}_0$ be the universal quotient sheaf on

$$\mathrm{Hilb}(R_0, d) \times R_0,$$

and define the descendent

$$\tau_c = \pi_*\Big(\mathrm{ch}_{2+c}(\mathbb{F}_0)\Big) \in A^c_T(\mathrm{Hilb}(R_0,d)) \tag{51}$$

where $\pi$ is the projection

$$\pi : \mathrm{Hilb}(R_0,d) \times R_0 \to \mathrm{Hilb}(R_0,d)\ .$$

The cotangent line comparisons, equation (49), and the vanishing (50) together yield the following result,

$$\begin{aligned} \Big\langle \mu \ \Big|\ \psi_0^\ell\ \tau_c \cdot \prod_{j=1}^k \tau_{i_j}\ \Big|\ \nu \Big\rangle^{\sim}_{n,d} = &\\ &\Big\langle \mu \ \Big|\ \psi_0^\ell\ \tau_c(\mathsf{p}) \cdot \prod_{j=1}^k \tau_{i_j}\ \Big|\ \nu \Big\rangle^{\mathsf{tube},T}_{n,d} \\ &\qquad - \Big\langle \tau_c \cdot \mu \ \Big|\ \psi_0^{\ell-1} \prod_{j=1}^k \tau_{i_j}\ \Big|\ \nu \Big\rangle^{\sim}_{n,d}\ . \end{aligned} \tag{52}$$

Equation (52) will be the main required property of the rubber calculus.

## 7. Capped 1-leg descendents: full

7.1. **Overview.** We complete the proof of Theorem 3 using the interplay between the **T**-equivariant localization of the cap and the theory of rubber integrals. A similar strategy was used in [18] to prove the Virasoro constraints for target curves. As a consequence, we will also obtain a special case of Theorem 2.

Let $N$ be a split rank 2 bundle on a nonsingular projective curve $C$ of genus $g$. Let $S \subset N$ be the relative divisor associated to the points $p_1, \ldots, p_r \in C$. We consider the $T$-equivariant stable pairs theory of $N/S$ with respect to the scaling action.

**Proposition 6.** *If $\gamma_j \in H^{2*}(C,\mathbb{Z})$ are* even *cohomology classes, then*

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j=1}^k \tau_{i_j}(\gamma_j)\right)^T$$

*is the Laurent expansion in $q$ of a rational function in $\mathbb{Q}(q,s_1,s_2)$.*

Proposition 6 is the restriction of Theorem 2 to even cohomology. The proof is given in Section 7.4. The proof of Theorem 2 will be completed with the inclusion of descendents of odd cohomology in Section 8.

7.2. **Induction II.** The first half of our induction argument was established in Lemma 8. The second half relates the $(0,0)$-tube back to the $(0,0)$-cap with an increase in depth.

**Lemma 10.** *The rationality of the $T$-depth $m$ theory of the $(0,0)$-tube implies the rationality of $\mathbf{T}$-depth $m+1$ theory of the $(0,0)$-cap.*

*Proof.* The result follows from the $\mathbf{T}$-equivariant localization formula for the $(0,0)$-cap and the rubber calculus of Section 6.3. To illustrate the method, consider first the $m=0$ case of Lemma 10.

The localization formula for $\mathbf{T}$-depth 1 series for the $(0,0)$-cap is the following:

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\cdot\tau_{i'_1}([\infty])\right)^{\mathbf{T}} =$$
$$\sum_{|\mu|=d}\mathsf{W}^{\mathsf{Vert}}_{\mu}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)\cdot\mathsf{W}^{(0,0)}_{\mu}\cdot\left(\mathsf{S}_{\eta}^{\tau_{i'_1}\cdot\mu}+\mathsf{S}^{\mu}_{\eta}(\tau_{i'_1})\right)\ ,$$

where the rubber terms on the right are

$$\mathsf{S}_{\eta}^{\tau_{i'_1}\cdot\mu} = \sum_{n\geq d}q^n\left\langle \tau_{i'_1}\cdot\mathsf{P}_{\mu}\ \middle|\ \frac{1}{s_3-\psi_0}\ \middle|\ \mathsf{C}_{\eta}\right\rangle^{\sim}_{n,d},$$
$$\mathsf{S}^{\mu}_{\eta}(\tau_{i'_1}) = \sum_{n\geq d}q^n\left\langle \mathsf{P}_{\mu}\ \middle|\ \frac{s_3\tau_{i'_1}}{s_3-\psi_0}\ \middle|\ \mathsf{C}_{\eta}\right\rangle^{\sim}_{n,d}.$$

In the first rubber term, $\tau_{i'_1}$ acts on the boundary condition $P_\mu$ via (51). The term arises from the distribution of the Chern character of the descendent $\tau_{i'_1}([\infty])$ away from the rubber.

The second rubber term simplifies via the topological recursion relation for $\psi_0$ after writing

$$\frac{s_3}{s_3-\psi_0} = 1+\frac{\psi_0}{s_3-\psi_0} \tag{53}$$

and the rubber calculus relation (52). We find

$$\mathsf{S}^{\mu}_{\eta}(\tau_{i'_1}) = \sum_{|\widehat{\eta}|=d}\mathsf{S}^{\mu}_{\widehat{\eta}}\cdot\frac{g^{\widehat{\eta}\widehat{\eta}}}{q^d}\cdot\mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\eta},\eta}\left(\tau_{i'_1}([\infty])\right)^{T} - \mathsf{S}_{\eta}^{\tau_{i'_1}\cdot\mu}.$$

The leading 1 on the right side of (53) corresponds to the degenerate leading term of $\mathsf{S}^{\mu}_{\widehat{\eta}}$. The topological recursion applied to the $\psi_0$ prefactor of the second term produces the rest of $\mathsf{S}^{\mu}_{\widehat{\eta}}$. The superscript

$\mathsf{tube}$ refers here to the $(0,0)$-tube. The rubber calculus produces the correction $-\mathsf{S}_\eta^{\tau_{i'_1}\cdot\mu}$.

After reassembling the localization formula, we find

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\cdot\tau_{i'_1}([\infty])\right)^{\mathbf{T}} =$$

$$\sum_{|\widehat{\eta}|=d}\mathsf{Z}^{\mathsf{cap}}_{d,\widehat{\eta}}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}}\cdot\frac{g^{\widehat{\eta}\widehat{\eta}}}{q^d}\cdot\mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\eta},\eta}\left(\tau_{i'_1}([\infty])\right)^{T}$$

which implies the $m=0$ case of Lemma 10.

The above method of expressing the $\mathbf{T}$-depth $m+1$ theory of the $(0,0)$-cap in terms of the $\mathbf{T}$-depth 0 theory of the $(0,0)$-cap and the $T$-depth $m$ theory of the $(0,0)$-tube is valid for all $m$.

Consider the $m=1$ case. The localization formula for $\mathbf{T}$-depth 2 series for the $(0,0)$-cap is the following:

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\cdot\tau_{i'_1}([\infty])\tau_{i_2}([\infty])\right)^{\mathbf{T}} =$$

$$\sum_{|\mu|=d}\mathsf{W}^{\mathsf{Vert}}_{\mu}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)\cdot\mathsf{W}^{(0,0)}_{\mu}$$

$$\cdot\left(\mathsf{S}_\eta^{\tau_{i'_1}\tau_{i'_2}\cdot\mu}+\mathsf{S}_\eta^{\tau_{i'_1}\cdot\mu}(\tau_{i'_2})+\mathsf{S}_\eta^{\tau_{i'_2}\cdot\mu}(\tau_{i'_1})+\mathsf{S}_\eta^{\mu}(\tau_{i'_1}\tau_{i'_2})\right),$$

where the rubber terms on the right are

$$\begin{aligned}
\mathsf{S}_\eta^{\tau_{i'_1}\tau_{i'_2}\cdot\mu} &= \sum_{n\geq d}q^n\left\langle \tau_{i'_1}\tau_{i'_2}\cdot\mathsf{P}_\mu\ \middle|\ \frac{1}{s_3-\psi_0}\ \middle|\ \mathsf{C}_\eta\right\rangle^{\sim}_{n,d},\\
\mathsf{S}_\eta^{\tau_{i'_1}\cdot\mu}(\tau_{i'_2}) &= \sum_{n\geq d}q^n\left\langle \tau_{i'_1}\cdot\mathsf{P}_\mu\ \middle|\ \frac{s_3\tau_{i'_2}}{s_3-\psi_0}\ \middle|\ \mathsf{C}_\eta\right\rangle^{\sim}_{n,d},\\
\mathsf{S}_\eta^{\tau_{i'_2}\cdot\mu}(\tau_{i'_1}) &= \sum_{n\geq d}q^n\left\langle \tau_{i'_2}\cdot\mathsf{P}_\mu\ \middle|\ \frac{s_3\tau_{i'_1}}{s_3-\psi_0}\ \middle|\ \mathsf{C}_\eta\right\rangle^{\sim}_{n,d},\\
\mathsf{S}_\eta^{\mu}(\tau_{i'_1}\tau_{i'_2}) &= \sum_{n\geq d}q^n\left\langle \mathsf{P}_\mu\ \middle|\ \frac{s_3^2\tau_{i'_1}\tau_{i'_2}}{s_3-\psi_0}\ \middle|\ \mathsf{C}_\eta\right\rangle^{\sim}_{n,d}.
\end{aligned}$$

Using (53) and the rubber calculus relation (52), we find

$$\begin{aligned}\mathsf{S}^{\mu}_{\eta}(\tau_{i'_1}\tau_{i'_2}) \;=\;& \sum_{|\widehat{\eta}|=d} \mathsf{S}^{\mu}_{\widehat{\eta}} \cdot \frac{g^{\widehat{\eta}\widehat{\eta}}}{q^d} \cdot \mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\eta},\eta}\left(\tau_{i'_1}([\infty])\cdot\tau_{i'_2}(1)\right)^T \;-\; \mathsf{S}^{\tau_{i'_1}\cdot\mu}_{\eta}(\tau_{i'_2}) \\ &+ \sum_{|\widehat{\eta}|=d} \mathsf{S}^{\mu}_{\widehat{\eta}}(\tau_{i'_2}) \cdot \frac{g^{\widehat{\eta}\widehat{\eta}}}{q^d} \mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\eta},\eta}\left(\tau_{i'_1}([\infty])\right)^T.\end{aligned}$$

As we have seen before,

$$\begin{aligned}\mathsf{S}^{\mu}_{\widehat{\eta}}(\tau_{i'_2}) \;=\;& \sum_{|\widehat{\mu}|=d} \mathsf{S}^{\mu}_{\widehat{\mu}} \cdot \frac{g^{\widehat{\mu}\widehat{\mu}}}{q^d} \cdot \mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\mu},\widehat{\eta}}\left(\tau_{i'_2}([\infty])\right)^T \;-\; \mathsf{S}^{\tau_{i'_2}\cdot\mu}_{\widehat{\eta}}, \\ \mathsf{S}^{\tau_{i'_2}\cdot\mu}_{\eta}(\tau_{i'_1}) \;=\;& \sum_{|\widehat{\eta}|=d} \mathsf{S}^{\tau_{i'_2}\cdot\mu}_{\widehat{\eta}} \cdot \frac{g^{\widehat{\eta}\widehat{\eta}}}{q^d} \cdot \mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\eta},\eta}\left(\tau_{i'_1}([\infty])\right)^T \;-\; \mathsf{S}^{\tau_{i'_1}\tau_{i'_2}\cdot\mu}_{\eta}.\end{aligned}$$

After adding everything together, we have for $m = 1$ the relation:

$$\begin{aligned}&\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\cdot\prod_{j'=1}^{2}\tau_{i'_{j'}}([\infty])\right)^{\mathbf{T}} = \\ &\quad + s_3 \sum_{|\widehat{\eta}|=d} \mathsf{Z}^{\mathsf{cap}}_{d,\widehat{\eta}}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}} \cdot \frac{g^{\widehat{\eta}\widehat{\eta}}}{q^d} \cdot \mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\eta},\eta}\left(\tau_{i'_1}([\infty])\cdot\tau_{i'_2}(1)\right)^T \\ &\quad + \sum_{|\widehat{\mu}|,|\widehat{\eta}|=d} \mathsf{Z}^{\mathsf{cap}}_{d,\widehat{\mu}}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}} \cdot \frac{g^{\widehat{\mu}\widehat{\mu}}}{q^d} \cdot \mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\mu},\widehat{\eta}}\left(\tau_{i'_2}([\infty])\right)^T \\ &\qquad\qquad \cdot \frac{g^{\widehat{\eta}\widehat{\eta}}}{q^d} \cdot \mathsf{Z}^{\mathsf{tube}}_{d,\widehat{\eta},\eta}\left(\tau_{i'_1}([\infty])\right)^T \;.\end{aligned}$$

We leave the derivation of the parallel formula for general $m$ (via elementary bookkeeping) to the reader. □

An identical argument yields the twisted version of Lemma 10 for the $(a_1, a_2)$-cap.

**Lemma 11.** *The rationality of the $T$-depth $m$ theory of the $(0,0)$-tube implies the rationality of the $\mathbf{T}$-depth $m+1$ theory of the $(a_1, a_2)$-cap.*

7.3. **Proof of Theorem 3.** Lemmas 8 and 10 together provide an induction which results in the rationality of the $\mathbf{T}$-depth $m$ theory of the $(0,0)$-cap for all $m$. Since the classes of the $\mathbf{T}$-fixed points $0, \infty \in \mathbb{P}^1$

generate $H^*_{\mathbf{T}}(\mathbb{P}^1,\mathbb{Z})$ after localization, all partition functions

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^{\mathbf{T}}, \qquad \gamma_j\in H^*_{\mathbf{T}}(\mathbb{P}^1,\mathbb{Z})$$

are Laurent series in $q$ of rational functions in $\mathbb{Q}(q,s_1,s_2,s_3)$. □

7.4. **Proof of Proposition 6.** Using Lemma 11, we obtain the extension of Theorem 3 to twisted $(a_1,a_2)$-caps.

**Proposition 7.** *For $\gamma_j\in H^*_{\mathbf{T}}(\mathbb{P}^1,\mathbb{Z})$, the descendent series*

$$\mathsf{Z}^{(a_1,a_2)}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^{\mathbf{T}}$$

*of the $(a_1,a_2)$-cap is the Laurent expansion in $q$ of a rational function in $\mathbb{Q}(q,s_1,s_2,s_3)$.*

By taking the $s_3=0$ specialization of Proposition 7, we obtain the rationality of the $T$-depth $m$ theory of the $(a_1,a_2)$-cap for all $m$. Proposition 6 then follows from Lemma 7. □

7.5. **T-equivariant tubes.** The $(a_1,a_2)$-tube is the total space of

$$\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\mathcal{O}_{\mathbb{P}^1}(a_2)\to\mathbb{P}^1$$

relative to the fibers over both $0,\infty\in\mathbb{P}^1$. We lift the $\mathbb{C}^*$-action on $\mathbb{P}^1$ to $\mathcal{O}_{\mathbb{P}^1}(a_i)$ with fiber weights 0 and $a_is_3$ over $0,\infty\in\mathbb{P}^1$. The 2-dimensional torus $T$ acts on the $(a_1,a_2)$-tube by scaling the line summands, so we obtain a $\mathbf{T}$-action on the $(a_1,a_2)$-tube.

**Proposition 8.** *For $\gamma_j\in H^*_{\mathbf{T}}(\mathbb{P}^1,\mathbb{Z})$, the descendent series*

$$\mathsf{Z}^{(a_1,a_2)}_{d,\eta_1\eta_2}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)\right)^{\mathbf{T}}$$

*of the $(a_1,a_2)$-tube is the Laurent expansion in $q$ of a rational function in $\mathbb{Q}(q,s_1,s_2,s_3)$.*

*Proof.* Consider the descendent series

$$\mathsf{Z}^{(a_1,a_2)}_{d,\eta_2}\left(L\prod_{j'=1}^{k'}\tau_{i'_{j'}}(1)\ \prod_{j=1}^{k}\tau_{i_j}([\infty])\right)^{\mathbf{T}} \tag{54}$$

of the $(a_1,a_2)$-cap where $L$ is a monomial in the descendents of $[0]$. The $(a_1,a_2)$-cap admits a $\mathbf{T}$-equivariant degeneration to a standard $(0,0)$-cap and an $(a_1,a_2)$-tube by bubbling off $0\in\mathbb{P}^1$. The insertions $\tau_{i_j}([0])$

of $L$ are sent $\mathbf{T}$-equivariantly to the non-relative point of the $(0,0)$-cap. Since (54) is rational by Proposition 7 and the matix $M_d$ of Lemma 7 is full rank, the rationality of

$$\mathsf{Z}^{(a_1,a_2)}_{d,\eta_1\eta_2}\left(\prod_{j'=1}^{k'} \tau_{i'_{j'}}(1) \prod_{j=1}^{k} \tau_{i_j}([\infty])\right)^{\mathbf{T}}$$

follows by induction on $k'$ from the degeneration formula. The classes $1$ and $[\infty]$ generate $H^*_{\mathbf{T}}(\mathbb{P}^1,\mathbb{Z})$ after localization. □

## 8. Descendents of odd cohomology

8.1. **Reduction to** $(0,0)$**.** Let $N/S$ be the relative geometry of a split rank 2 bundle on a nonsingular projective curve $C$ of genus $g$. Let

$$\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g \in H^1(C,\mathbb{Z})$$

be a standard symplectic basis of the odd cohomology of $C$. Proposition 6 establishes Theorem 2 in case only the descendents of the even classes $1, \mathsf{p} \in H^*(C,\mathbb{Z})$ are present. The descendents of $\alpha_i$ and $\beta_j$ will now be considered.

The relative geometry $N/S$ may be $T$-equivariantly degenerated to

$$\mathcal{O}_C \oplus \mathcal{O}_C \to C$$

and an $(a_1,a_2)$-cap. The relative points and the descendents $\tau_k(\alpha_i)$ and $\tau_k(\beta_j)$ in the integrand remain on $C$. Since the rationality of the $T$-equivariant descendent theory of the $(a_1,a_2)$-cap has been proven, we may restrict our study of the descendents of odd cohomology to the $(0,0)$ relative geometry over $C$.

8.2. **Proof of Theorem 2.** The full descendent theories of $(0,0)$ relative geometries of curves $C$ are uniquely determined by the even descendent theories of $(0,0)$ relative geometries by the following four properties:

(i) Algebraicity of the virtual class,
(ii) Degeneration formulas for the relative theory in the presence of odd cohomology,
(iii) Monodromy invariance of the relative theory,
(iv) Elliptic vanishing relations.

The properties (i)-(iv) were used in [18] to determine the full relative Gromov-Witten descendents of target curves in terms of the descendents of even classes.

The results of Section 5 of [18] are entirely formal and apply verbatim to the descendent theory of $(0,0)$ relative geometries of curves.

Moreover, the rationality of the even theory implies the rationality of the full descendent theory. □

## 9. Denominators

9.1. **Summary.** We prove the denominator claims of Conjecture 3 when only descendents of $\mathsf{1}$ and $\mathsf{p}$ are present.

**Theorem 5.** *If only descendents of even cohomology are considered, the denominators of the degree $d$ descendent partition functions $\mathsf{Z}$ of Theorems 1, 2, and 3 are products of factors of the form $q^k$ and*

$$1-(-q)^r$$

*for $1 \leq r \leq d$.*

Theorem 5 is proven by carefully tracing the denominators through the proofs of Theorems 1-3. When the descendents of odd cohomology are included, the strategy of Section 5 of [18] requires matrix inversions[11] for which we can not control the denominators.

Theorem 5 is new even when *no* descendents are present. For the trivial bundle

$$N = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathbb{P}^1 \ ,$$

the $T$-equivariant partition $\mathsf{Z}^{N/S}_{d,\eta^1,\eta^2,\eta^3}$ of Theorem 2 is (up to $q$ shifts) equal to the 3-point function

$$\langle \eta^1, \eta^2, \eta^3 \rangle$$

in the quantum cohomology of the Hilbert scheme of points of $\mathbb{C}^2$, see [19, 20].

**Corollary.** *The 3-point functions in the $T$-equivariant quantum cohomology of $Hilb(\mathbb{C}^2, d)$ have possible poles in -q only at the $r^{th}$ roots of unity for $r$ at most $d$.*

*Proof.* By Theorem 5, we see the possible poles in $-q$ of the 3-point functions are at 0 and the $r^{th}$ roots of unity for $r$ at most $d$. By definition, the 3-point functions have no poles at 0. □

[11] Specifically, the matrix associated to Lemma 5.6 of [18] has an inverse with denominators we cannot at present constrain.

9.2. **Denominators for Proposition 2.** We follow here the notation used in the proof of Proposition 2 in Section 3.

The matrix $\mathsf{S}^{\mu}_{\eta}$ is a fundamental solution of a linear differential equation with singularities only at 0 and $r^{th}$ roots of unity for $r$ at most $d$, see [21]. Hence, the poles in $-q$ of the evaluation

$$\mathsf{S}^{\mu}_{\eta}|_{s_3=\frac{1}{a}(s_1+s_2)}$$

can occur only at 0 and $r^{th}$ roots of unity for $r$ at most $d$. The denominator claim of Theorem 5 for Proposition 2 then follows directly from the proof in Section 3.5.

While only the rationality of Theorem 4 is needed in the proof of Proposition 2, the much stronger Laurent polynomiality of Theorem 4 is used here.

9.3. **Denominators for $T$-equivariant stationary theory.** Consider the denominators of

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j=1}^{k}\tau_{i_j}(\mathsf{p})\right)^T .$$

The denominator result for the $T$-equivariant stationary theory of the $(0,0)$-cap is obtained from the denominator result for Proposition 2 by the specialization $s_3=0$. By degenerating all the descendents $\tau_{i_j}(\mathsf{p})$ on to a $(0,0)$-cap, we need only study the denominators of $T$-equivariant partition functions $\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}$ with no descendent insertions.

The denominator result for the $T$-equivariant $(a,b)$-tube with no descendents is again a consequence of the study of the fundamental solution in [21]. By repeated degenerations (using the $(a,b)$-tube for the twists in $N$), we need only study the denominators of $T$-equivariant partition functions $\mathsf{Z}^{(0,0)}_{d,\eta^1,\eta^2,\eta^3}$ with 3 relative insertions.

9.4. **Relative/descendent correspondence.** Relative conditions in the theory of local curves were exchanged for descendents in the proof of Lemma 7. For the denominator result for $\mathsf{Z}^{(0,0)}_{d,\eta^1,\eta^2,\eta^3}$, we require a more efficient correspondence.

**Proposition 9.** *Let $d>0$ be an integer. The square matrix with coefficients*

$$\mathsf{Z}^{\mathsf{cap}}_{d,\lambda}\Big(\tau_{\mu_1-1}([0])\cdots\tau_{\mu_{\ell(\mu)}-1}([0])\Big)^T \tag{55}$$

*as $\lambda$ and $\mu$ vary among partitions of $d$*

(i) *is triangular with respect to the partial ordering by length,*

(ii) *has diagonal entries given by monomials in $q$,*

(iii) *and is of maximal rank.*

*Proof.* The Proposition follows from the results of Section 4.6 of [20] applied to the theory of stable pairs. Our relative conditions $\lambda$ are defined with identity weights in the $T$-equivariant cohomology of $\mathbb{C}^2$. For the proof, we weight all the parts of $\lambda$ with he $T$-equivariant class of the origin in $\mathbb{C}^2$. Then, by compactness and dimension constraints, the triangularity of the matrix is immediate for partitions of different lengths. On the diagonal, the expected dimension of the integrals are 0. Using the compactification

$$\mathbb{C}^2 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1 \tag{56}$$

as in Section 4.6 of [20], we obtain the triangularity of equal length partitions.

Consider the Hilbert scheme of points $\mathrm{Hilb}(\mathbb{C}^2, d)$ of the plane. Let $\mathbb{F}$ be the universal quotient sheaf on

$$\mathrm{Hilb}(\mathbb{C}^2, d) \times \mathbb{C}^2,$$

and define the descendent[12]

$$\tau_k = \pi_*\Big(\mathrm{ch}_{2+k}(\mathbb{F})\Big) \in A^k(\mathrm{Hilb}(\mathbb{C}^2, d), \mathbb{Q})$$

as before (51). Using the compactification (56), we reduce the calculation of the diagonal entries to the pairing

$$s_1 s_2 \left\langle \tau_{c-1} \,\Big|\, (c) \right\rangle_{\mathrm{Hilb}(\mathbb{C}^2,d)} = \frac{1}{c!} \tag{57}$$

which appears in [22].

We conclude the diagonal entries do not vanish. The diagonal entries are monomial in $q$ by the usual vanishing obtained by the holomorphic symplectic form on $\mathbb{C}^2$. □

The denominator result holds for the nonvanishing entries of the correspondence matrix (55). Since the matrix is triangular with monomials in $q$ on the diagonal, the denominator result holds for the *inverse* matrix.

We can now establish the denominator result for the $T$-equivariant 3-point function $\mathsf{Z}^{(0,0)}_{d,\eta^1,\eta^2,\eta^3}$. We start with the descendent series

$$\mathsf{Z}^{0,0}_{d,\eta^3}\Big(\tau_{\mu_1-1}(\mathsf{p})\cdots\tau_{\mu_{\ell(\mu)}-1}(\mathsf{p})\cdot\tau_{\widehat{\mu}_1-1}(\mathsf{p})\cdots\tau_{\widehat{\mu}_{\ell(\widehat{\mu})}-1}(\mathsf{p})\Big) \tag{58}$$

for partitions $\mu$ and $\widehat{\mu}$ of $d$. The denominator result holds for all series (58). By bubbling all the descendents $\tau_{\mu_i-1}(\mathsf{p})$ off of the point $0 \in \mathbb{P}^1$ and bubbling all the descendents $\tau_{\widehat{\mu}_i-1}(\mathsf{p})$ off of the point $1 \in \mathbb{P}^1$, we

[12]The Chern character of $\mathbb{F}$ is properly supported over $\mathrm{Hilb}(\mathbb{C}^2, d)$.

conclude the denominator result for $\mathsf{Z}^{(0,0)}_{d,\eta^1,\eta^2,\eta^3}$ from the denominator result for the inverse of the correspondence matrix (55).

9.5. **Denominators for Theorems 2-3.** The denominator result for Theorem 3 is obtained by following the proof given in Sections 5-7. An important point is to replace the matrix $M_d$ appearing in the proof of Lemma 7 with the correspondence matrix (55). The required matrix inversion then keeps the denominator form. The rest of the proof of Theorem 3 respects the denominators.

Proposition 6 is the statement of Theorem 2 for descendents of even cohomology. Again, the proof respects the denominators. The proof of Theorem 5 is complete. □

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