DESCENDENTS ON LOCAL CURVES: STATIONARY THEORY

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Abstract. The stable pairs theory of local curves in 3-folds (equivariant with respect to the scaling 2-torus) is studied with stationary descendent insertions. Reduction rules are found to lower descendents when higher than the degree. Factorization then yields a simple proof of rationality in the stationary case and a proof of the functional equation related to inverting q. The method yields an effective determination of stationary descendent integrals. The series $\mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))$ plays a special role and is calculated exactly using the stable pairs vertex and an analysis of the solution of the quantum differential equation for the Hilbert scheme of points of the plane.

Contents

0. Introduction

0.1. **Relative local curves.** The geometry of a 3-fold *local curve* consists of a split rank 2 bundle N on a nonsingular projective curve C of genus g,

$$N = L_1 \oplus L_2. \tag{1}$$

The splitting determines a scaling action of a 2-dimensional torus

$$T = \mathbb{C}^* \times \mathbb{C}^*$$

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on N. The *level* of the splitting is the pair of integers (k_1, k_2) where,

$$k_i = \deg(L_i).$$

Of course, the scaling action and the level depend upon the choice of splitting (1).

The fiber of N over a point $p \in C$ determines a T-invariant divisor

$$N_p \subset N$$

isomorphic to $\mathbb{C}^2$ with the standard T-action. We will consider the local stable pairs theory of N relative to the divisor

$$S = \bigcup_{i=1}^{r} N_{p_i} \subset N$$

determined by the fibers over $p_1, \ldots, p_r \in C$. Let $P_n(N/S, d)$ denote the relative moduli space of stable pairs[1], see [22].

For each p_i, let η^i be a partition of d weighted by the equivariant Chow ring,

$$A_T^*(N_{p_i}, \mathbb{Q}) \stackrel{\sim}{=} \mathbb{Q}[s_1, s_2],$$

of the fiber N_{p_i}. By Nakajima's construction, a weighted partition η^i determines a T-equivariant class

$$\mathsf{C}_{\eta^i} \in A_T^*(\mathrm{Hilb}(N_{p_i}, d), \mathbb{Q})$$

in the Chow ring of the Hilbert scheme of points. In the theory of stable pairs, the weighted partition η^i specifies relative conditions via the boundary map

$$\epsilon_i : P_n(N/S, d) \to \mathrm{Hilb}(N_{p_i}, d).$$

An element $\eta \in \mathcal{P}(d)$ of the set of partitions of d may be viewed as a weighted partition with all weights set to the identity class

$$1 \in A_T^*(N_{p_i}, \mathbb{Q}) \ .$$

The Nakajima basis of $A_T^*(\mathrm{Hilb}(N_{p_i}, d), \mathbb{Q})$ consists of identity weighted partitions indexed by $\mathcal{P}(d)$.

Let $s_1, s_2 \in H_{\mathbf{T}}^*(\bullet)$ be the first Chern classes of the standard representations of the first and second $\mathbb{C}^*$-factors of T respectively. The T-equivariant intersection pairing in the Nakajima basis is

$$g_{\mu\nu} = \int_{\mathrm{Hilb}(N_{p_i}, d)} \mathsf{C}_\mu \cup \mathsf{C}_\nu = \frac{1}{(s_1 s_2)^{\ell(\mu)}} \frac{(-1)^{d-\ell(\mu)}}{\mathfrak{z}(\mu)} \ \delta_{\mu,\nu},$$

[1] The curve class is d times the zero section $C \subset N$.

where

$$\mathfrak{z}(\mu) = \prod_{i=1}^{\ell(\mu)} \mu_i \cdot |\mathrm{Aut}(\mu)|.$$

Let $g^{\mu\nu}$ be the inverse matrix.

0.2. **Descendents.** We define descendents in the relative stable pairs theory of local curves by the slant products with universal sheaf following [20].

There exists a universal sheaf on the universal 3-fold $\mathcal{N}$ over the moduli space $P_n(N/S,d)$,

$$\mathbb{F} \to \mathcal{N}\ .$$

For a stable pair $[\mathcal{O} \to F] \in P_n(N/S,d)$, the restriction of $\mathbb{F}$ to the fiber

$$\mathcal{N}_{[\mathcal{O}\to F]} \subset \mathcal{N}$$

is canonically isomorphic to F. Let

$$\pi_N \colon \mathcal{N} \to N,$$
$$\pi_P \colon \mathcal{N} \to P_n(N/S,d)$$

be the canonical projections.

By the stability conditions for the relative theory of stable pairs, $\mathbb{F}$ has a finite resolution by locally free sheaves. Hence, the Chern character of the universal sheaf $\mathbb{F}$ is well-defined. By definition, the operation

$$\pi_{P*}\big(\pi_N^*(\gamma)\cdot \mathrm{ch}_{2+i}(\mathbb{F}) \cap (\pi_P^*(\ \cdot\ )\big) \colon H_*(P_n(N/S,d)) \to H_*(P_n(N/S,d))$$

is the action of the descendent $\tau_i(\gamma)$, where $\gamma \in H^*(C,\mathbb{Z})$. The push-forwards are defined by T-equivariant residues as in [3, 18].

We will use bracket notation for descendents,

$$\left\langle \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j) \right\rangle_{n,d}^{N,\eta^1,\ldots,\eta^r} = \int_{[P_n(N/S,d)]^{vir}} \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\ \prod_{i=1}^{r} \epsilon_i^*(\mathsf{C}_{\eta^i})\ . \tag{2}$$

The partition function is denoted by

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^T = \sum_n \left\langle \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j) \right\rangle_{n,d}^{N,\eta^1,\ldots,\eta^r} q^n.$$

The following basic result is proved in [20].

Theorem. $\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\big(\prod_{j=1}^{k} \tau_{i_j}(\gamma_j)\big)^T$ *is the Laurent expansion in q of a rational function in $\mathbb{Q}(q,s_1,s_2)$.*

0.3. **Stationary theory.** Our main results here concern stationary descendents in the stable pairs theory of local curves. Let

$$\mathsf{p} \in H^2(C, \mathbb{Z})$$

be the class of a point. The *stationary descendents* are $\tau_k(\mathsf{p})$. The methods of the paper, while not fully applicable to other descendents, are much simpler and more effective than the techniques of [20, 21].

Our first result concerns reduction rules for stationary descendents in the theory of local curves.

Theorem 1. *For $k > d$, there exist universal polynomials*

$$f_{k,d}(x_1, \ldots, x_d) \in \mathbb{Q}(s_1, s_2)[x_1, \ldots, x_d]$$

for which the degree d descendent theory of local curves satisfies the reduction rule

$$\tau_k(\mathsf{p}) \mapsto f_{k,d}(\tau_1(\mathsf{p}), \ldots, \tau_d(\mathsf{p}))\ .$$

Explicitly, Theorem 1 yields the following equality for T-equivariant integrals:

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\tau_k(\mathsf{p}) \cdot \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^T =$$

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(f_{k,d}(\tau_1(\mathsf{p}), \ldots \tau_d(\mathsf{p})) \cdot \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^T$$

for $k > d$ and all $\gamma_j \in H^*(C, \mathbb{Z})$. Theorem 1 is proven in Section 1.

Via Theorem 1, factorization properties of the relative conditions, and the established rationality of the stable pairs theory of local curves without insertions, we obtain our second result in Section 2.

Theorem 2. *The stationary series $\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\big(\prod_{j=1}^{k} \tau_{i_j}(\mathsf{p})\big)^T$ is the Laurent expansion in q of a rational function $F(q, s_1, s_2) \in \mathbb{Q}(q, s_1, s_2)$ satisfying the functional equation*

$$F(q^{-1}, s_1, s_2) = (-1)^{\Delta + |\eta| - \ell(\eta) + \sum_{j=1}^{k} i_j} q^{-\Delta} F(q, s_1, s_2),$$

where the constants are defined by

$$\Delta = \int_\beta c_1(T_N), \quad |\eta| = \sum_{i=1}^{r} |\eta^i|, \quad \text{and} \quad \ell(\eta) = \sum_{i=1}^{r} \ell(\eta^i)\ .$$

Here, T_N is the tangent bundle of the 3-fold N, and β is the curve class given by d times the 0-section. Our proof of Theorem 2 is much easier than the rationality results of [20]. Moreover, we do not know how to derive the functional equation from the methods of [20].

As a step in the proof of Theorem 2, we show the entire stationary descendent theory is *determined* from the theory of local curves without insertions and the set of series

$$\mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))^T = \sum_n \Big\langle \tau_d(\mathsf{p}) \Big\rangle^{N,(d)}_{n,d} q^n \ , \qquad d > 0 \ .$$

Here, the cap geometry is $\mathbb{P}^1$ relative to $\infty \in \mathbb{P}^1$. A central result of the paper is the following calculation.

Theorem 3. *We have*

$$\mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))^T = \frac{q^d}{d!}\left(\frac{s_1+s_2}{s_1 s_2}\right)\frac{1}{2}\sum_{i=1}^{d}\frac{1+(-q)^i}{1-(-q)^i} \ .$$

In the above formula, the coefficient of q^d,

$$\big\langle \tau_d, (d) \big\rangle_{\mathrm{Hilb}(\mathbb{C}^2,d)} = \frac{1}{2\cdot(d-1)!}\left(\frac{s_1+s_2}{s_1 s_2}\right),$$

is the classical T-equivariant pairing on the Hilbert scheme of d points on $\mathbb{C}^2$. The proof of Theorem 3 is given in Section 4.

Very few exact calculations for descendents in 3-fold sheaf theories have previously been found. Theorem 3 provides a closed form for the most fundamental descendent series in the stationary theory of local curves. The derivation uses the localization methods of [23] together with an analysis of the fundamental solution of the quantum differential equation of the Hilbert scheme of points of the plane.

The descendent partition functions for the stable pairs theory of local curves have very restricted denominators when considered as rational functions in q with coefficients in $\mathbb{Q}(s_1, s_2)$. A basic result proven in Section 9 of [20] is the following.

Theorem. *The denominators of the degree d descendent partition functions* $\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j=1}^k \tau_{i_j}(\mathsf{p})\right)^T$ *are products of factors of the form* q^s *and*

$$1-(-q)^r$$

for $1 \le r \le d$.

Certainly the calculation of Theorem 3 is consistent with the denominator result.

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about the $q \leftrightarrow q^{-1}$ symmetry for descendents prompted us to work out the proof of the functional equation.

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1. Reduction for stationary descendents

1.1. **Cap geometry.** The capped 1-leg geometry concerns the trivial bundle,

$$N = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathbb{P}^1 \ ,$$

relative to the fiber

$$N_\infty \subset N$$

over $\infty \in \mathbb{P}^1$. The total space N naturally carries an action of a 3-dimensional torus

$$\mathbf{T} = T \times \mathbb{C}^* \ .$$

Here, T acts as before by scaling the factors of N and preserving the relative divisor N_∞. The $\mathbb{C}^*$-action on the base $\mathbb{P}^1$ which fixes the points $0, \infty \in \mathbb{P}^1$ lifts to an additional $\mathbb{C}^*$-action on N fixing N_∞.

The equivariant cohomology ring $H^*_{\mathbf{T}}(\bullet)$ is generated by the Chern classes s_1, s_2, and s_3 of the standard representation of the three $\mathbb{C}^*$-factors. At the $\mathbf{T}$-fixed point of N over $0 \in \mathbb{P}^1$ the tangent weights are specified as follows

- (i) tangent weights of $-s_1$ and $-s_2$ along the fiber directions for the action of T,
- (ii) tangent weight $-s_3$ along $\mathbb{P}^1$ for the action on $\mathbb{C}^*$.

For the $\mathbf{T}$-fixed point of N over $\infty \in \mathbb{P}^1$, the weights are $-s_1, -s_2, s_3$. We define

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^{\mathbf{T}} = \sum_{n\in\mathbb{Z}} q^n \int_{[P_n(N/N_\infty,d)]^{vir}} \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j) \ \cup \epsilon^*_\infty(\mathsf{C}_\eta), \tag{3}$$

by $\mathbf{T}$-equivariant residues where $\gamma_j \in H^*_{\mathbf{T}}(\mathbb{P}^1, \mathbb{Z})$.

1.2. **Reduction for the cap.** Consider the following partition function for the cap

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_k([0]) \cdot \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^{\mathbf{T}} , \tag{4}$$

where $\gamma_j \in H^*_{\mathbf{T}}(\mathbb{P}^1, \mathbb{Z})$.

The $\mathbf{T}$-equivariant localization formula for (4) has two sides. The contribution over $0 \in \mathbb{P}^1$ yields the descendent vertex $\mathsf{W}_\mu^{\mathsf{Vert}}$ of Section 2.6 of [20]. We will follow here exactly the terminology of the $\mathbf{T}$-fixed point analysis of Sections 2.1-2.7 of [20]. The contribution over $\infty \in \mathbb{P}^1$ yields rubber integrals discussed in Section 3.3 of [20]. While only the descendent vertex is required for the proof of Theorem 1, the rubber theory plays an essential role in the proof of Theorem 3.

Let Q_U determine a $\mathbf{T}$-fixed point of the moduli space of stable pairs on the affine chart associated to $0 \in \mathbb{P}^1$. For each $x_1^a x_2^b \in \mu[x_1, x_2]$, let $c_{a,b}$ be the *largest* integer satisfying

$$x_1^a x_2^b x_3^{-c_{a,b}} \in Q_U \ .$$

The length of Q_U is the sum of the $c_{a,b}$,

$$\ell(Q_U) = \sum_{(a,b)\in\mu} c_{a,b} \ .$$

The Laurent polynomial

$$\mathsf{F}_U = \frac{1}{1-t_3} \sum_{(a,b)\in\mu} t_1^a t_2^b t_3^{-c_{a,b}} \tag{5}$$

plays a basic role.

In the formula in Section 2.6 of [20] for the descendent vertex $\mathsf{W}_\mu^{\mathsf{Vert}}(\tau_k([0]))$, the descendent[2] $\tau_k([0])$ enters via

$$\frac{1}{s_1 s_2}\mathrm{ch}_{2+k}\big(\mathsf{F}_U \cdot (1-t_1)(1-t_2)(1-t_3)\big) =$$
$$\frac{1}{s_1 s_2}\mathrm{ch}_{2+k}\left((1-t_1)(1-t_2)\sum_{(a,b)\in\mu} t_1^a t_2^b t_3^{-c_{a,b}}\right) =$$
$$\frac{1}{s_1 s_2}\mathrm{Coeff}_{z^{2+k}}\left((1-e^{zs_1})(1-e^{zs_2})\sum_{(a,b)\in\mu} e^{z(as_1+bs_2-c_{a,b}s_3)}\right) \ .$$

The third line exhibits the action of the descendent on Q_U as a symmetric function of the $d = |\mu|$ variables

$$\{\ as_1 + bs_2 - c_{a,b}s_3 \mid (a,b) \in \mu\ \} \tag{6}$$

with coefficients in $\mathbb{Q}[s_1, s_2]$.

In fact, the descendent $\tau_k([0])$ is a symmetric function of degree k in the variables (6). The symmetric function is inhomogeneous with degree k part equal to $\frac{\mathfrak{p}_k}{k!}$ where $\mathfrak{p}_k$ is the power sum. Since the ring of

[2]Here, the class $[0]$ is the pull-back to N of the fixed point $0 \in \mathbb{P}^1$.

symmetric functions in d variables is generated by $\mathfrak{p}_1, \ldots, \mathfrak{p}_d$, we obtain *universal reduction rules*.

Let $\mathfrak{t}_k$ be the symmetric function in d variables with coefficients in $\mathbb{Q}[s_1, s_2]$ defined by

$$\sum_{k=0}^{\infty} \mathfrak{t}_k z^{k+2} = \frac{1}{s_1 s_2}(1 - e^{zs_1})(1 - e^{zs_2}) \sum_{n=0}^{\infty} \mathfrak{p}_n \frac{z^n}{n!} \ .$$

For $k > d$, there are unique polynomials $f_{k,d}$ with coefficients in $\mathbb{Q}(s_1, s_2)$ satisfying

$$\mathfrak{t}_k = f_{k,d}(\mathfrak{t}_1, \ldots, \mathfrak{t}_d) \ . \tag{7}$$

We have proven the following result.

Proposition 1. *In the degree d theory of the $\mathbf{T}$-equivariant cap, the reduction rule*

$$\tau_k([0]) \mapsto f_{k,d}(\tau_1([0]), \ldots, \tau_d([0]))$$

holds universally when $k > d$.

Explicitly, Proposition 1 yields the following equality for $\mathbf{T}$-equivariant integrals:

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_k([0]) \cdot \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^{\mathbf{T}} = \mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(f_{k,d}(\tau_1(\mathsf{p}), \ldots, \tau_d(\mathsf{p})) \cdot \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^{\mathbf{T}}$$

for $k > d$. Of course, Proposition 1 implies the same result for the T-equivariant theory of the cap.

1.3. **Proof of Theorem 1.** Consider the partition function for the relative geometry N/S over a curve C,

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r}\left(\tau_k(\mathsf{p}) \cdot \prod_{j=1}^{\ell} \tau_{i_j}(\gamma_j)\right)^{T}, \qquad \gamma_j \in H^*(C, \mathbb{Z}) \ .$$

Since the insertion $\tau_k(\mathsf{p})$ may be degenerated to lie on a cap, Proposition 1 implies Theorem 1. □

1.4. **Parity considerations.** We will need the following property of the reduction polynomials $f_{k,d}$ to obtain the functional equation of Theorem 2.

Lemma 1. *For every $k > d > 0$, the reduction polynomial*

$$f_{k,d} \in \mathbb{Q}(s_1, s_2)[x_1, \ldots, x_d]$$

lies in the span of the monomials of the form $x_1^{\sigma_1} \cdots x_d^{\sigma_d}$ *where*

$$\sum_{i=1}^{d} i\sigma_i \equiv k \mod 2 \ .$$

Proof. Using the homogeneity of t_i, we see from (7) the coefficient of $x_1^{\sigma_1} \cdots x_d^{\sigma_d}$ in $f_{k,d}$ is homogeneous as a rational function in s_1 and s_2. Moreover the degree of the coefficient is congruent mod 2 to $k - \sum_{i=1}^{d} i\sigma_i$. We need only show that these degrees are all even.

We write the descendent $\tau_k([0])$ as a symmetric function in the adjusted variables

$$\{ \ as_1 + bs_2 - c_{a,b}s_3 + \frac{s_1+s_2}{2} \ | \ (a,b) \in \mu \ \}.$$

If we let $\mathfrak{p}'_k$ denote the kth power sum of these d variables, then we have

$$\sum_{k=0}^{\infty} \mathsf{t}_k z^{k+2} = \frac{1}{s_1 s_2}(e^{zs_1/2} - e^{-zs_1/2})(e^{zs_2/2} - e^{-zs_2/2}) \sum_{n=0}^{\infty} \mathfrak{p}'_n \frac{z^n}{n!} \ ,$$

where t_k is as in the proof of Proposition 1. Since

$$(e^{zs_1/2} - e^{-zs_1/2})(e^{zs_2/2} - e^{-zs_2/2})$$

is an even function of s_1 and s_2, the coefficients of the monomial of $f_{k,d}$ must have even degree. □

2. Factorization and rationality

2.1. Dependence upon the cap. Consider the stationary series

$$\mathsf{Z}^{N/S}_{d,\eta^1,\ldots,\eta^r} \left(\prod_{j=1}^{\ell} \tau_{i_j}(\mathsf{p}) \right)^T \ . \tag{8}$$

If $\ell = 0$, then no descendents appear and the rationality of the partition function (8) has been proven in [15, 18]. If $\ell > 0$, each stationary descendent $\tau_i(\mathsf{p})$ can be degenerated to a distinct cap. Hence, the series (8) is determined by:

- the stable pairs theory of local curves (without insertions),
- the 1-pointed caps $\mathsf{Z}^{\mathsf{cap}}_{d,\eta} \left(\tau_k(\mathsf{p})\right)^T$.

In fact, we can do much better by using Theorem 1.

2.2. **Factorization I.** If $k > d$, then we have

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\Big(\tau_k(\mathsf{p})\Big)^T = \mathsf{Z}^{\mathsf{cap}}_{d,\eta}\Big(f_{k,d}(\tau_1(\mathsf{p}),\ldots,\tau_d(\mathsf{p}))\Big)^T \tag{9}$$

by Theorem 1. After expanding $f_{k,d}(\tau_1(\mathsf{p}),\ldots,\tau_d(\mathsf{p}))$ and degenerating each stationary descendent $\tau_i(\mathsf{p})$ to a distinct cap, we find the series (9) is determined by:

- the stable pairs theory of local curves (without insertions),
- the 1-pointed caps $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_{k\le d}(\mathsf{p})\right)^T$.

2.3. **Factorization II.** We can further restrict the descendents $\tau_k(\mathsf{p})$ which occur on the caps by geometrically factoring the parts of the relative condition η.

Proposition 2. *The series* $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_{k\le d}(\mathsf{p})\right)^T$ *are determined by*

- *the stable pairs theory of local curves (without insertions),*
- *the 1-pointed caps* $\mathsf{Z}^{\mathsf{cap}}_{c,(c)}\left(\tau_c(\mathsf{p})\right)^T$ *for* $1 \le c \le d$.

Proof. We proceed by induction on d. If $d = 1$, there is nothing to prove. Assume Proposition 2 holds for all degrees less than d and consider

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_k(\mathsf{p})\right)^T \ .$$

There are two main cases.

Case $k < d$.

We consider the geometry of $\mathbb{P}^2 \times \mathbb{P}^1$ relative to the fiber

$$\mathbb{P}^2_\infty = \mathbb{P}^2 \times \{\infty\} \subset \mathbb{P}^2 \times \mathbb{P}^1 \ .$$

Let $\beta \in H_2(\mathbb{P}^2 \times \mathbb{P}^1, \mathbb{Z})$ be the class of the section $\mathbb{P}^1$ contracted over $\mathbb{P}^2$. The 2-dimensional torus T acts on $\mathbb{P}^2$ with fixed points $\xi_0, \xi_1, \xi_2 \in \mathbb{P}^2$. The tangent weights can be chosen as follows:

$$-s_1, -s_2 \ \text{ for } \xi_0, \quad s_1, s_1 - s_2 \ \text{ for } \xi_1, \quad s_2 - s_1, s_2 \ \text{ for } \xi_2 \ .$$

Let the partition η have parts $\eta_1, \ldots, \eta_\ell$. Let $\widetilde{\eta}$ be the cohomology weighted partition with η_1 of weight $[\xi_0] \in H^*(\mathbb{P}^2, \mathbb{Z})$ and all of the other parts assigned weight $1 \in H^*_T(\mathbb{P}^2, \mathbb{Z})$. The series

$$\mathsf{Z}^{\mathbb{P}^2\times\mathbb{P}^1/\mathbb{P}^2_\infty}_{d\beta,\widetilde{\eta}}\left(\tau_k([0])\right)^T \in \mathbb{Q}[s_1, s_2][[q]] \tag{10}$$

is well-defined.

The virtual dimension of the moduli space $P_n(\mathbb{P}^2 \times \mathbb{P}^1/\mathbb{P}^2_\infty, d\beta)$ after the imposition of the boundary condition $\widetilde{\eta}$ is

$$2d - 2 - \sum_{i=1}^{\ell}(\eta_i - 1) = d + \ell - 2 \geq d - 1 \ .$$

The dimension of the integrand $\tau_k([0])$ is $k < d$. Hence, the integrals

$$\langle \tau_k([0]) \rangle^{\mathbb{P}^2 \times \mathbb{P}^1, \widetilde{\eta}}_{n,d\beta} = \int_{[P_n(\mathbb{P}^2 \times \mathbb{P}^1/\mathbb{P}^2_\infty, d\beta)]^{vir}} \tau_k([0]) \cup \epsilon^*(\mathsf{C}_{\widetilde{\eta}})$$

arising as coefficients of (10) have degree at most 0 in $\mathbb{Q}[s_1, s_2]$. If the degree is negative, then the series (10) vanishes.

The degree of (10) is 0 only when $k = d-1$ and $\eta = (d)$. The moduli space then lies entirely in

$$\mathbb{C}^2 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$$

where $\mathbb{C}^2 \subset \mathbb{P}^2$ is the T-invariant affine containing ξ_0 (corresponding to the cohomology weight $[\xi_0]$ on the part d). By the basic divisibility results of [15, 18], the linear factor $s_1 + s_2$ must divide the q^n coefficient of (10) for all $n > d$. Since the invariant is of degree 0, the divisibility by $s_1 + s_2$ is impossible unless all such coefficients vanish. Since the leading term of (10) is q^d, we conclude (10) is a monomial in q.

If $k < d$, we have calculated the series (10). Direct calculation of (10) by T-equivariant localization yields a single term equal to

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_k(\mathsf{p})\right)^T \tag{11}$$

up to an $s_1 s_2$ factor. The T-equivariant localization formula for the relative geometry $\mathbb{P}^2 \times \mathbb{P}^1/\mathbb{P}^2_\infty$ in the class $d\beta$ distributes the parts of $\widetilde{\eta}$ among the T-fixed points

$$\xi_0, \xi_1, \xi_2 \in \mathbb{P}^2 \ .$$

The term equal to (11) arises when all parts are distributed to ξ_0. Since the first part of $\widetilde{\eta}$ must be distributed to ξ_0, the remaining terms are known by the induction hypothesis. Hence, we have calculated (11).

Case $\ell > 1$.

The dimension estimates as above show the series

$$\mathsf{Z}^{\mathbb{P}^2 \times \mathbb{P}^1/\mathbb{P}^2_\infty}_{d\beta,\widetilde{\eta}}\left(\tau_k([0])\right)^T \in \mathbb{Q}[s_1, s_2][[q]] \tag{12}$$

is degree at most 0 in s_1 and s_2. The series (12) must vanish in the negative degree case.

The degree of (12) is 0 only when $k = d$ and $\eta = (d_1, d_2)$. In the degree 0 case, the invariant (12) is independent of s_1 and s_2, so we may calculate (12) in the specialization $s_1 + s_2 = 0$. In the T-equivariant localization of (12), the terms at ξ_0 all have vanishing coefficients of $q^{n>d}$ in the specialization $s_1 + s_2 = 0$. The terms away from ξ_0 are known inductively. Hence, (12) is determined.

If $\ell > 1$, we have calculated the series (12). As before, the T-equivariant localization formula for (12) yields a single term equal to $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_k(\mathsf{p})\right)^T$ up to an $s_1 s_2$ factor. The remaining terms are known by the induction hypothesis. We have calculated $\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\tau_k(\mathsf{p})\right)^T$.

The only possibility not covered by the two above cases is the 1-pointed cap

$$\mathsf{Z}^{\mathsf{cap}}_{d,(d)}\left(\tau_d(\mathsf{p})\right)^T \in \mathbb{Q}[s_1, s_2][[q]] \ . \tag{13}$$

The factorization methods do not inductively determine (13). □

2.4. **Proof of Theorem 2.** The methods of Sections 2.2-2.3 provide an effective algorithm for calculating an arbitrary degree d stationary series (8) in terms of

- the stable pairs theory of local curves (without insertions),
- the 1-pointed caps $\mathsf{Z}^{\mathsf{cap}}_{c,(c)}\left(\tau_c(\mathsf{p})\right)^T$ for $1 \leq c \leq d$.

The partition functions of the stable pairs theory of local curves (without insertions) are rational and satisfy the functional equation of Theorem 2, see Theorems 2 and 3 of [18]. The steps in the effective algorithm preserve the functional equation. For the Factorization I step, Lemma 1 is needed to ensure that the total weight of the descendent insertions does not change parity. Theorem 2 then follows from Theorem 3 proven in Section 4 below together with the observation that the rational functions appearing there satisfy the functional equation. □

3. Localization formalism

3.1. **Formula.** The $\mathbf{T}$-equivariant localization formula for the capped 1-leg descendent vertex is the following:

$$\mathsf{Z}^{\mathsf{cap}}_{d,\eta}\left(\prod_{i=1}^{k}\tau_{i_j}([0])\right)^{\mathbf{T}} = \sum_{|\mu|=d}\mathsf{W}^{\mathsf{Vert}}_{\mu}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)\cdot \mathsf{W}^{(0,0)}_{\mu}\cdot \mathsf{S}^{\mu}_{\eta} \ . \tag{14}$$

The result is a consequence of [6] applied to stable pairs theory of the cap [23] — see Section 3.4 of [20]. The form is the same as the Donaldson-Thomas localization formulas used in [11, 18].

The right side of localization formula is expressed in term of three parts of different geometric origins:

- the vertex term $\mathsf{W}^{\mathsf{Vert}}_{\mu}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)$ over $0 \in \mathbb{P}^1$,
- the edge term $\mathsf{W}^{(0,0)}_{\mu}$,
- the rubber integrals S^{μ}_{η} over $\infty \in \mathbb{P}^1$.

The vertex term has been explained (for $i = 1$) already in Section 1.2. The edge term $\mathsf{W}^{(0,0)}_{\mu}$ is simply the inverse product of the tangent weights of the Hilbert scheme of points of $\mathbb{C}^2$ at the T-fixed point corresponding to the partition μ. We review the rubber integrals here.

3.2. **Rubber theory.** The stable pairs theory of *rubber*[3] naturally arises at the boundary of $P_n(N/N_\infty, d)$. Let R be a rank 2 bundle of level $(0,0)$ over $\mathbb{P}^1$. Let

$$R_0, R_\infty \subset R$$

denote the fibers over $0, \infty \in \mathbb{P}^1$. The 1-dimensional torus $\mathbb{C}^*$ acts on R via the symmetries of $\mathbb{P}^1$. Let $P_n(R/R_0 \cup R_\infty, d)$ be the relative moduli space of ideal sheaves, and let

$$P_n(R/R_0 \cup R_\infty, d)^\circ \subset P_n(R/R_0 \cup R_\infty, d)$$

denote the open set with finite stabilizers for the $\mathbb{C}^*$-action and *no* destabilization over $\infty \in \mathbb{P}^1$. The rubber moduli space,

$$P_n(R/R_0 \cup R_\infty, d)^\sim = P_n(R/R_0 \cup R_\infty, d)^\circ/\mathbb{C}^*,$$

denoted by a superscripted tilde, is determined by the (stack) quotient. The moduli space is empty unless $n > d$. The rubber theory of R is defined by integration against the rubber virtual class,

$$[P_n(R/R_0 \cup R_\infty, d)^\sim]^{vir}.$$

All of the above rubber constructions are T-equivariant for the scaling action on the fibers of R with weights s_1 and s_2.

The rubber moduli space $P_n(R/R_0 \cup R_\infty, d)^\sim$ carries a cotangent line at the dynamical point $0 \in \mathbb{P}^1$. Let

$$\psi_0 \in A^1_T(P_n(R/R_0 \cup R_\infty, d)^\sim, \mathbb{Q})$$

denote the associated cotangent line class. Let

$$\mathsf{P}_\mu \in A^{2d}_T(\mathrm{Hilb}(\mathbb{C}^2, d), \mathbb{Z})$$

be the class corresponding to the T-fixed point determined by the monomial ideal $\mu[x_1, x_2] \subset \mathbb{C}[x_1, x_2]$.

[3]We follow the terminology and conventions of the parallel rubber discussion for the local Donaldson-Thomas theory of curves treated in [18].

In the localization formula for the cap, special rubber integrals with relative conditions P_μ over 0 and C_η (in the Nakajima basis) over ∞ arise. Let

$$\mathsf{S}^\mu_\eta = \sum_{n\geq d} q^n \left\langle \mathsf{P}_\mu \ \middle| \ \frac{1}{s_3-\psi_0} \ \middle| \ \mathsf{C}_\eta \right\rangle^\sim_{n,d} \in \mathbb{Q}(s_1,s_2,s_3)((q)) \ .$$

The bracket on the right is the rubber integral defined by T-equivariant residues. If $n=d$, the rubber moduli space in undefined — the bracket is then taken to be the T-equivariant intersection pairing between the classes P_μ and C_η in $\mathrm{Hilb}(\mathbb{C}^2,d)$.

4. Calculation of $\mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))^{\mathbf{T}}$

4.1. **Dimension.** The notation $(d[0])$ will be used to assign the weight $[0]\in A^*_T(\mathbb{C}^2,\mathbb{Q})$ to the part d. Since

$$[0]=s_1s_2\in A^*_T(\mathbb{C}^2,\mathbb{Q}),$$

we see[4]

$$\mathsf{Z}^{\mathsf{cap}}_{d,(d)}\left(\tau_d(\mathsf{p})\right)^{\mathbf{T}} = \left(\frac{1}{s_1s_2}\right)\mathsf{Z}^{\mathsf{cap}}_{d,(d[0])}\left(\tau_d(\mathsf{p})\right)^{\mathbf{T}}$$

After imposing the boundary condition $(d[0])$, the moduli space

$$P_n(\mathbb{P}^2\times\mathbb{P}^1/\mathbb{P}^2_\infty, d\beta)$$

is compact of virtual dimension $d-1$.

The moduli space $P_n(\mathbb{P}^2\times\mathbb{P}^1/\mathbb{P}^2_\infty, d\beta)$ is empty for $n<d$ and isomorphic to $\mathrm{Hilb}(\mathbb{C}^2,d)$ for $n=d$. Hence, the leading term of the series $\mathsf{Z}^{\mathsf{cap}}_{d,(d[0])}\left(\tau_d(\mathsf{p})\right)^{\mathbf{T}}$ is the classical pairing

$$q^d\Big\langle \tau_d, \mathsf{C}_{(d[0])}\Big\rangle_{\mathrm{Hilb}(\mathbb{C}^2,d)} \in \mathbb{Q}[s_1,s_2] \ . \tag{15}$$

The class τ_d is defined as follows. Let $\mathbb{F}_0$ be the universal quotient sheaf on $\mathrm{Hilb}(\mathbb{C}^2,d)\times\mathbb{C}^2$. Then,

$$\tau_d = \pi_*\Big(\mathrm{ch}_{2+d}(\mathbb{F}_0)\Big) \in A^d_T(\mathrm{Hilb}(\mathbb{C}^2,d)) \tag{16}$$

where π is the projection

$$\pi : \mathrm{Hilb}(\mathbb{C}^2,d)\times\mathbb{C}^2 \to \mathrm{Hilb}(\mathbb{C}^2,d) \ .$$

Lemma 2. $\mathsf{Z}^{\mathsf{cap}}_{d,(d[0])}\left(\tau_d(\mathsf{p})\right)^{\mathbf{T}} = (s_1+s_2)\cdot q^d F(d)(q)$ *for* $F(d)\in\mathbb{Q}[[q]]$.

[4]We will consider descendents here equivariant with respect to the 3-torus $\mathbf{T}$ of Section 1.

Proof. By compactness of the underlying moduli spaces of pairs, we see the series $\mathsf{Z}^{\mathsf{cap}}_{d,(d[0])}(\tau_d(\mathsf{p}))^{\mathbf{T}}$ must lie in $\mathbb{Q}[s_1,s_2,s_3][[q]]$. The leading q^d coefficient certainly has no s_3 dependence by (15). By dimension considerations, the leading q^d coefficient must be linear and thus, by symmetry, a multiple of s_1+s_2. For the coefficient of $q^{n>d}$, divisibility by s_1+s_2 is obtained from [15, 18]. □

4.2. **Localization.** We wish to compute the series

$$F(d) = \frac{s_1s_2}{s_1+s_2} q^{-d}\ \mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d([0]))^{\mathbf{T}} \in \mathbb{Q}[[q]] \subset \mathbb{Q}(s_1,s_2,s_3)[[q]]$$

introduced in Lemma 2. Via the localization formula (14), we have

$$F(d) = \frac{s_1s_2}{s_1+s_2} q^{-d} \sum_{|\mu|=d} \mathsf{W}^{\mathsf{Vert}}_{\mu}(\tau_d([0])) \cdot \mathsf{W}^{(0,0)}_{\mu} \cdot \mathsf{S}^{\mu}_{(d)}\ .$$

We will separate the classical terms occuring on the right side.

By definition, the classical term of $\mathsf{W}^{\mathsf{Vert}}_{\mu}(\tau_d([0])$ is the leading q^d term. Let

$$\mathsf{F}_\mu = \sum_{(a,b)\in\mu} t_1^a t_2^b.$$

We write the vertex as

$$\mathsf{W}^{\mathsf{Vert}}_{\mu}(\tau_d([0])) = \frac{q^d}{s_1s_2}\ \mathrm{ch}_{d+2}\big(\mathsf{F}_\mu\cdot(1-t_1)(1-t_2)\big) + \widehat{\mathsf{W}}^{\mathsf{Vert}}_{\mu}(\tau_d([0]))$$

where $\widehat{\mathsf{W}}^{\mathsf{Vert}}_{\mu}(\tau_d([0]))$ represents all the higher order terms in q. Similarly, we write

$$\mathsf{S}^{\mu}_{(d)} = \langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\rangle + \widehat{\mathsf{S}}^{\mu}_{(d)}$$

where the leading term $\langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\rangle$ is the T-equivariant pairing on $\mathrm{Hilb}(\mathbb{C}^2,d)$.

Using the above formulas with the leading classical terms, we rewrite the result of the localization formula as

$$\begin{aligned} F(d) \quad = \quad & \sum_{|\mu|=d} \frac{1}{s_1+s_2}\mathrm{ch}_{d+2}(\mathsf{F}_\mu\cdot(1-t_1)(1-t_2))\cdot \mathsf{W}^{(0,0)}_{\mu}\cdot\langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\rangle \\ & + \sum_{|\mu|=d} \frac{s_1s_2}{s_1+s_2} q^{-d}\ \widehat{\mathsf{W}}^{\mathsf{Vert}}_{\mu}(\tau_d([0]))\cdot \mathsf{W}^{(0,0)}_{\mu}\cdot\langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\rangle \\ & + \sum_{|\mu|=d} \frac{1}{s_1+s_2}\mathrm{ch}_{d+2}(\mathsf{F}_\mu\cdot(1-t_1)(1-t_2))\cdot \mathsf{W}^{(0,0)}_{\mu}\cdot \widehat{\mathsf{S}}^{\mu}_{(d)} \\ & + \sum_{|\mu|=d} \frac{s_1s_2}{s_1+s_2} q^{-d}\ \widehat{\mathsf{W}}^{\mathsf{Vert}}_{\mu}(\tau_d([0]))\cdot \mathsf{W}^{(0,0)}_{\mu}\cdot \widehat{\mathsf{S}}^{\mu}_{(d)}. \end{aligned}$$

The first line on the right is the classical pairing

$$F_0(d) = \frac{1}{s_1+s_2}\Big\langle \tau_d, \mathsf{C}_{(d[0])}\Big\rangle \in \mathbb{Q}$$

which we will compute in Proposition 3 below. We will compute the difference

$$\widehat{F}(d) = F(d) - F_0(d)$$

by evaluating each of the other three terms at $s_2 = -s_1$, expanding as a Laurent series in $\frac{s_3}{s_1}$, and taking the constant term.

Both $\widehat{\mathsf{W}}^{\mathsf{Vert}}_\mu$ and $\widehat{\mathsf{S}}^\mu_{(d)}$ are divisible by s_1+s_2. Therefore, the fourth term in the formula for $F(d)$ vanishes after the substitution $s_2 = -s_1$. Only two terms,

$$\widehat{F}(d) = \sum_{|\mu|=d}\left(\frac{s_1s_2}{s_1+s_2}q^{-d}\ \widehat{\mathsf{W}}^{\mathsf{Vert}}_\mu(\tau_d([0]))\cdot \mathsf{W}^{(0,0)}_\mu \cdot \langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\rangle\right)\Big|_{s_2=-s_1}$$
$$+\sum_{|\mu|=d}\left(\frac{1}{s_1+s_2}\,\mathrm{ch}_{d+2}(\mathsf{F}_\mu\cdot(1-t_1)(1-t_2))\cdot \mathsf{W}^{(0,0)}_\mu\cdot \widehat{\mathsf{S}}^\mu_{(d)}\right)\Big|_{s_2=-s_1},$$

remain.

We evaluate the two above terms separately. The first requires detailed knowledge of the vertex factor

$$\frac{s_1s_2}{s_1+s_2}q^{-d}\ \widehat{\mathsf{W}}^{\mathsf{Vert}}_\mu(\tau_d([0]))\Big|_{s_2=-s_1}$$

and is evaluated in Section 4.3. The second requires detailed knowledge of the rubber factor

$$\frac{1}{s_1+s_2}\widehat{\mathsf{S}}^\mu_{(d)}\Big|_{s_2=-s_1}$$

and is evaluated in Section 4.4.

4.3. **Vertex calculation.** We begin with the first term

$$\sum_{|\mu|=d}\left(\frac{s_1s_2}{s_1+s_2}q^{-d}\ \widehat{\mathsf{W}}^{\mathsf{Vert}}_\mu(\tau_d([0]))\cdot \mathsf{W}^{(0,0)}_\mu \cdot \langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\rangle\right)\Big|_{s_2=-s_1} \tag{17}$$

of $\widehat{F}(d)$. The pairing $\langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\rangle$ has a simple expression mod s_1+s_2,

$$\Big\langle \mathsf{P}_\mu, \mathsf{C}_{(d)}\Big\rangle\Big|_{s_2=-s_1} = \frac{(-1)^{d-1}(d-1)!}{\dim\mu}\chi^\mu\big((d)\big)s_1^{d-1}. \tag{18}$$

Here, $\dim\mu$ is the dimension of the irreducible representation of the symmetric group Σ_d corresponding to the partition μ, and χ^μ is the associated character. The proof of (18) is obtained directly from the

Jack polynomial expression for the T-fixed points of $\mathrm{Hilb}(\mathbb{C}^2, d)$, see Section 3.7 of [17].[5]

The character χ^μ vanishes on a d-cycle unless μ is of the following simple form

$$\alpha_a = (a+1, 1, \ldots, 1)$$

for $0 \leq a \leq d-1$. We have

$$\chi^{\alpha_a}\big((d)\big) = (-1)^{d-1-a} .$$

We will restrict to the case $\mu = \alpha_a$ and replace the sum over μ with a sum over a. The dimension formula

$$\dim \alpha_a = \binom{d-1}{a}$$

holds. The constant

$$b = d - 1 - a$$

will occur often below.

The edge factor $\mathsf{W}^{(0,0)}_\mu$ is also easy to compute after the evaluation $s_2 = -s_1$:

$$\mathsf{W}^{(0,0)}_\mu\Big|_{s_2=-s_1} = \frac{(-1)^d(\dim \mu)^2}{(d!)^2} s_1^{-2d}. \tag{19}$$

The dimension of μ here enters via the hook length formula.

The most complicated part of the calculation is the vertex factor $\widehat{\mathsf{W}}^{\mathsf{Vert}}_\mu(\tau_d([0]))$ for $\mu = \alpha_a$. From Section 2.6 of [20],

$$\widehat{\mathsf{W}}^{\mathsf{Vert}}_\mu(\tau_d([0])) = \sum_{Q_U:\ \ l(Q_U)>0} \frac{q^{d+l(Q_U)}}{s_1 s_2} \mathrm{ch}_{d+2}\big(\mathsf{F}_U \cdot (1-t_1)(1-t_2)(1-t_3)\big) \cdot e(-\mathsf{V}_U),$$

where the sum runs over $\mathbf{T}$-fixed loci Q_U of positive length. Recall, the the $\mathbf{T}$-fixed loci correspond to box configurations defined by height functions $c_{a,b}$ on the partition μ determining F_U by formula (5). The term V_U is expressed in terms of F_U in Section 2.5 of [20].

In the case $\mu = \alpha_a$, a straightforward calculation shows the vertex weight $e(-\mathsf{V}_U)$ is divisible by $(s_1+s_2)^2$ unless the box configuration is a cylinder (of height $h > 0$) under a rim hook η of μ.[6] We break the

[5] Our variable conventions here differ slightly from [17]. Specifically, our s_i correspond to $-t_i$ in [17].

[6] The divisibility statement is actually true for any μ.

sum into terms by the size r of η. When $r = d$, the only possibility for the rim hook is $\eta = \mu$. The corresponding vertex weight is

$$\frac{1}{s_1+s_2} e(-\mathsf{V}_U)\Big|_{s_2=-s_1} = \frac{(-1)^{dh+1}}{hs_3}\left(1+\frac{hs_3}{s_1}\sum_{\substack{i=-b\\ i\neq 0}}^{a}\frac{1}{i}+\ldots\right).$$

Here and below, the dots on the right stand for terms of order 2 and higher in $\frac{s_3}{s_1}$. For each $r < d$, there are at most two such rim hooks, depending on whether $a \geq r$ and whether $d-1-a \geq r$. For $a \geq r$, we find

$$\frac{1}{s_1+s_2} e(-\mathsf{V}_U)\Big|_{s_2=-s_1} = \frac{(-1)^{rh+1}}{hs_3}\left(1+\frac{hs_3}{s_1}\left(\frac{1}{d}-\frac{1}{d-r}-\frac{1}{r}+\sum_{i=a-r+1}^{a}\frac{1}{i}\right)+\ldots\right).$$

For $d-1-a \geq r$, the answer is obtained by symmetry by interchanging s_1 and s_2. The symmetry propagates through the entire calculation of (17). After setting $s_2 = -s_1$, we will take the constant term of the Laurent expansion in

$$\frac{s_3}{s_1} = -\frac{s_3}{s_2}\ .$$

Hence, we can treat the symmetry as exact.

After putting all the terms together and inserting the descendent factors, we obtain for

$$\frac{s_1 s_2}{s_1+s_2} q^{-d}\mathsf{W}_\mu^{\mathsf{Vert}+}(\tau_d([0]))\Big|_{s_2=-s_1}$$

the following formula:

$$\sum_{h=1}^{\infty}\frac{(-1)^{dh+1}q^{dh}}{hs_3}\left(1+\frac{hs_3}{s_1}\sum_{\substack{i=-b\\ i\neq 0}}^{a}\frac{1}{i}+\ldots\right) \cdot \mathrm{ch}_{d+2}\left(-t_1^{a+1}t_3^{-h}+t_1^{a}t_3^{-h}+t_1^{-b}t_3^{-h}-t_1^{-b-1}t_3^{-h}\right)$$

$$+2\sum_{r=1}^{a}\sum_{h=1}^{\infty}\frac{(-1)^{rh+1}q^{rh}}{hs_3}\left(1+\frac{hs_3}{s_1}\left(\frac{1}{d}-\frac{1}{d-r}-\frac{1}{r}+\sum_{i=a-r+1}^{a}\frac{1}{i}\right)+\ldots\right) \cdot \mathrm{ch}_{d+2}\left(-t_1^{a+1}t_3^{-h}+t_1^{a}t_3^{-h}+t_1^{a-r+1}t_3^{-h}-t_1^{a-r}t_3^{-h}\right)$$

Here, we have included the symmetry discussed above.

After combining all of the parts of (17), summing over $\mu = \alpha_a$, and taking the constant term when expanded as a Laurent series in $\frac{s_3}{s_1}$, we obtain an expression of the form

$$\sum_{r=1}^{d} A_r \frac{(-q)^r}{1-(-q)^r} \ .$$

The explicit formulas for A_r depend upon two cases. For A_d, we have

$$(-1)^{d-1} d \cdot d!(d+2)!\ A_d =$$
$$\sum_{a+b=d-1} (-1)^a \binom{d-1}{a} \cdot \sum_{\substack{i=-b\\ i\neq 0}}^{a} \frac{1}{i} \left(-(-b-1)^{d+2} + (-b)^{d+2} + a^{d+2} - (a+1)^{d+2}\right)$$
$$-(d+2) \sum_{a+b=d-1} (-1)^a \binom{d-1}{a} \left(-(-b-1)^{d+1} + (-b)^{d+1} + a^{d+1} - (a+1)^{d+1}\right),$$

and for $r < d$, we have

$$\frac{(-1)^{d-1}}{2} d \cdot d!(d+2)!\ A_r =$$
$$\sum_{\substack{a+b=d-1\\ a\geq r}} (-1)^a \binom{d-1}{a} \left(\frac{1}{d} - \frac{1}{d-r} - \frac{1}{r} + \sum_{i=a-r+1}^{a} \frac{1}{i}\right)$$
$$\cdot \left(-(-b-1)^{d+2} + (-b)^{d+2} + a^{d+2} - (a+1)^{d+2}\right)$$
$$-(d+2) \sum_{a=r}^{d-1} (-1)^a \binom{d-1}{a} \left(-(a-r)^{d+1} + (a-r+1)^{d+1} + a^{d+1} - (a+1)^{d+1}\right).$$

While the above formulas for A_d and $A_{r<d}$ look unpleasantly complicated, a remarkable cancellation will occur in Section 4.6.

4.4. **Rubber calculation.** Evaluating the second term

$$\sum_{|\mu|=d} \left(\frac{1}{s_1+s_2} \mathrm{ch}_{d+2}(\mathsf{F}_\mu \cdot (1-t_1)(1-t_2)) \cdot \mathsf{W}_\mu^{(0,0)} \cdot \widehat{\mathsf{S}}^{\mu}_{(d)}\right)\Big|_{s_2=-s_1} \tag{20}$$

requires care in moving between two bases for the equivariant cohomology of the Hilbert scheme of $\mathbb{C}^2$ (which we identify with Fock space): the Nakajima basis $\{\mathsf{C}_\lambda\}$ and the T-fixed point basis $\{\mathsf{P}_\lambda\}$. The change of basis formula is simple mod s_1+s_2:

$$\mathsf{P}_\lambda = \sum_{\mu} \frac{(-1)^{\ell(\mu)} d!}{\dim \lambda} \chi^\lambda(\mu) s_1^{d+\ell(\mu)} \mathsf{C}_\mu. \tag{21}$$

Our $\widehat{\mathsf{S}}^{\mu}_{(d)}$ should be viewed as having upper index given in the T-fixed point basis but lower index in the Nakajima basis.

The main tool for evaluating S is the quantum differential equation of [18] valid also for stable pairs [15],

$$s_3 q \frac{d}{dq} \mathsf{S} = \mathsf{M}\mathsf{S} - \mathsf{S}\mathsf{M}(0) \ . \tag{22}$$

Here, S has both components indexed by the Nakajima basis and is viewed as an operator on Fock space, see [18]. The operator M is defined on Fock space by[7]

$$\begin{aligned} \mathsf{M}(q, s_1, s_2) = (s_1+s_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k+1}{(-q)^k-1}\, \alpha_{-k}\, \alpha_k + \\ \frac{1}{2} \sum_{k,l>0} \Big[s_1 s_2\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} - \alpha_{-k-l}\, \alpha_k\, \alpha_l \Big] \ . \end{aligned} \tag{23}$$

The q-dependence of M is only in the first sum in (23). The operator $\mathsf{M}(0)$ is the q^0-coefficient of M.

From the differential equation (22), we find

$$s_3 q \frac{d}{dq} \mathsf{S} = (s_1+s_2)\mathsf{A} + [\mathsf{B}, \mathsf{S}] \qquad \text{mod } (s_1+s_2)^2 \ .$$

The first term is

$$\mathsf{A} = \left(\sum_{k>0} \frac{k}{2} \frac{(-q)^k+1}{(-q)^k-1}\, \alpha_{-k}\, \alpha_k \right) \circ \mathsf{S}(0) + \mathsf{S}(0) \circ \left(\sum_{k>0} \frac{k}{2}\, \alpha_{-k}\, \alpha_k \right) .$$

The operator in the second term is

$$\mathsf{B} = \frac{1}{2} \sum_{k,l>0} \Big[s_1 s_2\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} - \alpha_{-k-l}\, \alpha_k\, \alpha_l \Big] \ .$$

Since we are interested now in $\widehat{\mathsf{S}}$, we can simplify the differential equation:

$$s_3 q \frac{d}{dq} \widehat{\mathsf{S}} = (s_1+s_2)\widehat{\mathsf{A}} + [\mathsf{B}, \widehat{\mathsf{S}}] \qquad \text{mod } (s_1+s_2)^2 \ . \tag{24}$$

for

$$\widehat{\mathsf{A}} = \left(\sum_{k>0} k \frac{(-q)^k}{(-q)^k-1}\, \alpha_{-k}\, \alpha_k \right) \circ \mathsf{S}(0).$$

[7]The operator M was found earlier in the quantum cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ [17]. A parallel occurance appears in the local Gromov-Witten theory of curves [3].

The eigenvectors for B (mod $s_1 + s_2$) are the classes P_λ with eigenvalues

$$w_\lambda = \sum_{(i,j)\in\lambda} (i-j)s_1.$$

Equation (24) then gives a simple relationship between the entries of $\widehat{\mathsf{S}}$ and of $\widehat{\mathsf{A}}$ in the P_λ basis.

The operator $\widehat{\mathsf{A}}$ is diagonal in the Nakajima basis C_λ with entries

$$\widehat{\mathsf{A}}^{\mathsf{C}}_{\lambda\lambda} = \sum_{k \text{ part of } \lambda} k^2 \frac{(-q)^k}{(-q)^k - 1}.$$

Applying the change of basis formula (21), we obtain the entries in the P_λ basis (mod $s_1 + s_2$):

$$\widehat{\mathsf{A}}^{\mathsf{P}}_{\mu'\mu} = \sum_\lambda \frac{\dim\mu'}{\dim\mu} \frac{\chi^\mu(\lambda)\chi^{\mu'}(\lambda)}{z(\lambda)} \sum_{k \text{ part of } \lambda} k^2 \frac{(-q)^k}{(-q)^k - 1}.$$

If we use the notation λ_r for the number of parts in a partition λ of size r, then we can rewrite the entries as:

$$\widehat{\mathsf{A}}^{\mathsf{P}}_{\mu'\mu} = \frac{\dim\mu'}{\dim\mu} \sum_{r=1}^{d} r \frac{(-q)^r}{(-q)^r - 1} \sum_\lambda \frac{\chi^\mu(\lambda)\chi^{\mu'}(\lambda) r\lambda_r}{z(\lambda)}.$$

The following Lemma (easily proven using the Murnaghan-Nakayama rule) gives a simpler expression for the innermost sum in the above expression.

Lemma 3. *Let μ, μ' be partitions of the same size and let $r > 0$. Then*

$$\sum_\lambda \frac{\chi^\mu(\lambda)\chi^{\mu'}(\lambda) r\lambda_r}{z(\lambda)} = \sum_{\substack{\gamma,\gamma' \ r\text{-hooks} \\ \mu\backslash\gamma = \mu'\backslash\gamma'}} (-1)^{h(\gamma)+h(\gamma')}$$

where $h(\gamma)$ is the number of rows in a rim hook γ.

In the calculations below, we denote by $\theta_r(\mu,\mu')$ the quantity appearing in Lemma 3.

We now are able to compute the restriction

$$\frac{1}{s_1+s_2} \widehat{\mathsf{S}}^\mu_{(d)}\Big|_{s_2=-s_1}$$

in terms of θ_r and the eigenvalues $w(\lambda)$:

$$\sum_{r=1}^{d} \left(\frac{(-1)^{d-1} s_1^{d-1} (d-1)!}{\dim\mu} \sum_\nu \chi^\nu((d)) \theta_r(\mu,\nu) \frac{r}{w(\mu) - w(\nu) + n_{\mu,\nu} s_3} \right) \frac{(-q)^r}{1-(-q)^r}.$$

As before, $\chi^\nu((d)) = 0$ unless $\nu = \alpha_a$ is a hook. For $\theta_r(\mu,\nu)$ to be nonzero, we must have μ be the union of two hooks, of sizes $d-r$ and r. The integers $n_{\mu,\nu}$ which arise will not affect the answer.

If we multiply by the descendent and edge factors and take the constant term in $\frac{s_3}{s_1}$, we obtain an expression for (20) of the form

$$\sum_{r=1}^{d} B_r \frac{(-q)^r}{1-(-q)^r}.$$

The explicit formulas for B_r depend upon two cases. For B_d, we have

$$(-1)^d d \cdot d!(d+2)!\ B_d =$$
$$\sum_{a+b=d-1} (-1)^a \binom{d-1}{a} \sum_{\substack{i=-b \\ i\neq 0}}^{a} \frac{1}{i} \left(-(-b-1)^{d+2} + (-b)^{d+2} + a^{d+2} - (a+1)^{d+2}\right) ,$$

and for $r < d$, we have

$$\frac{(-1)^{d+r}}{2} d \cdot d!(d+2)!\ B_r =$$
$$\sum_{\substack{a+b=d-1 \\ a\geq r \\ 0\leq c\leq r-1}} (-1)^{a+c} \binom{d-r-1}{a-r}\binom{r-1}{c} \frac{(a-r-c)(b+c+1-r)}{(a-c)^2(b+c+1)}$$
$$\cdot \Bigg(-(-b-1)^{d+2} + (-b)^{d+2} + (a-r)^{d+2} - (a-r+1)^{d+2}$$
$$- (c-r)^{d+2} + (c-r+1)^{d+2} + c^{d+2} - (c+1)^{d+2} \Bigg) .$$

4.5. **Classical pairing.** We compute the classical pairing $\langle \tau_d, \mathsf{C}_{(d[0])}\rangle$. The simpler pairing $\langle \tau_{d-1}, \mathsf{C}_{(d[0])}\rangle$ is needed for the calculation and is addressed first.

Lemma 4. $\langle \tau_{d-1}, \mathsf{C}_{(d[0])}\rangle = \frac{1}{d!}$

Proof. By dimension counting, the pairing has no dependence on s_1 and s_2, so we can work mod $s_1 + s_2$. Localization then yields

$$(-1)^d d \cdot d!(d+1)! \left\langle \tau_{d-1}, \mathsf{C}_{(d[0])} \right\rangle$$
$$= \sum_{a+b=d-1} (-1)^a \binom{d-1}{a} \left(-(-b-1)^{d+1} + (-b)^{d+1} + a^{d+1} - (a+1)^{d+1}\right)$$

If we rewrite $-(-b-1)^{d+1}+(-b)^{d+1}+a^{d+1}-(a+1)^{d+1}$ as a polynomial in a alone, the leading term $-(d+1)d^2a^{d-1}$. Then,

$$\left\langle \tau_{d-1}, \mathsf{C}_{(d[0])} \right\rangle = \frac{(-1)^d}{d \cdot d!(d+1)!}(-(d+1)d^2)(-1)^{d-1}(d-1)! = \frac{1}{d!}$$

since the contributions of all the lower terms are 0. □

We cannot compute $\langle \tau_d, \mathsf{C}_{(d[0])} \rangle$ in the same way, since we cannot work mod $s_1 + s_2$ (as we know the answer is a multiple of $s_1 + s_2$). Instead we work mod s_2 and consider the function

$$f(k) = (k+1)! s_1^{d-1-k} \Big\langle \tau_k, \mathsf{C}_{(d[0])} \Big\rangle \Big|_{s_2=0} .$$

We can compute by localization that f is of the form

$$\sum_{i=1}^{d} c_i i^{k+1}$$

for some constants $c_i \in \mathbb{Q}$. We also know

$$f(0) = f(1) = \cdots = f(d-2) = 0$$

by dimension constraints. By Lemma 4, we have $f(d-1) = 1$. Interpolation then gives $f(d) = \frac{d(d+1)}{2}$. We conclude the following result.

Proposition 3.

$$\Big\langle \tau_d, \mathsf{C}_{(d[0])} \Big\rangle = \frac{s_1 + s_2}{2 \cdot (d-1)!}.$$

4.6. **Proof of Theorem 3.** We have

$$F(d) = F_0(d) + \sum_{r=1}^{d} (A_r + B_r) \frac{(-q)^r}{1 - (-q)^r} .$$

Although the formulas for A_r and B_r calculated in Sections 4.3 - 4.4 are very complicated, a wonderful combinatorial identity holds:

$$A_r + B_r = \frac{1}{d!} \tag{25}$$

for all $1 \le r \le d$. The proof of (25) is by straightforward manipulation using a few standard binomial sum identities. In the case $r = d$, all that is needed is the identity

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} (a_m i^m + a_{m-1} i^{m-1} + \cdots + a_0) = (-1)^m m! \cdot a_m.$$

For $r < d$, the following two identities must also be used to compute the sum over c in the expression for B_r:

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{1}{x+i} = \frac{1}{(m+1)\binom{x+m}{m+1}}$$

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{1}{(x+i)^2} = \frac{1}{(m+1)\binom{x+m}{m+1}} \sum_{i=0}^{m} \frac{1}{x+i}.$$

Combined with Proposition 3, the identity (25) yields

$$F(d) = \frac{1}{2 \cdot d!} \sum_{r=1}^{d} \frac{1 + (-q)^r}{1 - (-q)^r},$$

which completes the proof of Theorem 3. □

The same method of computation actually yields a relatively simple formula for a larger family of invariants. Suppose that $m_1, \ldots, m_k$ are positive integers satisfying

$$\sum_{i=1}^{k} m_i = d\ .$$

Lemma 2 relied only on a dimension analysis which also applies to $\mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_{m_1}([0]) \cdots \tau_{m_k}([0]))^{\mathbf{T}}$, so we can expect the series to be relatively simple. In fact, we can prove

$$\mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_{m_1}([0]) \cdots \tau_{m_k}([0]))^{\mathbf{T}} = \frac{q^d}{m_1! \cdots m_k!} \left(\frac{s_1 + s_2}{s_1 s_2} \right) \frac{1}{2} \sum_{r=1}^{d} C_r(m_1, \ldots, m_k) \frac{1 + (-q)^r}{1 - (-q)^r} \tag{26}$$

for coefficients

$$C_r(m_1, \ldots, m_k) = \sum_{\substack{I \subset \{1, \ldots, k\} \\ \sum_{i \in I} m_i < r}} r^{|I|-1} (d - r)^{k-|I|-1} \left(r - \sum_{i \in I} m_i \right).$$

For example, we have

$$\mathsf{Z}^{\mathsf{cap}}_{3,(3)}(\tau_1([0])\tau_2([0]))^{\mathbf{T}} = \frac{q^3}{2} \left(\frac{s_1 + s_2}{s_1 s_2} \right) \frac{1}{2} \left(2 \cdot \frac{1 - q}{1 + q} + 2 \cdot \frac{1 + q^2}{1 - q^2} + 3 \cdot \frac{1 - q^3}{1 + q^3} \right).$$

When proving (26) by the method used for Theorem 3, the leading classical term requires the somewhat surprising identity

$$\sum_{r=1}^{d} C_r(m_1, \ldots, m_k) = \sum_{\substack{f : \{1, \ldots, k\} \to \{1, \ldots, k\} \\ f \text{ has only one periodic orbit}}} \prod_{i=1}^{k} m_{f(i)}\ .$$

We can prove this identity by showing that both sides satisfy the same recurrence equation as a function of $m_1, \ldots, m_k$:

$$\begin{aligned} F(m_1,\ldots,m_k) = m_k^2(m_1+\cdots+m_k)^{k-2} &+ m_1F(m_1+m_k,m_2,\ldots,m_{k-1}) \\ &+ m_2F(m_1,m_2+m_k,\ldots,m_{k-1}) \\ &+ \cdots \\ &+ m_{k-1}F(m_1,m_2,\ldots,m_{k-1}+m_k). \end{aligned}$$

Finally, we can also use the same method of computation to obtain the analogous normalized Donaldson-Thomas partition functions. Surprisingly, these are equal to the stable pairs partition functions except in degree one:

$$\mathsf{Z}^{\mathsf{DT},\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))^T = \begin{cases} \mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))^T & \text{if } d>1 \\ \mathsf{Z}^{\mathsf{cap}}_{d,(d)}(\tau_d(\mathsf{p}))^T + q\left(\frac{s_1+s_2}{s_1s_2}\right) q\frac{d}{dq}\log(M(-q)) & \text{if } d=1. \end{cases}$$

Here $M(q) = \prod(1-q^r)^{-r}$ is the MacMahon function.

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