Tautological relations via r-spin structures

R. Pandharipande, A. Pixton, D. Zvonkine

February 2019

Abstract

Relations among tautological classes on $\overline{\mathcal{M}}_{g,n}$ are obtained via the study of Witten's r-spin theory for higher r. In order to calculate the quantum product, a new formula relating the r-spin correlators in genus 0 to the representation theory of $\mathsf{sl}_2(\mathbb{C})$ is proven. The Givental-Teleman classification of CohFTs is used at two special semisimple points of the associated Frobenius manifold. At the first semisimple point, the R-matrix is exactly solved in terms of hypergeometric series. As a result, an explicit formula for Witten's r-spin class is obtained (along with tautological relations in higher degrees). As an application, the $r = 4$ relations are used to bound the Betti numbers of $R^*(\mathcal{M}_g)$. At the second semisimple point, the form of the R-matrix implies a polynomiality property in r of Witten's r-spin class.

In the Appendix (with F. Janda), a conjecture relating the $r = 0$ limit of Witten's r-spin class to the class of the moduli space of holomorphic differentials is presented.

Contents

0 Introduction

0.1 Overview

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable genus g curves with n markings. Let

$$RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$$

be the subring of tautological classes in cohomology[1]. The subrings

$$\left\{ RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n}) \right\}_{g,n}$$

are defined together as *the smallest system of $\mathbb{Q}$-subalgebras closed under push-forward via all boundary and forgetful maps*, see [6, 7, 12]. There has been substantial progress in the understanding of $RH^*(\overline{\mathcal{M}}_{g,n})$ since the study began in the 1980s [21]. The subject took a new turn in 2012 with the family of relations conjectured in [26]. We refer the reader to [23] for a survey of recent developments.

Witten's r-spin class defines a Cohomological Field Theory (CohFT) for each integer $r \geq 2$. Witten's 2-spin theory concerns only the fundamental classes of the moduli spaces of curves (and leads to no new geometry). In our previous paper [25], we used Witten's 3-spin theory to construct a family of relations among tautological classes of $\overline{\mathcal{M}}_{g,n}$ equivalent (in cohomology) to the relations proposed in [26]. Our goal here is to extend our study of tautological relations to Witten's r-spin theory for all $r \geq 3$.

Taking [25] as a starting point, Janda has completed a formal study of tautological relations obtained from CohFTs. Two results of Janda are directly relevant here:

(i) The relations for $r = 3$ are valid in Chow [14, 16].

(ii) The relations for $r \geq 4$ are implied by the relations for $r = 3$ [14, 15].

By (i) and (ii) together, all of the r-spin relations that we find will be valid in Chow. However, since our methods here are cohomological, we will use the language of cohomology throughout the paper.

[1] All cohomology and Chow groups will be taken with $\mathbb{Q}$-coefficients. The tautological ring in Chow is denoted by

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}) .$$

We will use the complex grading for $RH^*(\overline{\mathcal{M}}_{g,n})$, so $R^i(\overline{\mathcal{M}}_{g,n}) \to RH^i(\overline{\mathcal{M}}_{g,n})$.

Given Janda's results, why proceed with the higher r-spin analysis? There are three basic reasons:

- The $r = 4$ relations are simpler and easier to use than the 3-spin relations when restricted to $\mathcal{M}_g$. From the 4-spin relations, we derive a new bound on the rank of $R^d(\mathcal{M}_g)$ which specializes in case $d \geq g-2$ to the basic results

  $$\dim_{\mathbb{Q}} R^{g-2}(\mathcal{M}_g) \leq 1\,, \qquad \dim_{\mathbb{Q}} R^{>g-2}(\mathcal{M}_g) = 0 \tag{1}$$

  of Looijenga [19]. By (ii) above, we conclude that both statements in (1) follow from the restriction of the 3-spin relations to $\mathcal{M}_g$. The latter restriction equals the Faber-Zagier relations[2].

  The outcome is a proof that Looijenga's results (1) follow from the Faber-Zagier relations. Since Faber's conjectures [5] governing the proportionalities of κ monomials in $R^{g-2}(\mathcal{M}_g)$ are known[3] to be compatible with the Faber-Zagier relations, we can also conclude that the Faber-Zagier relations *imply* these proportionalities.

  Relations, by themselves, cannot prove non-vanishing results. The non-vanishing

  $$\dim_{\mathbb{Q}} R^{g-2}(\mathcal{M}_g) \geq 1 \tag{2}$$

  is proven [5, 6] by Hodge integral evaluations. The results (1) and (2) together prove

  $$R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}\,.$$

- Another outcome is a much better understanding of Witten's r-spin class for higher r. We obtain an exact formula for Witten's r-spin correlators in genus 0 in terms of the representation theory of $\mathsf{sl}_2(\mathbb{C})$. The genus 0 results and the Givental-Teleman classification of semisimple CohFTs together provide two explicit approaches to Witten's r-spin class in all genera. The first leads to a formula for all r parallel to the 3-spin formula of [25]. As an application of the second approach, we prove a new *polynomiality* property in r of Witten's r-spin class.

[2]See [23] for a survey of the Faber-Zagier relations and related topics.

[3]A proof (unpublished) was found by Faber and Zagier in 2002. The result is also derived in [27, Section 3].

- In the Appendix with Janda, we present a new conjecture relating an appropriate limit (defined by polynomiality) of Witten's r-spin class to the class of the moduli space of holomorphic differentials with prescribed zero multiplicities.

These are unexpected developments. There is no *a priori* reason to believe the 4-spin relations would be algebraically simpler or that polynomiality in r holds for Witten's class. The very simple connection with the class of the moduli space of differentials leads to a much more direct calculation than in [8, Appendix], but is available only in the holomorphic case.

0.2 Cohomological field theories

We recall here the basic definitions of a cohomological field theory by Kontsevich and Manin [18], see also [24] for a survey.

Let V be a finite dimensional $\mathbb{Q}$-vector space with a non-degenerate symmetric 2-form η and a distinguished element $\mathbf{1} \in V$. The data $(V, \eta, \mathbf{1})$ is the starting point for defining a cohomological field theory. Given a basis $\{e_i\}$ of V, we write the symmetric form as a matrix

$$\eta_{jk} = \eta(e_j, e_k) \ .$$

The inverse matrix is denoted by η^{jk} as usual.

A cohomological field theory consists of a system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of elements

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n}.$$

We view $\Omega_{g,n}$ as associating a cohomology class on $\overline{\mathcal{M}}_{g,n}$ to elements of V assigned to the n markings. The CohFT axioms imposed on Ω are:

(i) Each $\Omega_{g,n}$ is S_n-invariant, where the action of the symmetric group S_n permutes both the marked points of $\overline{\mathcal{M}}_{g,n}$ and the copies of V^*.

(ii) Denote the basic gluing maps by

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n} \ ,$$

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n} \ .$$

The pull-backs $q^*(\Omega_{g,n})$ and $r^*(\Omega_{g,n})$ are equal to the contractions of $\Omega_{g-1,n+2}$ and $\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1}$ by the bi-vector

$$\sum_{j,k} \eta^{jk} e_j \otimes e_k$$

inserted at the two identified points.

(iii) Let $v_1, \ldots, v_n \in V$ be any vectors, and let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. We require

$$\Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbf{1}) = p^*\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) \ ,$$

$$\Omega_{0,3}(v_1 \otimes v_2 \otimes \mathbf{1}) = \eta(v_1, v_2) \ .$$

Definition 0.1. A system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of elements

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n}$$

satisfying properties (i) and (ii) is a *cohomological field theory* or a *CohFT*. If (iii) is also satisfied, Ω is a *CohFT with unit*.

A CohFT Ω yields a *quantum product* $\bullet$ on V via

$$\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) \ .$$

Associativity of $\bullet$ follows from (ii). The element $\mathbf{1} \in V$ is the identity for $\bullet$ by (iii).

A CohFT ω composed only of degree 0 classes,

$$\omega_{g,n} \in H^0(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \ ,$$

is called a *topological field theory*. Via property (ii), $\omega_{g,n}(v_1, \ldots, v_n)$ is determined by considering stable curves with a maximal number of nodes. Such a curve is obtained by identifying several rational curves with three marked points. The value of $\omega_{g,n}(v_1 \otimes \cdots \otimes v_n)$ is thus uniquely specified by the values of $\omega_{0,3}$ and by the quadratic form η. In other words, given V and η, a topological field theory is uniquely determined by the associated quantum product.

0.3 Witten's r-spin class

For every integer $r \geq 2$, there is a beautiful CohFT obtained from Witten's r-spin class. We review here the basic properties of the construction. The integer r is fixed once and for all.

Let V_r be an $(r-1)$-dimensional $\mathbb{Q}$-vector space with basis $e_0, \ldots, e_{r-2}$, bilinear form

$$\eta_{ab} = \eta(e_a, e_b) = \delta_{a+b,r-2}\ ,$$

and unit vector $\mathbf{1} = e_0$. Witten's r-spin theory provides a family of classes

$$W^r_{g,n}(a_1, \ldots, a_n) \in H^*(\overline{\mathcal{M}}_{g,n}).$$

for $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$. These define a CohFT by

$$\mathsf{W}^r_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}), \quad \mathsf{W}^r_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) = W^r_{g,n}(a_1, \ldots, a_n)\ .$$

Witten's class $W^r_{g,n}(a_1, \ldots, a_n)$ has (complex) degree given by the formula

$$\begin{aligned} \deg_{\mathbb{C}} W^r_{g,n}(a_1, \ldots, a_n) &= \mathsf{D}^r_{g,n}(a_1, \ldots, a_n) \\ &= \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}\ . \end{aligned} \tag{3}$$

If $\mathsf{D}^r_{g,n}(a_1, \ldots, a_n)$ is not an integer, the corresponding Witten's class vanishes.

In genus 0, the construction was first carried out by Witten [34] using r-spin structures (r^{th} roots of the canonical bundle) and satisfies the following initial conditions:

$$W^r_{0,3}(a_1, a_2, a_3) = \left|\begin{array}{ll} 1 & \text{if } a_1 + a_2 + a_3 = r-2, \\ 0 & \text{otherwise.} \end{array}\right. \tag{4}$$

$$W^r_{0,4}(1, 1, r-2, r-2) = \frac{1}{r} \cdot [\mathrm{pt}]\ \in H^2(\overline{M}_{0,4})\ .$$

Uniqueness of Witten's r-spin theory in genus 0 follows from the initial conditions (4) and the axioms of a CohFT with unit.

The genus 0 sector defines a quantum product $\bullet$ on V with unit e_0,

$$\eta(e_a \bullet e_b, e_c) = W^r_{0,3}(a, b, c)\ .$$

The resulting algebra, even after extension to $\mathbb{C}$, is not semisimple.

The existence of Witten's class in higher genus is both remarkable and highly non-trivial. An algebraic construction was first obtained by Polishchuk and Vaintrob [28] defining

$$W^r_{g,n}(a_1,\ldots,a_n) \in A^*(\overline{\mathcal{M}}_{g,n})$$

as a cycle class. The algebraic approach was later simplified by Chiodo [4]. Analytic constructions have been given by Mochizuki [20] and later by Fan, Jarvis, and Ruan [13]. As a consequence of the following result, the analytic and algebraic approaches coincide and yield tautological classes in cohomology.

Theorem 1 ([25])**.** *For every $r \geq 2$, there is a unique CohFT which extends Witten's r-spin theory in genus 0 and has pure dimension* (3)*. The unique extension takes values in the tautological ring*

$$RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n}).$$

Whether Witten's r-spin theory as an algebraic cycle takes values in

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$$

is an open question.

0.4 Witten's r-spin class and representations of $\mathsf{sl}_2(\mathbb{C})$.

Consider the Lie algebra $\mathsf{sl}_2 = \mathsf{sl}_2(\mathbb{C})$. Denote by ρ_k the k-th symmetric power of the standard 2-dimensional representation of sl_2,

$$\rho_k = \mathrm{Sym}^k(\rho_1)\,, \quad \dim \rho_k = k+1\,.$$

The complete list of irreducible representations of sl_2 is $\{\rho_k\}_{k\geq 0}$, see [10]. Let

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathsf{sl}_2\,.$$

The trace of the exponential is

$$\mathrm{tr}_k e^{tH} = \frac{e^{(k+1)t/2} - e^{-(k+1)t/2}}{e^{t/2} - e^{-t/2}}\,.$$

The formula for the tensor product of two irreducible representations is then easily obtained:

$$\rho_k \otimes \rho_l = \rho_{|k-l|} \oplus \rho_{|k-l|+2} \oplus \cdots \oplus \rho_{k+l}\,.$$

Our first result relates Witten's r-spin class in genus 0 with the representation theory of sl_2.

Theorem 2. *Let* $\mathbf{a} = (a_1, \ldots, a_{n\geq 3})$ *with* $a_i \in \{0, \ldots, r-2\}$ *satisfy the degree constraint* $\mathsf{D}^r_{0,n}(\mathbf{a}) = n-3$*. Then,*

$$W^r_{0,n}(\mathbf{a}) = \frac{(n-3)!}{r^{n-3}} \dim\Big[\rho_{r-2-a_1} \otimes \cdots \otimes \rho_{r-2-a_n}\Big]^{\mathsf{sl}_2} \cdot [\mathrm{pt}] \;\in H^{2(n-3)}(\overline{\mathcal{M}}_{0,n}),$$

where the superscript sl_2 *denotes the* sl_2*-invariant subspace and the class*

$$[\mathrm{pt}] \in H^{2(n-3)}(\overline{\mathcal{M}}_{0,n})$$

is Poincaré dual to a point.

The degree constraint $\mathsf{D}^r_{0,n}(\mathbf{a}) = n-3$ in the statement of Theorem 2 can be written equivalently (using (3)) as

$$\sum_{i=1}^{n} a_i = (n-2)r - 2\,.$$

Since $a_i \leq r-2$, the bound $n \leq r+1$ is a simple consequence.

0.5 Shifted Witten class

Given a vector $\gamma \in V_r$, the *shifted Witten class* is defined by

$$\mathsf{W}^{r,\gamma}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum_{m\geq 0} \frac{1}{m!}\, p_{m*}\, \mathsf{W}^r_{g,n+m}(v_1 \otimes \cdots \otimes v_n \otimes \gamma^{\otimes m}),$$

where $p_m \colon \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map. The shifted Witten class $\mathsf{W}^{r,\gamma}$ determines a CohFT, see [25, Section 1.1].

The vector space V_r carries a *Gromov-Witten potential* F satisfying

$$\frac{\partial^3 \mathsf{F}}{\partial t^a \partial t^b \partial t^c}(\gamma) = \mathsf{W}^{r,\gamma}_{0,3}(e_a \otimes e_b \otimes e_c)$$

which defines a Frobenius manifold structure on V_r.

Example 0.2. *For $r = 3$, the Gromov-Witten potential obtained from Witten's class equals*

$$\mathsf{F}(x, y) = \frac{1}{2}x^2y + \frac{1}{72}y^4,$$

where $x = t^0$ and $y = t^1$.

For $r = 4$, the potential is

$$\mathsf{F}(x, y, z) = \frac{1}{2}x^2z + \frac{1}{2}xy^2 + \frac{1}{16}y^2z^2 + \frac{1}{960}z^5,$$

where $x = t^0$, $y = t^1$, and $z = t^2$.

The tangent vector space to $\gamma \in V_r$ has a natural Frobenius (or fusion) algebra structure $\Phi^{r,\gamma}$ given by the structure constants

$$\eta(e_a \bullet_\gamma e_b, e_c) = \frac{\partial^3 \mathsf{F}}{\partial t^a \partial t^b \partial t^c}(\gamma).$$

Theorem 3. *For $(0, \ldots, 0, r) \in V_r$, the algebra $\Phi^{r,(0,\ldots,0,r)}$ is isomorphic to the Verlinde algebra of level r for sl_2.*

Since the Verlinde algebra is semisimple, the algebra $\Phi^{r,(0,\ldots,0,r)}$ is semisimple as a consequence of Theorem 3. Proposition 2.3 of Section 2.4 provides a basis of idempotents for the algebras $\Phi^{(0,\ldots,0,r\phi)}$ for all $0 \neq \phi \in \mathbb{Q}$.

Theorems 2 and 3 relate Witten's r-spin class and the corresponding Frobenius manifold to the representations of sl_2 at level r. On the other hand, the Frobenius manifold associated to Witten's r-spin theory is usually constructed from the A_{r-1} singularity and thus is related to the Lie algebra sl_r Lie algebra. Perhaps there is some form of rank-level duality for Frobenius manifolds, but we are not aware of other examples.

0.6 Euler field and Hodge grading operator.

The Frobenius manifold structure on V_r includes an Euler field and a conformal dimension which determine a Hodge grading operator.

The *Euler field* on the Frobenius manifold V_r is

$$E = \sum_{a=0}^{r-2} \left(1 - \frac{a}{r}\right) t^a \partial_a \,.$$

The Lie derivatives with respect to E of the basis vectors fields are easily calculated:

$$L_E(\partial_a) = [E, \partial_a] = -\left(1 - \frac{a}{r}\right)\partial_a\,.$$

The *conformal dimension* equals

$$\delta = \frac{r-2}{r}\,.$$

Let v be a tangent vector at a point of the Frobenius manifold. We define the *shifted degree operator* $\mu(v)$, also called the *Hodge grading operator*, by

$$\mu(v) = [E, v] + \left(1 - \frac{\delta}{2}\right) v\ .$$

Here, the vector v is extended to a flat tangent vector field in order to compute the commutator. We have

$$\mu(\partial_a) = \frac{2a+2-r}{2r}\partial_a\,.$$

0.7 Tautological relations

We will construct tautological relations using Givental's R-matrix action on CohFTs.[4] The relations will be proven by studying Witten's r-spin class. The point

$$\tau = (0, \ldots, 0, r\phi) \in V_r$$

with respect to a nonzero parameter $\phi \in \mathbb{Q}$ will play a special role. Let

$$\widehat{\partial}_a = \phi^{-(2a-r+2)/4}\partial_a$$

be a new tangent frame on the Frobenius manifold V_r at the point τ.

We define a multilinear map

$$\omega_{g,n}^{r,\tau} : V_r^{\otimes n} \to \mathbb{Q}$$

[4]A review of Givental's R-matrix action on CohFTS in the form we require can be found in [25, Section 2].

by the trigonometric formula

$$\omega^{r,\tau}_{g,n}\left(\widehat{\partial}_{a_1}\otimes\cdots\otimes\widehat{\partial}_{a_n}\right) = \left(\frac{r}{2}\right)^{g-1}\phi^{(r-2)(2g-2+n)/4}\sum_{k=1}^{r-1}\frac{(-1)^{(k-1)(g-1)}\prod_{i=1}^{n}\sin\left(\frac{(a_i+1)k\pi}{r}\right)}{\left(\sin(\frac{k\pi}{r})\right)^{2g-2+n}}. \quad (5)$$

We will prove $\omega^{r,\tau}$ is a CohFT with the right side of (5) interpreted as a multiple of the identity $1\in H^0(\overline{\mathcal{M}}_{g,n})$. In fact, $\omega^{r,\tau}$ is the topological part of the τ-shifted r-spin CohFT $\mathsf{W}^{r,\tau}$ defined in Section 0.5.

Our construction of tautological relations depends upon the following hypergeometric series. For every $a\in\{0,\ldots,r-2\}$, we define

$$\boldsymbol{B}_{r,a}(T)=\sum_{m=0}^{\infty}\left[\prod_{i=1}^{m}\frac{\big((2i-1)r-2(a+1)\big)\big((2i-1)r+2(a+1)\big)}{i}\right]\left(-\frac{T}{16r^2}\right)^m.$$

For r even and $a=\frac{r}{2}-1$, we have $\boldsymbol{B}_{r,a}=1$. Otherwise, $\boldsymbol{B}_{r,a}$ is a power series with all coefficients nonzero. We denote by $\boldsymbol{B}^{\mathrm{even}}_{r,a}$ and $\boldsymbol{B}^{\mathrm{odd}}_{r,a}$ the even and odd parts of the power series $\boldsymbol{B}_{r,a}$,

$$\boldsymbol{B}_{r,a}(T)=\boldsymbol{B}^{\mathrm{even}}_{r,a}(T)+\boldsymbol{B}^{\mathrm{odd}}_{r,a}(T).$$

Example 0.3. *For $r=3$, we obtain a slight variation of the series occurring in the Faber-Zagier relations:*

$$\boldsymbol{B}_{3,0}(T)=\sum_{m\geq 0}\frac{(6m)!}{(2m)!\,(3m)!}\left(-\frac{T}{1728}\right)^m,$$
$$\boldsymbol{B}_{3,1}(T)=\sum_{m\geq 0}\frac{1+6m}{1-6m}\frac{(6m)!}{(2m)!\,(3m)!}\left(-\frac{T}{1728}\right)^m.$$

For $r=4$, we obtain:

$$\boldsymbol{B}_{4,0}(T)=\sum_{m\geq 0}\frac{(4m)!}{m!\,(2m)!}\left(-\frac{T}{256}\right)^m,$$
$$\boldsymbol{B}_{4,1}(T)=1,$$
$$\boldsymbol{B}_{4,2}(T)=\sum_{m\geq 0}\frac{1+4m}{1-4m}\frac{(4m)!}{m!\,(2m)!}\left(-\frac{T}{256}\right)^m.$$

Consider the matrix-valued power series $R(z) \in \mathrm{End}(V_r)[[z]]$ with coefficients given by

$$R_a^a = \boldsymbol{B}^{\mathrm{even}}_{r,r-2-a}(\phi^{-r/2}z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the main diagonal,

$$R_a^{r-2-a} = -\boldsymbol{B}^{\mathrm{odd}}_{r,a}(\phi^{-r/2}z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the antidiagonal (if r is even, the coefficient at the intersection of both diagonals is 1), and 0 everywhere else.

Example 0.4. *For $r = 3$, the R matrix is*

$$\begin{pmatrix} \boldsymbol{B}^{\mathrm{even}}_{3,1}(\phi^{-3/2}z) & -\boldsymbol{B}^{\mathrm{odd}}_{3,1}(\phi^{-3/2}z) \\ -\boldsymbol{B}^{\mathrm{odd}}_{3,0}(\phi^{-3/2}z) & \boldsymbol{B}^{\mathrm{even}}_{3,0}(\phi^{-3/2}z) \end{pmatrix}.$$

For $r = 4$, the R matrix is

$$\begin{pmatrix} \boldsymbol{B}^{\mathrm{even}}_{4,2}(\phi^{-2}z) & 0 & -\boldsymbol{B}^{\mathrm{odd}}_{4,2}(\phi^{-2}z) \\ 0 & 1 & 0 \\ -\boldsymbol{B}^{\mathrm{odd}}_{4,0}(\phi^{-2}z) & 0 & \boldsymbol{B}^{\mathrm{even}}_{4,0}(\phi^{-2}z) \end{pmatrix}.$$

We will prove that the inverse matrix $R^{-1}(z)$ has coefficients

$$(R^{-1})_a^a = \boldsymbol{B}^{\mathrm{even}}_{r,a}(\phi^{-r/2}z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the main diagonal,

$$(R^{-1})_a^{r-2-a} = \boldsymbol{B}^{\mathrm{odd}}_{r,a}(\phi^{-r/2}z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the anti-diagonal (if r is even, the coefficient at the intersection of both diagonals is 1), and 0 everywhere else.

Let $\Omega^{r,\tau}$ be the stable graph expression for the CohFT obtained by the action $\omega^{r,\tau}$ of the above R-matrix,

$$\Omega^{r,\tau} = R.\omega^{r,\tau},$$

see [25, Definition 2.13].

Theorem 4. *For every $d > \mathsf{D}^r_{g,n}(a_1,\ldots,a_n)$, the degree d part of*

$$\Omega^{r,\tau}_{g,n}(e_{a_1}\otimes\cdots\otimes e_{a_n}) \in H^*(\overline{\mathcal{M}}_{g,n})$$

vanishes.

The complexity of the topological field theory $\omega^{r,\tau}$ for higher r leads to complicated relations on $\overline{\mathcal{M}}_{g,n}$. However, by multiplying by ψ classes, pushing forward by forgetful maps, and then restricting to the interior, we obtain much simpler relations on $\mathcal{M}_{g,n}$. In order to write the resulting relations, we extend the definition of the power series $\boldsymbol{B}_{r,a}(T)$ to all

$$a \geq 0 \quad \text{satisfying} \quad a \not\equiv r-1 \mod r$$

by the formula

$$\boldsymbol{B}_{r,a+rb}(T) = T^b \boldsymbol{B}_{r,a}(T)\,.$$

The relations depend upon on a partition[5]

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{\ell(\sigma)})$$

with *no* part σ_i congruent to $r-1$ mod r, and a vector of non-negative integers

$$\mathbf{a} = (a_1,\ldots,a_n)$$

with *no* a_i congruent to $r-1$ mod r.

Theorem 5. *Let σ and $\mathbf{a}$ avoid $r-1$ mod r, and let d satisfy*

$$rd > (r-2)(g-1) + |\sigma| + \sum_{i=1}^n a_i$$

and the parity condition

$$rd \equiv (r-2)(g-1) + |\sigma| + \sum_{i=1}^n a_i \mod 2.$$

Then, the degree d part of

$$\prod_{i=1}^n \boldsymbol{B}_{r,a_i}(\psi_i)\left(\sum_{m\geq 0}\frac{1}{m!}p_{\ell(\sigma)+m*}\prod_{j=1}^{\ell(\sigma)}\boldsymbol{B}_{r,\sigma_j+r}(\psi_{n+j})\prod_{k=1}^{m}(T-T\boldsymbol{B}_{r,0})(\psi_{n+\ell(\sigma)+k})\right)$$

vanishes in $H^{2d}(\mathcal{M}_{g,n})$.

[5] A partition has positive integer parts. The empty partition is permitted.

In the statement of Theorem 5, $p_{\ell(\sigma)+m}$ is (as before) the forgetful map

$$p_{\ell(\sigma)+m} : \overline{\mathcal{M}}_{g,n+\ell(\sigma)+m} \to \overline{\mathcal{M}}_{g,n}$$

forgetting the last $\ell(\sigma)+m$ points.

We use the relations of Theorem 5 to bound the Betti numbers of the tautological ring of $\mathcal{M}_g$. Let $P(n,k)$ denote the set of partitions of n of length at most k.

Theorem 6. *For $g \geq 2$ and $d \geq 0$,* $\dim_{\mathbb{Q}} RH^d(\mathcal{M}_g) \leq |P(d, g-1-d)|$.

The bound of Theorem 6 implies the results

$$\dim_{\mathbb{Q}} RH^{g-2}(\mathcal{M}_g) \leq 1\,, \qquad \dim_{\mathbb{Q}} RH^{>g-2}(\mathcal{M}_g) = 0\,. \tag{6}$$

For $d < g-2$, Theorem 6 is new (no non-trivial bounds were known before), but is not expected to be sharp. For example, Theorem 6 yields

$$\dim_{\mathbb{Q}} RH^{g-3}(\mathcal{M}_g) \leq 1 + \lfloor \frac{g-3}{2} \rfloor\,,$$

while the expectation based on calculations is

$$\dim_{\mathbb{Q}} RH^{g-3}(\mathcal{M}_g) = 1\,.$$

Though our proofs use cohomological methods, results (i) and (ii) of Janda discussed in Section 0.1 imply Theorems 4-6 all are valid in Chow. In particular, Theorem 6 yields the bound

$$\dim_{\mathbb{Q}} R^d(\mathcal{M}_g) \leq |P(d, g-1-d)|\,,$$

and (6) specializes to Looijenga's result (1).

0.8 Polynomiality

Let $a_1, \ldots, a_n$ be non-negative integers satisfying the condition

$$\sum_{i=1}^{n} a_i = 2g-2\,.$$

If $a_i \leq r-2$ for all i, then Witten's r-spin class $W^r_{g,n}(a_1, \ldots, a_n)$ is well-defined and of degree *independent* of the choice of r,

$$\mathsf{D}^r_{g,n}(a_, \ldots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^{n} a_i}{r} = g-1\,.$$

We may reasonably ask[6] here about the dependence of $W^r_{g,n}(a_1, \ldots, a_n)$ on r.

[6] The question was posed to us by P. Rossi.

Theorem 7. *For* $\sum_{i=1}^n a_i = 2g-2$,

$$r^{g-1} \cdot W^r_{g,n}(a_1, \ldots, a_n) \in RH^{g-1}(\overline{\mathcal{M}}_{g,n})$$

is a polynomial in r *for all sufficiently large* r.

Our proof of Theorem 7 is obtained by the analysis of the shifted CohFT $\mathsf{W}^{r,\widetilde{\tau}}$ at the point

$$\widetilde{\tau} = (0, r\phi, 0, \ldots, 0) \in V_r \, .$$

The CohFTs $\mathsf{W}^{r,\tau}$ and $\mathsf{W}^{r,\widetilde{\tau}}$ behave differently and yield different insights. While our knowledge of the R-matrix at $\widetilde{\tau}$ is *not* as explicit as at τ, the property of polynomiality is easier to see at $\widetilde{\tau}$.

In the Appendix (with F. Janda), we conjecture the constant term of the polynomial of Theorem 7 is $(-1)^g$ times the class of the closure of the locus of holomorphic differentials with zero multiplicities given by $(a_1, \ldots, a_n)$. In addition to a precise formulation, the evidence for the conjecture and the connection to the conjectures of [8, Appendix] are discussed in the Appendix.

0.9 Plan of the paper

We start in Section 1 with the proof of Theorem 2. The result plays a basic role in our analysis of the r-spin theory in genus 0. The study of the shift

$$\tau = (0, \ldots, 0, r\phi) \in V_r$$

is presented in Section 2. Theorem 3 is proven in Section 2.2, and the corresponding R-matrix is solved in terms of hypergeometric series in Section 2.3. Section 3 concerns the tautological relations obtained from the shifted CohFT $\mathsf{W}^{r,\tau}$. Theorems 4, 5, and 6 are proven in Sections 3.1, 3.3, and 3.4 respectively. The study of the shift

$$\widetilde{\tau} = (0, r\phi, 0, \ldots, 0) \in V_r$$

is presented in Section 4. The polynomiality of the R-matrix is derived in Section 4.5, and Theorem 7 is proven in Section 4.6.

Two formulas for Witten's r-spin class are given: Theorem 8 of Section 3.1 via the τ-shift and Theorem 9 of Section 4.5 via the $\widetilde{\tau}$-shift. The Appendix (with F. Janda) conjectures a connection between Witten's r-spin class and the class of the moduli space of holomorphic differentials.

0.10 Acknowledgments

We are grateful to A. Buryak, A. Chiodo, E. Clader, C. Faber, G. Farkas, J. Guéré, F. Janda, Y.-P. Lee, P. Rossi, Y. Ruan, and S. Shadrin for discussions about Witten's r-spin theory. The *Workshop on Pixton's conjectures* at ETH Zürich in October 2014 played in an important role in our collaboration. A. P. and D. Z. have been frequent guests of the Forschungsinstitut für Mathematik (FIM) at ETHZ. The conjecture of the Appendix was formulated at the conference *Moduli spaces of holomorphic differentials* at Humboldt University in Berlin in February 2016.

R. P. was partially supported by SNF-200021143274, SNF-200020162928, ERC-2012-AdG-320368-MCSK, ERC-2017-AdG-786580-MACI, SwissMAP, and the Einstein Stiftung. A. P. was supported by a fellowship from the Clay Mathematics Institute. D. Z. was supported by the grant ANR-09-JCJC-0104-01. This project has received funding from the European Research council (ERC) under the European Union Horizon 2020 research and innovation program (grant agreement No. 786580).

1 Representations of $\mathsf{sl}_2(\mathbb{C})$

1.1 Correlators

Our goal here is to prove Theorem 2 relating Witten's class and the representation theory of $\mathsf{sl}_2 = \mathsf{sl}_2(\mathbb{C})$.

Definition 1.1. Let $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$ satisfy

$$\sum_{i=1}^{n} a_i = (n-2)r - 2\,.$$

The associated genus 0 *correlator* is

$$\big\langle a_1, \ldots, a_n \big\rangle^r = \int_{\overline{\mathcal{M}}_{0,n}} W^r_{0,n}(a_1, \ldots, a_n) \;\in \mathbb{Q}\,.$$

Since r can be deduced from $a_1, \ldots, a_n$, we will often drop the superscript and denote the correlator by $\langle a_1, \ldots, a_n \rangle$.

1.2 The WDVV equation.

Let $a, b, c, d, x_1, \dots, x_k \in \{0, \dots, r-2\}$ satisfy

$$a + b + c + d + \sum_{i=1}^{k} x_i = (k+1)r - 2\,,$$

so $W^r_{0,4+k}(a, b, c, d, x_1, \dots, x_k)$ is a class of degree k on $\overline{\mathcal{M}}_{0,4+k}$. Since the dimension of $\overline{\mathcal{M}}_{0,4+k}$ is $k+1$, we can cut $W^r_{0,k+4}(a, b, c, d, x_1, \dots, x_k)$ once with boundary divisors. If we pull-back the WDVV relation from $\overline{\mathcal{M}}_{0,4}$ to $\overline{\mathcal{M}}_{4+k}$ via the map forgetting the last k markings,

$$p_k : \overline{\mathcal{M}}_{4+k} \to \overline{\mathcal{M}}_4\,,$$

and apply the CohFT splitting axiom for Witten's class, we obtain

$$\sum_{\substack{I \sqcup J = \{1,\dots,k\} \\ * + \widehat{*} = r-2}} \big\langle x_I, a, c, * \big\rangle \big\langle x_J, b, d, \widehat{*} \big\rangle = \sum_{\substack{I \sqcup J = \{1,\dots,k\} \\ * + \widehat{*} = r-2}} \big\langle x_I, a, d, * \big\rangle \big\langle x_J, b, c, \widehat{*} \big\rangle\,, \quad (7)$$

where x_I and x_J are the insertions

$$(x_i : i \in I) \quad \text{and} \quad (x_j : j \in J)\,,$$

and $*$ and $\widehat{*}$ are non-negative integers with sum $r-2$.

Using the values (4) of the 3-points correlators, we can rewrite the WDVV equation (7) as

$$\begin{aligned}\langle a, c, b+d, x_1, \dots, x_k \rangle &+ \langle b, d, a+c, x_1, \dots, x_k \rangle \\ &= \langle a, d, b+c, x_1, \dots, x_k \rangle + \langle b, c, a+d, x_1, \dots, x_k \rangle + \mathsf{Q},\end{aligned}$$

where, by convention, a correlator vanishes if any insertion exceeds $r-2$. Here, Q is a sum of products of correlators with 4 to $k+2$ insertions each. In particular, Q vanishes unless $k \geq 2$.

Lemma 1.2. *The* 3*-point evaluations*

$$\big\langle a_1, a_2, a_3 \big\rangle = \left| \begin{array}{ll} 1 & \text{if } a_1 + a_2 + a_3 = r-2, \\ 0 & \text{otherwise} \end{array} \right.$$

together with the WDVV equation (7) *force the vanishing of all genus 0 correlators*

$$\big\langle a_1, \dots, a_n \big\rangle$$

with $n \geq 4$ *and at least one* 0 *insertion.*

Proof. Apply the WDVV equation to the class $W^r_{0,n+1}(a,0,b,0,c_1,\ldots,c_{n-3})$ with nonzero a and b. We obtain

$$\begin{aligned}\langle a+b,0,0,c_1,\ldots,c_{n-3}\rangle &+ \langle a,b,0,c_1,\ldots,c_{n-3}\rangle \\ &= \langle a,b,0,c_1,\ldots,c_{n-3}\rangle + \langle a,b,0,c_1,\ldots,c_{n-3}\rangle + \mathsf{Q}\end{aligned}$$

or, equivalently,

$$\langle a,b,0,c_1,\ldots,c_{n-3}\rangle = \langle a+b,0,0,c_1,\ldots,c_{n-3}\rangle - \mathsf{Q}. \tag{8}$$

If $n=4$, the quadratic term Q vanishes. Hence, we can use (8) to increase the number of 0 insertions from 1 to 2 and then 2 to 3. The nonzero insertion of the correlator $\big\langle x,0,0,0\big\rangle$ must satisfy

$$x = 2r-2 > r-2\,.$$

Hence $\big\langle x,0,0,0\big\rangle = 0$, and the Lemma is proved for $n=4$.

Assume $n \geq 5$. In every term of Q in (8), there is a correlator with fewer than n insertions and with a 0 insertion. By induction, such correlators vanish, so Q again vanishes. We may use (8) as before to increase the number of 0 entries until we arrive at

$$\big\langle x,\underbrace{0,0,0}_{n-1}\big\rangle \tag{9}$$

Then $x=(n-2)r-2$ exceeds $r-2$, and the correlator (9) vanishes. $\square$

Lemma 1.3. *The correlators*

$$\big\langle a_1,a_2,a_3\big\rangle = \left|\begin{array}{ll} 1 & \text{if } a_1+a_2+a_3=r-2, \\ 0 & \text{otherwise}\end{array}\right.$$

$$\big\langle r-2,r-2,1,1\big\rangle = \frac{1}{r}$$

and the WDVV equation (7) *uniquely determine the values of* all *the other correlators in genus 0.*

Proof. Consider a correlator with $n \geq 4$ insertions. If the correlator has at least one 0 insertion, the correlator vanishes by Lemma 1.2. If not, all insertions are *positive.*

Let $\langle a, b, c, d_1, \ldots, d_{n-3} \rangle$ be a correlator with positive insertions ordered in a non-increasing sequence. Assume also $c \geq 2$. For $n \geq 5$, the condition $c \geq 2$ is automatic, while for $n = 4$, the condition excludes only the single correlator

$$\langle r-2, r-2, 1, 1 \rangle = \frac{1}{r}\,.$$

By applying the WDVV equation (7) to Witten's r-spin class

$$W^r_{0,n+1}(a, 1, c-1, b, d_1, \ldots, d_{n-3})\,,$$

we obtain the relation

$$\begin{aligned} &\langle a+c-1, b, 1, d_1, \ldots, d_{n-3} \rangle + \langle a, b+1, c-1, d_1, \ldots, d_{n-3} \rangle \\ &\qquad = \langle a+b, c-1, 1, d_1, \ldots, d_{n-3} \rangle + \langle a, b, c, d_1, \ldots, d_{n-3} \rangle + \mathsf{Q}\,. \end{aligned} \tag{10}$$

If we order the insertions of the correlators in non-increasing order, then our original correlator $\langle a, b, c, d_1, \ldots, d_{n-3} \rangle$ is lesser than the three others principal correlators of (10) in lexicographic order. Thus every correlator except $\langle r-2, r-2, 1, 1 \rangle$ can be expressed via correlators greater in lexicographic order and correlators with fewer entries. $\square$

1.3 Proof of Theorem 2.

A straightforward calculation of the sl_2 side shows that the formula of Theorem 2 is true for $n \leq 4$. By Lemma 1.3, the WDVV equation (7) uniquely determines the correlators for $n \geq 5$. To finish the proof, we need only to show that the sl_2 side of the formula of Theorem 2 satisfies the WDVV equation (7).

The WDVV equation (7) starts with a choice of insertions

$$a, b, c, d, x_1, \ldots, x_k\,.$$

For convenience, we replace the above insertions by their $r-2$ complements:

$$a \mapsto r-2-a\,, \quad b \mapsto r-2-b\,, \quad c \mapsto r-2-c\,, \quad d \mapsto r-2-d\,, \quad x_i \mapsto r-2-x_i\,.$$

Then, we have

$$a + b + c + d + \sum_{i=1}^{k} x_i = 3r - 6 - 2k\,.$$

We must prove the identity

$$\begin{aligned}\sum_{I\sqcup J=\{1,\dots,k\}} &|I|!|J|!\Big\langle x_I, a, c, 2r-4-a-c-2|I|-\sum_{i\in I} x_i\Big\rangle^{\mathsf{sl}_2}\\ &\qquad\cdot\Big\langle x_J, b, d, 2r-4-b-d-2|J|-\sum_{j\in J} x_j\Big\rangle^{\mathsf{sl}_2}\\ =\sum_{I\sqcup J=\{1,\dots,k\}} &|I|!|J|!\Big\langle x_I, a, d, 2r-4-a-d-2|I|-\sum_{i\in I} x_i\Big\rangle^{\mathsf{sl}_2}\\ &\qquad\cdot\Big\langle x_J, b, c, 2r-4-b-c-2|J|-\sum_{j\in J} x_j\Big\rangle^{\mathsf{sl}_2},\end{aligned}$$

where we define

$$\Big\langle a_1,\dots,a_k\Big\rangle^{\mathsf{sl}_2} = \dim\Big[\rho_{a_1}\otimes\cdots\otimes\rho_{a_k}\Big]^{\mathsf{sl}_2}.$$

The following formula for the dimensions of the sl_2-invariants follows easily from the multiplication rule for the ρ_a:

$$\Big\langle a_1,\dots,a_n, 2s-a_1-\cdots-a_n\Big\rangle^{\mathsf{sl}_2} = \left[(1-t^{-1})\prod_{i=1}^{n}\frac{1-t^{a_i+1}}{1-t}\right]_{t^s}.$$

Applying the above formula, we can rewrite the desired identity as

$$\begin{aligned}&\sum_{I\sqcup J=\{1,\dots,k\}}|I|!|J|!\left[\frac{(1-t^{a+1})(1-t^{c+1})(1-u^{b+1})(1-u^{d+1})}{(1-t)(1-u)}\prod_{i\in I}\frac{t-t^{x_i+2}}{1-t}\prod_{j\in J}\frac{u-u^{x_j+2}}{1-u}\right]_{t^{r-1}u^{r-1}}\\ &=\sum_{I\sqcup J=\{1,\dots,k\}}|I|!|J|!\left[\frac{(1-t^{a+1})(1-t^{d+1})(1-u^{b+1})(1-u^{c+1})}{(1-t)(1-u)}\prod_{i\in I}\frac{t-t^{x_i+2}}{1-t}\prod_{j\in J}\frac{u-u^{x_j+2}}{1-u}\right]_{t^{r-1}u^{r-1}}.\end{aligned}$$

Subtracting one side from the other and moving factors outside the sum, we must, equivalently, show that the coefficient of $t^{r-1}u^{r-1}$ in

$$\begin{aligned}(1-t^{a+1})(1-u^{b+1})&\frac{(1-t^{c+1})(1-u^{d+1})-(1-t^{d+1})(1-u^{c+1})}{(1-t)(1-u)}\\ &\cdot\sum_{I\sqcup J=\{1,\dots,k\}}|I|!|J|!\prod_{i\in I}\frac{t-t^{x_i+2}}{1-t}\prod_{j\in J}\frac{u-u^{x_j+2}}{1-u}\end{aligned}$$

vanishes. Interchanging t and u and adding, we can replace the latter polynomial by a symmetric one:

$$\left((1-t^{a+1})(1-u^{b+1})-(1-t^{b+1})(1-u^{a+1})\right)\frac{(1-t^{c+1})(1-u^{d+1})-(1-t^{d+1})(1-u^{c+1})}{(1-t)(1-u)}$$

$$\cdot\sum_{I\sqcup J=\{1,\dots,k\}}|I|!|J|!\prod_{i\in I}\frac{t-t^{x_i+2}}{1-t}\prod_{j\in J}\frac{u-u^{x_j+2}}{1-u}\,.$$

Next, we apply the change of variables

$$t\mapsto\frac{t}{v}\quad\text{and}\quad u\mapsto\frac{u}{v}$$

and multiply the outcome by v^m where

$$m=a+b+c+d+3+\sum_{i=1}^{k}(x_i+2)=3r-3\,.$$

Then, we replace a,b,c,d by $a-1,b-1,c-1,d-1$ and x_i by x_i-2. After these transformations, the required identity is that the coefficient of $t^{r-1}u^{r-1}v^{r-1}$ vanishes in the polynomial

$$\frac{(t^au^b-u^at^b-t^av^b+v^at^b+u^av^b-v^au^b)(t^cu^d-u^dt^c-t^cv^d+v^ct^d+u^cv^d-v^cu^d)}{t^2u-u^2t-t^2v+v^2t+u^2v-v^2u}$$

$$\cdot(t-u)v\sum_{I\sqcup J=\{1,\dots,k\}}|I|!|J|!\prod_{i\in I}\frac{t^{x_i}v-v^{x_i}t}{t-v}\prod_{j\in J}\frac{u^{x_j}v-v^{x_j}u}{u-v}\,.$$

The initial factor is invariant under cyclically permuting t,u,v. So we can instead prove the vanishing of the coefficient of $t^{r-1}u^{r-1}v^{r-1}$ in the polynomial

$$\frac{(t^au^b-u^at^b-t^av^b+v^at^b+u^av^b-v^au^b)(t^cu^d-u^dt^c-t^cv^d+v^ct^d+u^cv^d-v^cu^d)}{t^2u-u^2t-t^2v+v^2t+u^2v-v^2u}$$

$$\cdot\sum_{\text{cyc}}(t-u)v\sum_{I\sqcup J=\{1,\dots,k\}}|I|!|J|!\prod_{i\in I}\frac{t^{x_i}v-v^{x_i}t}{t-v}\prod_{j\in J}\frac{u^{x_j}v-v^{x_j}u}{u-v}\,,$$

where the sum in the second term is over the three cyclic permutations of t,u,v. In fact, we claim a stronger vanishing:

$$\sum_{\text{cyc}}(t-u)v\sum_{I\sqcup J=\{1,\dots,k\}}|I|!|J|!\prod_{i\in I}\frac{t^{x_i}v-v^{x_i}t}{t-v}\prod_{j\in J}\frac{u^{x_j}v-v^{x_j}u}{u-v}=0 \qquad (11)$$

for any $x_i \in \mathbb{Z}$.

The last step in the proof of Theorem 2 is to show the vanishing (11). Let $A \sqcup B \sqcup C = \{1, \ldots, k\}$ be a partition of the $\{x_i\}$ into three sets of sizes

$$a = |A|\,, \quad b = |B|\,, \quad c = |C|\,.$$

We can compute the coefficient of $\prod_{i\in A} t^{x_i} \prod_{i\in B} u^{x_i} \prod_{i\in C} v^{x_i}$ on the left side (11):

$$\sum_{\text{cyc}} (t-u)v \sum_{i+j=c} \binom{c}{i} (i+a)!(j+b)! \left(\frac{v}{t-v}\right)^a \left(\frac{v}{u-v}\right)^b \left(\frac{-t}{t-v}\right)^i \left(\frac{-u}{u-v}\right)^j ,$$

where now a, b, c are also cyclically permuted in correspondence with t, u, v. Multiplying by $(-1)^{a+b+c}(t-u)^{a+b}(u-v)^{b+c}(v-t)^{c+a}$ and dividing by $a!b!c!$ then yields

$$\sum_{\text{cyc}} \sum_{i+j=c} (-1)^{b+i} \binom{a+i}{i} \binom{b+j}{j} t^i u^j v^{a+b+1} (t-u)^{a+b+1} (u-v)^i (v-t)^j \,. \tag{12}$$

Now set $z_1 = t(u-v)$, $z_2 = u(v-t)$, $z_3 = v(t-u)$, multiply by $y_1^a y_2^b y_3^c$, and sum (12) over all non-negative integers a, b, c. The result is

$$\sum_{\text{cyc}} \frac{z_3}{(1 + z_1 y_3 - z_3 y_1)(1 + z_3 y_2 - z_2 y_3)}\,,$$

which expands to

$$\frac{z_1 + z_2 + z_3}{(1 + z_1 y_3 - z_3 y_1)(1 + z_2 y_1 - z_1 y_2)(1 + z_3 y_2 - z_2 y_3)}\,. \tag{13}$$

Since $z_1 + z_2 + z_3 = tu - tv + uv - ut + vt - vu = 0$, so (13) vanishes, and the identity (11) holds. $\square$

Proposition 1.4. *A genus 0, n-point correlator vanishes if there is an insertion less than $n-3$.*

Proof. Consider the correlator

$$\left\langle a_1, \ldots, a_n \right\rangle, \qquad \sum_{i=1}^n a_i = r(n-2) - 2\,. \tag{14}$$

Let $b_i = r-2-a_i$, so $\sum_{i=1}^n b_i = 2(r+1-n)$. If one of the b_i is greater than the sum of all others, then the representation $\otimes_{i=1}^n \rho_{b_i}$ has no invariant vectors and the correlator (14) vanishes by Theorem 2. Thus the greatest possible value of b_i in a nonzero correlator is equal to $r+1-n$. In other words, the smallest possible value of a_i is $n-3$. $\square$

2 The semisimple point $\tau = (0, \ldots, 0, r\phi)$

2.1 Special shift

The last basis vector $e_{r-2} \in V_r$ plays a special role in Witten's r-spin theory: e_{r-2} corresponds, via the formula of Theorem 2, to the trivial representation of sl_2. The shift along the last coordinate yields simpler expressions for the quantum product and the R-matrix which are calculated in Sections 2.2 and 2.3 respectively. The associated topological field theory is calculated in Sections 2.4 and 2.5.

2.2 The quantum product (proof of Theorem 3)

We study the Frobenius algebra at the point

$$\tau = (0, \ldots, 0, r\phi) \in V_r$$

and find an isomorphism with the Verlinde fusion algebra for sl_2 of level r.

Let $a+b+c = r-2+2k$. By Theorem 2, we have

$$W^r_{0,3+k}(a,b,c,\underbrace{r-2,\ldots,r-2}_{k}) = \left| \begin{array}{ll} \frac{k!}{r^k} & \text{if} \quad \min(a,b,c) \geq k, \\ 0 & \text{otherwise.} \end{array} \right. \tag{15}$$

The *quantum product* at τ is given by

$$\partial_a \bullet_\tau \partial_b = \sum_{k=\max(0,a+b-r+2)}^{\min(a,b)} \phi^k \partial_{a+b-2k} \,.$$

The $k!$ in the evaluation (15) is cancelled by the $k!$ in the denominator of the Gromov-Witten potential, and the r^k in the evaluation is cancelled by r factor in the last coordinate of τ.

To simplify the computations, we introduce the new frame[7]

$$\widehat{\partial}_a = \phi^{-(2a-r+2)/4} \partial_a \,.$$

Then, the quantum product assumes a slightly simpler form:

$$\widehat{\partial}_a \bullet_\tau \widehat{\partial}_b = \phi^{(r-2)/4} \sum_{k=\max(0,a+b-r+2)}^{\min(a,b)} \widehat{\partial}_{a+b-2k}. \tag{16}$$

We have proven the following result.

Proposition 2.1. *The coefficient of $\widehat{\partial}_c$ in the quantum product $\widehat{\partial}_a \bullet_\tau \widehat{\partial}_b$ at the point τ is equal to either to $\phi^{(r-2)/4}$ or to 0. The coefficient is equal to $\phi^{(r-2)/4}$ if and only if $a+b+c$ is even and the point (a,b,c) lies inside the tetrahedron*

$$\begin{aligned} &a+b \geq c, \\ &a+c \geq b, \\ &b+c \geq a, \\ &a+b+c \leq 2r-4. \end{aligned}$$

The tetrahedron is represented in Figure 1. To help the reader visualize the constraint in Propositions 2.1, we have represented on the right the values of a and b for which the quantum product $\widehat{\partial}_a \bullet \widehat{\partial}_b$ contains the term $\widehat{\partial}_c$ for a fixed c (a horizontal section of the tetrahedron with the parity condition taken into account).

The *Verlinde algebra* of level r for sl_2 is spanned by the weights of sl_2 from 0 to $r-2$. The coefficient of c in the product $a \bullet b$ is equal to the dimension of the sl_2-invariant subspace of the representation $\rho_a \otimes \rho_b \otimes \rho_c$ provided the inequality

$$a+b+c \leq 2r-4$$

is satisfied. The invariant subspace is easily seen to always have dimension 0 or 1. It is of dimension 1 if and only if $a+b+c$ is even and (a,b,c) satisfy the triangle inequalities. The structure constants of the Verlinde algebra therefore coincide with those of the Frobenius algebra at $\tau = (0,\ldots,0,r)$. We have completed the proof of Theorem 3. $\square$

[7] The metric is $\eta(\widehat{\partial}_a, \widehat{\partial}_b) = \delta_{a+b,r+2}$.

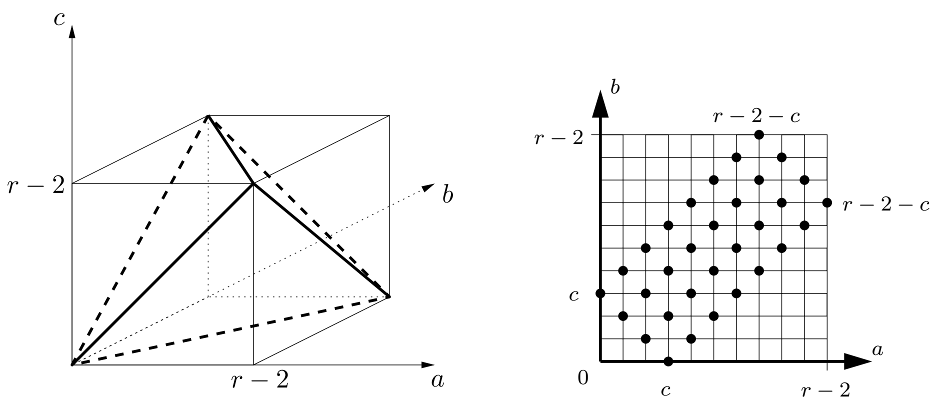


Figure 1: Quantum product: nonzero coefficients

Although the algebra structure of the Verlinde and Frobenius algebras are the same, the quadratic forms are different. The quadratic form for the Verlinde algebra is given by

$$\eta^{\mathrm{Ver}}_{a,b} = \delta_{a,b}\,,$$

while for the Frobenius manifold V_r we have

$$\eta^{\mathrm{Frob}}_{a,b} = \delta_{a+b,r-2}\,.$$

The difference is related to the large automorphism group of the tetrahedron. The symmetry under permutations of a, b, c of the structure constants of the algebra is expected. However, the tetrahedron has an extra symmetry obtained by replacing every a by $r-2-a$ (central symmetry) and then changing the metric from η^{Ver} to η^{Frob} (a vertical flip).

2.3 The R-matrix

We compute here the R-matrix of the Frobenius manifold V_r at the point

$$\tau = (0, \dots, 0, r\phi)\,.$$

We do not know closed form expressions for the coefficients of the R-matrix at *any* other point of V_r.

The operator of quantum multiplication by the Euler field at τ,

$$E = 2\phi^{(r+2)/4}\,\widehat{\partial}_{r-2}\,,$$

is given in the frame $\{\widehat{\partial}_a\}$ by the matrix

$$\xi = 2\phi^{r/2}\begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 0 & & 0 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & 0 & & 0 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

The nonvanishing coefficients[8] are

$$\xi^a_{r-2-a} = 2\phi^{r/2}\,.$$

In the same frame, the shifted degree operator is

$$\mu = \frac{1}{2r}\begin{pmatrix} -(r-2) & 0 & \cdots & \cdots & 0 \\ 0 & -(r-4) & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & r-4 & 0 \\ 0 & \cdots & \cdots & 0 & r-2 \end{pmatrix}.$$

The nonvanishing coefficients are

$$\mu^a_a = \frac{2a-r+2}{2r}\,.$$

Recall the hypergeometric series defined in Section 0.7 for every integer $a \in \{0, \ldots, r-2\}$,

$$\boldsymbol{B}_{r,a}(T) = \sum_{m=0}^{\infty}\left[\prod_{i=1}^{m}\frac{\big((2i-1)r-2(a+1)\big)\big((2i-1)r+2(a+1)\big)}{i}\right]\left(-\frac{T}{16r^2}\right)^m.$$

[8]In our convention for matrix coefficients, the superscript is the row index and the subscript is the column index.

We denote by $\boldsymbol{B}^{\text{even}}_{r,a}$ and $\boldsymbol{B}^{\text{odd}}_{r,a}$ the even and odd parts of the series $\boldsymbol{B}_{r,a}$.

Proposition 2.2. *The unique solution $R(z) = \sum_{m=0}^{\infty} R_m z^m \in \mathrm{End}(V_r)[[z]]$ of the equations*

$$[R_{m+1}, \xi] = (m+\mu) R_m$$

with the initial condition $R_0 = 1$ has coefficients

$$R_a^a = \boldsymbol{B}^{\text{even}}_{r,r-2-a}(\phi^{-r/2} z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the main diagonal,

$$R_a^{r-2-a} = -\boldsymbol{B}^{\text{odd}}_{r,a}(\phi^{-r/2} z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the antidiagonal (if r is even, the coefficient at the intersection of both diagonals is 1), and 0 everywhere else.

The inverse matrix $R^{-1}(z)$ has coefficients

$$(R^{-1})_a^a = \boldsymbol{B}^{\text{even}}_{r,a}(\phi^{-r/2} z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the main diagonal,

$$(R^{-1})_a^{r-2-a} = \boldsymbol{B}^{\text{odd}}_{r,a}(\phi^{-r/2} z)\,, \qquad a \in \{0, \dots, r-2\}$$

on the antidiagonal (if r is even, the coefficient at the intersection of both diagonals is 1), and 0 everywhere else.

Proof. The *uniqueness* of the solution follows from the semisimplicity of the Frobenius manifold V_r at τ (proven in Section 2.4). We verify here that the R-matrix described in the Proposition is indeed a solution of the recursion

$$[R_{m+1}, \xi] = (m+\mu) R_m \tag{17}$$

with initial condition[9] $R_0 = 1$.

The coefficients of the commutator on the left side of (17) are given by

$$[R_{m+1}, \xi]_b^a = 2\phi^{r/2} \Big((R_{m+1})^a_{r-2-b} - (R_{m+1})_b^{r-2-a} \Big)\,. \tag{18}$$

The right side of (18) vanishes unless $a = b$ or $a + b = r - 2$. The same is true of R_m and therefore of $(m+\mu)R_m$, since μ is a diagonal matrix. Thus, we have two cases to consider.

[9] Here, $R_0 = 1$ denotes the identity in $\mathrm{End}(V_r)$.

Case $a = b$**.** We have

$$\begin{aligned}
[R_{m+1},\xi]_a^a &= 2\phi^{r/2}\big((R_{m+1})^a_{r-2-a} - (R_{m+1})_a^{r-2-a}\big)\\
&= 2\phi^{r/2}[z^{m+1}]\big(-\boldsymbol{B}^{\mathrm{odd}}_{r,r-2-a}(\phi^{-r/2}z) + \boldsymbol{B}^{\mathrm{odd}}_{r,a}(\phi^{-r/2}z)\big)\,.
\end{aligned}$$

Using the definition of the $\boldsymbol{B}^{\mathrm{odd}}$ series, the last expression is

$$\delta_m^{\mathrm{even}}\, 2\phi^{r/2} \prod_{i=1}^{m+1} \frac{\big((2i-1)r - 2(a+1)\big)\big((2i-1)r + 2(a+1)\big)}{i} \cdot \left(\frac{(3+2m)r - 2(a+1)}{(2m+1)r + 2(a+1)} + 1\right)\left(-\frac{1}{16r^2\,\phi^{r/2}}\right)^{m+1},$$

or, equivalently,

$$\delta_m^{\mathrm{even}}\, 2\phi^{r/2} \prod_{i=1}^{m+1} \frac{\big((2i-1)r - 2(a+1)\big)\big((2i-1)r + 2(a+1)\big)}{i} \cdot \left(\frac{4(m+1)r}{(2m+1)r + 2(a+1)}\right)\left(-\frac{1}{16r^2\,\phi^{r/2}}\right)^{m+1}.$$

After further simplification, we conclude

$$[R_{m+1},\xi]_a^a = -\delta_m^{\mathrm{even}}\left(\frac{(2m+1)r - 2(a+1)}{2r}\right)\cdot [z^m]\boldsymbol{B}_{r,a}\,.$$

On the other hand, we have

$$\begin{aligned}
&[(m+\mu)R_m]_a^a = \\
&\quad= \left(m + \frac{2a - r + 2}{2r}\right)(R_m)_a^a\\
&\quad= \delta_m^{\mathrm{even}}\left(\frac{(2m-1)r + 2a + 2}{2r}\right)\cdot [z^m]\boldsymbol{B}_{r,r-2-a}\\
&\quad= \delta_m^{\mathrm{even}}\left(\frac{(2m-1)r + 2a + 2}{2r}\right)\left(\frac{2(a+1) - (1+2m)r}{2(a+1) - (1-2m)r}\right)\cdot [z^m]\boldsymbol{B}_{r,a}\\
&\quad= -\delta_m^{\mathrm{even}}\left(\frac{(2m+1)r - 2(a+1)}{2r}\right)\cdot [z^m]\boldsymbol{B}_{r,a}\,.
\end{aligned}$$

Therefore, equation (17) is satisfied.

Case $a+b=r-2$. We have

$$\begin{aligned}[][R_{m+1},\xi]_a^{r-2-a} &= 2\phi^{r/2}\big((R_{m+1})_{r-2-a}^{r-2-a}-(R_{m+1})_a^a\big)\\ &= 2\phi^{r/2}[z^{m+1}]\big(\boldsymbol{B}^{\mathrm{even}}_{r,r-2-a}(\phi^{-r/2}z)-\boldsymbol{B}^{\mathrm{even}}_{r,a}(\phi^{-r/2}z)\big)\,.\end{aligned}$$

Using the definition of the $\boldsymbol{B}^{\mathrm{odd}}$ series, the last expression is

$$\delta_m^{\mathrm{odd}}\,2\phi^{r/2}\prod_{i=1}^{m+1}\frac{\Big((2i-1)r-2(a+1)\Big)\Big((2i-1)r+2(a+1)\Big)}{i}\\ \cdot\left(\frac{2(a+1)-(3+2m)r}{2(a+1)+(2m+1)r}-1\right)\left(-\frac{1}{16r^2\,\phi^{r/2}}\right)^{m+1}\,,$$

or, equivalently,

$$-\,\delta_m^{\mathrm{odd}}\,2\phi^{r/2}\prod_{i=1}^{m+1}\frac{\Big((2i-1)r-2(a+1)\Big)\Big((2i-1)r+2(a+1)\Big)}{i}\\ \cdot\left(\frac{4(m+1)r}{(2m+1)r+2(a+1)}\right)\left(-\frac{1}{16r^2\,\phi^{r/2}}\right)^{m+1}\,.$$

After further simplification, we conclude

$$[R_{m+1},\xi]_a^a=\delta_m^{\mathrm{odd}}\,\left(\frac{(2m+1)r-2(a+1)}{2r}\right)\cdot[z^m]\boldsymbol{B}_{r,a}\,.$$

On the other hand, we have

$$\begin{aligned}[][(m+\mu)R_m]_a^{r-2-a} &= \left(m-\frac{2a-r+2}{2r}\right)(R_m)_a^{r-2-a}\\ &= \delta_m^{\mathrm{odd}}\frac{(2m+1)r-2(a+1)}{2r}\cdot[z^m]\boldsymbol{B}_{r,a}\,.\end{aligned}$$

Therefore, equation (17) is satisfied.

The expression for R^{-1} is obtained from the symplectic condition[10]

$$R^{-1}(z)=R^*(-z)$$

[10] R^* denotes the adjoint with respect to the metric η on V_r.

for the R-matrix. The symplectic condition implies the identities

$$\boldsymbol{B}_{r,a}(T)\boldsymbol{B}_{r,r-2-a}(-T) + \boldsymbol{B}_{r,a}(-T)\boldsymbol{B}_{r,r-2-a}(T) = 2,$$

or, equivalently,

$$\boldsymbol{B}^{\mathrm{even}}_{r,a}(T)\boldsymbol{B}^{\mathrm{even}}_{r,r-2-a}(T) - \boldsymbol{B}^{\mathrm{odd}}_{r,a}(T)\boldsymbol{B}^{\mathrm{odd}}_{r,r-2-a}(T) = 1.$$

Of course, these identities can also be proved directly. □

2.4 The topological field theory

We study next the topological field theory arising from the Frobenius algebra of V_r at the point

$$\tau = (0, \ldots, 0, r\phi)\,.$$

Proposition 2.3. *The basis of normalized idempotents of the quantum product* (16) *is given by*

$$v_k = \sqrt{\frac{2}{r}}\,\sum_{a=0}^{r-2} \sin\left(\frac{(a+1)k\pi}{r}\right)\,\widehat{\partial}_a, \qquad k \in \{1, \ldots, r-1\}\,.$$

More precisely, we have

$$\eta(v_k, v_l) = (-1)^{k-1}\delta_{k,l}\,, \qquad v_k \bullet_\tau v_l = \phi^{(r-2)/4}\frac{\sqrt{r/2}}{\sin(\frac{k\pi}{r})} v_k\,\delta_{k,l}\,.$$

Proof. The idempotents of the Verlinde algebras are known for all semisimple Lie algebras and all levels [33]. The idempotents are automatically orthogonal to each other, so we only need to compute their scalar squares. Since the extraction of the elementary formulas for sl_2 from the general case is not so simple, we check the statements of the Proposition independently by a series of elementary computations with trigonometric functions in Section 2.5. □

Proposition 2.4. *For $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$, we have*

$$\omega^{r,\tau}_{g,n}(\widehat{\partial}_{a_1} \otimes \cdots \otimes \widehat{\partial}_{a_n}) =$$

$$\left(\frac{r}{2}\right)^{g-1} \phi^{(r-2)(2g-2+n)/4} \sum_{k=1}^{r-1} \frac{(-1)^{(k-1)(g-1)} \prod_{i=1}^{n} \sin\left(\frac{(a_i+1)k\pi}{r}\right)}{\left(\sin(\frac{k\pi}{r})\right)^{2g-2+n}}\,.$$

Proof. The topological field theory $\omega_{g,n}^{r,\tau}$ can be computed by restricting the τ-shifted r-spin theory $\mathsf{W}_{g,n}^{r,\tau}$ to

$$[C] \in \overline{\mathcal{M}}_{g,n}\,,$$

where C is a completely degenerate curve with $2g-2+n$ rational components and $3g-3+n$ nodes.

By Proposition 2.3, $\omega_{0,3}^{r,\tau}(v_i \otimes v_j \otimes v_k)$ vanishes unless the three indices coincide. For equal indices, we have

$$\omega_{0,3}^{r,\tau}(v_k \otimes v_k \otimes v_k) = (-1)^{k-1}\phi^{(r-2)/4}\frac{\sqrt{r/2}}{\sin(\frac{k\pi}{r})}\,. \tag{19}$$

After applying the splitting axioms of the CohFT to

$$\omega_{g,n}^{r,\tau}(v_k \otimes \cdots \otimes v_k)\Big|_{[C]}\,,$$

each node contributes the sign $(-1)^{k-1}$. Including the sign in (19) associated to each rational component, the total sign is

$$(-1)^{(k-1)(2g-2+n)} \cdot (-1)^{(k-1)(3g-3+n)} = (-1)^{(k-1)(g-1)}\,.$$

Hence, for $k \in \{1, \ldots, r-1\}$,

$$\omega_{g,n}^{r,\tau}(v_k \otimes \cdots \otimes v_k) = \left(\sqrt{\frac{r}{2}}\right)^{2g-2+n} \phi^{(r-2)(2g-2+n)/4}\,\frac{(-1)^{(k-1)(g-1)}}{\left(\sin(\frac{k\pi}{r})\right)^{2g-2+n}}\,. \tag{20}$$

The change of basis matrix from $\widehat{\partial}_a$ to v_k is self-inverse.[11] Therefore, we have

$$\widehat{\partial}_a = \sqrt{\frac{2}{r}}\,\sum_{k=1}^{r-1}\sin\left(\frac{(a+1)k\pi}{r}\right)\,v_k\,.$$

The Proposition then follows from (20). □

2.5 Proof of Proposition 2.3.

We provide here direct proofs of the claims of Proposition 2.3. The methods are via simple manipulation of trigonometric functions, but we were not able to find an elementary reference.

[11]The self-inverse property follows easily from the trigonometric identities used to calculate $\eta(v_k, v_l)$ in Section 2.3.

The scalar square $\eta(v_k, v_k)$. We have

$$\begin{aligned}
\eta(v_k, v_k) &= \frac{2}{r}\sum_{a=0}^{r-2} \sin\left(\frac{(a+1)k\pi}{r}\right) \sin\left(\frac{(r-1-a)k\pi}{r}\right) \\
&= (-1)^{k-1}\,\frac{2}{r}\sum_{a=0}^{r-2} \sin^2\left(\frac{(a+1)k\pi}{r}\right) \\
&= (-1)^{k-1}\,\frac{1}{r}\sum_{a=0}^{r-2} \left(1-\cos\left(\frac{2(a+1)k\pi}{r}\right)\right) \\
&= (-1)^{k-1}\,\frac{1}{r}\Big((r-1)-(-1)\Big) = (-1)^{k-1}\,.
\end{aligned}$$

The scalar product $\eta(v_k, v_l)$ for $k \neq l$. We have

$$\begin{aligned}
\eta(v_k, v_l) &= \frac{2}{r}\sum_{a=0}^{r-2} \sin\left(\frac{(a+1)k\pi}{r}\right) \sin\left(\frac{(r-1-a)l\pi}{r}\right) \\
&= (-1)^{l-1}\,\frac{2}{r}\sum_{a=0}^{r-2} \sin\left(\frac{(a+1)k\pi}{r}\right) \sin\left(\frac{(a+1)l\pi}{r}\right) \\
&= (-1)^{l-1}\,\frac{1}{r}\sum_{a=0}^{r-2} \left[\cos\left(\frac{(a+1)(k-l)\pi}{r}\right) - \cos\left(\frac{(a+1)(k+l)\pi}{r}\right)\right] \\
&= (-1)^{l-1}\,\frac{1}{r}\Big((-1)-(-1)\Big) = 0\,.
\end{aligned}$$

The quantum square $v_k \bullet_\tau v_k$. The coefficient of $\widehat{\partial}_c$ in $v_k \bullet_\tau v_k$ is equal to

$$\frac{2\phi^{(r-2)/4}}{r}\sum_{a,b} \sin\left(\frac{(a+1)k\pi}{r}\right) \sin\left(\frac{(b+1)k\pi}{r}\right),$$

where the sum runs over the set shown in Figure 1. The sum can be conveniently reparameterized by the change of variables

$$p = a+b+2\,, \quad q = a-b\,.$$

We obtain the following expression for the coefficient of $\widehat{\partial}_c$ in $v_k \bullet_\tau v_k$:

$$\frac{2\phi^{(r-2)/4}}{r}\sum_{p=c+2}^{2r-2-c}\sum_{q=-c}^{c}\sin\left(\frac{(p+q)k\pi}{2r}\right)\sin\left(\frac{(p-q)k\pi}{2r}\right)$$
$$=\frac{\phi^{(r-2)/4}}{r}\sum_{p=c+2}^{2r-2-c}\sum_{q=-c}^{c}\left(\cos\left(\frac{qk\pi}{r}\right)-\cos\left(\frac{pk\pi}{r}\right)\right), \quad (21)$$

where the sums run over only over integers p and q which have the same parity as c. The second line of (21) can be rewritten as

$$(r-c-1)\frac{\phi^{(r-2)/4}}{r}\sum_{q=-c}^{c}\Re\exp\left(\frac{iqk\pi}{r}\right)-(c+1)\frac{\phi^{(r-2)/4}}{r}\sum_{p=c+2}^{2r-2-c}\Re\exp\left(\frac{ipk\pi}{r}\right),$$

where $\Re$ denotes the real part. The first term equals

$$(r-c-1)\frac{\phi^{(r-2)/4}}{r}\,\Re\frac{\exp\left(\frac{i(c+1)k\pi}{r}\right)-\exp\left(-\frac{i(c+1)k\pi}{r}\right)}{\exp(\frac{ik\pi}{r})-\exp(-\frac{ik\pi}{r})},$$

and the second term equals

$$-(c+1)\frac{\phi^{(r-2)/4}}{r}\,\Re\frac{\exp\left(\frac{i(2r-1-c)k\pi}{r}\right)-\exp\left(\frac{i(c+1)k\pi}{r}\right)}{\exp(\frac{ik\pi}{r})-\exp(-\frac{ik\pi}{r})}.$$

Since $\exp\left(\frac{i(2r-1-c)k\pi}{r}\right)$ and $\exp\left(-\frac{i(c+1)k\pi}{r}\right)$ are equal, the first and second terms combine as

$$\phi^{(r-2)/4}\,\Re\frac{\exp\left(\frac{i(c+1)k\pi}{r}\right)-\exp\left(-\frac{i(c+1)k\pi}{r}\right)}{\exp(\frac{ik\pi}{r})-\exp(-\frac{ik\pi}{r})}=\phi^{(r-2)/4}\frac{\sin\frac{(c+1)k\pi}{r}}{\sin(\frac{k\pi}{r})}.$$

Hence, we obtain

$$v_k \bullet_\tau v_k=\frac{\phi^{(r-2)/4}}{\sin(\frac{k\pi}{r})}\sum_{c=0}^{r-2}\sin\left(\frac{(c+1)k\pi}{r}\right)\widehat{\partial}_c=\phi^{(r-2)/4}\frac{\sqrt{r/2}}{\sin(\frac{k\pi}{r})}v_k.$$

The quantum product $v_k \bullet_\tau v_l$ for $k \neq l$. The coefficient of $\widehat{\partial}_c$ in $v_k \bullet_\tau v_l$ is

$$\frac{2\phi^{(r-2)/4}}{r} \sum_{a,b} \sin\left(\frac{(a+1)k\pi}{r}\right) \sin\left(\frac{(b+1)l\pi}{r}\right) , \tag{22}$$

where the sum runs over the set shown in Figure 1. As before, we set

$$p = a + b + 2 \, , \quad q = a - b$$

and write (22) as

$$\frac{2\phi^{\frac{r-2}{4}}}{r} \sum_{p=c+2}^{2r-2-c} \sum_{q=-c}^{c} \sin\left(\frac{(p+q)k\pi}{2r}\right) \sin\left(\frac{(p-q)l\pi}{2r}\right) =$$
$$\frac{\phi^{\frac{r-2}{4}}}{r} \sum_{p=c+2}^{2r-2-c} \sum_{q=-c}^{c} \left[\cos\left(\frac{q(k+l)+p(k-l)}{2r}\pi\right) - \cos\left(\frac{p(k+l)+q(k-l)}{2r}\pi\right)\right] ,$$

where the sums run over only over integers p and q which have the same parity (mod 2) as c. The result can be rewritten as

$$\frac{\phi^{\frac{r-2}{4}}}{r} \sum_{p=c+2}^{2r-2-c} \sum_{q=-c}^{c} \Re\left[\exp\left(\frac{q(k+l)+p(k-l)}{2r} i\pi\right) - \exp\left(\frac{p(k+l)+q(k-l)}{2r} i\pi\right)\right] ,$$

where $\Re$ denotes the real part as before. After further transformation, we obtain

$$\frac{\phi^{(r-2)/4}}{r} \Re \sum_{p=c+2}^{2r-2-c} \sum_{q=-c}^{c} \exp\left(\frac{q(k+l)}{2r} i\pi\right) \exp\left(\frac{p(k-l)}{2r} i\pi\right)$$
$$- \frac{\phi^{(r-2)/4}}{r} \Re \sum_{p=c+2}^{2r-2-c} \sum_{q=-c}^{c} \exp\left(\frac{p(k+l)}{2r} i\pi\right) \exp\left(\frac{q(k-l)}{2r} i\pi\right) .$$

Consider the following four functions:

$$
\begin{aligned}
A = \sum_{q=-c}^{c} \exp\left(\frac{q(k+l)}{2r} i\pi\right) &= \frac{\exp\left(\frac{(c+1)(k+l)i\pi}{2r}\right) - \exp\left(-\frac{(c+1)(k+l)i\pi}{2r}\right)}{\exp\left(\frac{(k+l)i\pi}{2r}\right) - \exp\left(-\frac{(k+l)i\pi}{2r}\right)}, \\
B = \sum_{p=c+2}^{2r-2-c} \exp\left(\frac{p(k-l)}{2r} i\pi\right) &= \frac{\exp\left(\frac{(2r-1-c)(k-l)i\pi}{2r}\right) - \exp\left(\frac{(c+1)(k-l)i\pi}{2r}\right)}{\exp\left(\frac{(k-l)i\pi}{2r}\right) - \exp\left(-\frac{(k-l)i\pi}{2r}\right)} \\
&= -\frac{\exp\left(\frac{(c+1)(k-l)i\pi}{2r}\right) - \exp\left(-\frac{(c+1)(k-l)i\pi}{2r}\right)}{\exp\left(\frac{(k-l)i\pi}{2r}\right) - \exp\left(-\frac{(k-l)i\pi}{2r}\right)}, \\
C = \sum_{q=-c}^{c} \exp\left(\frac{q(k-l)}{2r} i\pi\right) &= \frac{\exp\left(\frac{(c+1)(k-l)i\pi}{2r}\right) - \exp\left(-\frac{(c+1)(k-l)i\pi}{2r}\right)}{\exp\left(\frac{(k-l)i\pi}{2r}\right) - \exp\left(-\frac{(k-l)i\pi}{2r}\right)}, \\
D = \sum_{p=c+2}^{2r-2-c} \exp\left(\frac{p(k+l)}{2r} i\pi\right) &= \frac{\exp\left(\frac{(2r-1-c)(k+l)i\pi}{2r}\right) - \exp\left(\frac{(c+1)(k+l)i\pi}{2r}\right)}{\exp\left(\frac{(k+l)i\pi}{2r}\right) - \exp\left(-\frac{(k+l)i\pi}{2r}\right)} \\
&= -\frac{\exp\left(\frac{(c+1)(k+l)i\pi}{2r}\right) - \exp\left(-\frac{(c+1)(k+l)i\pi}{2r}\right)}{\exp\left(\frac{(k+l)i\pi}{2r}\right) - \exp\left(-\frac{(k+l)i\pi}{2r}\right)}.
\end{aligned}
$$

Since $AB - CD = 0$, we conclude

$$v_k \bullet_\tau v_l = 0$$

for $k \neq l$.

3 Tautological relations

3.1 Proof of Theorem 4

By Teleman's classification [32], the semisimple CohFT $\mathsf{W}^{r,\tau}$ for

$$\tau = (0, \ldots, 0, r\phi) \in V_r$$

defined in Section 0.5 is given by Givental's action [11] of the R-matrix computed in Section 2.3 on the topological field theory $\omega^{r,\tau}$ computed in

Section 2.4. By the degree bound,

$$\begin{aligned}\deg\Big[p_{m*}\mathsf{W}^r_{g,n+m}(e_{a_1}\otimes\cdots\otimes e_{a_n}\otimes\tau\otimes\cdots\otimes\tau)\Big]&\\ \leq&\ \frac{(g-1)(r-2)+\sum a_i+m(r-2)}{r}-m\\ =&\ \mathsf{D}_{g,n}(a_1,\ldots,a_n)-\frac{2m}{r}\,,\end{aligned}$$

$\mathsf{W}^{r,\tau}_{g,n}(e_{a_1}\otimes\cdots\otimes e_{a_n})\in H^*(\overline{\mathcal{M}}_{g,n})$ is a cohomology class with top degree

$$\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)=\frac{(g-1)(r-2)+\sum a_i}{r}$$

equal to $W^r_{g,n}(a_1,\ldots,a_n)\in H^{2\mathsf{D}^r_{g,n}(\mathbf{a})}(\overline{\mathcal{M}}_{g,n})$ and lower degree terms.

Hence, for any $d>\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)$, the degree d part of the stable graph expression of

$$\Omega^{r,\tau}=R.\omega^{r,\tau}$$

vanishes. The proof of Theorem 4 is complete. □

The proof of Theorem 4 also yields an explicit calculation of Witten's r-spin class.

Theorem 8. *$W^r_{g,n}(a_1,\ldots,a_n)$ equals the degree $\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)$ part of the stable graph expression of*

$$\Omega^{r,\tau}=R.\omega^{r,\tau}$$

in $H^*(\overline{\mathcal{M}}_{g,n})$.

As a consequence of Theorem 8, we see

$$W^r_{g,n}(a_1,\ldots,a_n)\in RH^*(\overline{\mathcal{M}}_{g,n})\,. \tag{23}$$

Since Theorem 8 and the implication (23) do *not* concern relations, Janda's results do *not* apply: we do *not* have a proof of the lifts to Chow of these two statements.

3.2 An example: $g = 1$, $n = 1$

Let $r \geq 2$. Let $g = 1$, $n = 1$, $a_1 = a \in \{0, \ldots, r-2\}$, $d = 1$. We have

$$\mathsf{D}^r_{1,1}(a) = \frac{a}{r} < 1\,.$$

Thus, the degree 1 part of the stable graph expression of

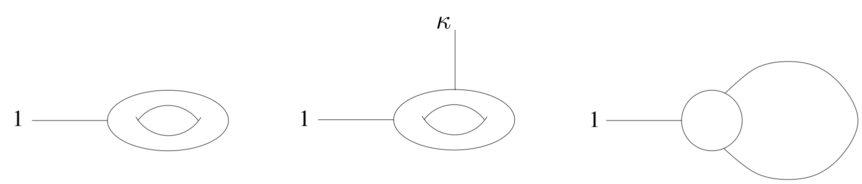
$$\Omega^{r,\tau}_{1,1}(e_a) = \phi^{(2a-r+2)/4}\,\Omega^{r,\tau}_{1,1}(\widehat{\partial}_a)$$

is a tautological relation for every a. To write the relations, we will require the coefficient

$$\boldsymbol{b}_{r,a} = \frac{(2a+2+r)(2a+2-r)}{16r^2}$$

of T in $\boldsymbol{B}_{r,a}$.

There are exactly three stable graphs[12] which contribute to the degree 1 part of $\Omega^{r,\tau}_{1,1}(\widehat{\partial}_a)$: the graph Γ_1 with one genus 1 vertex and one marked leg, the graph Γ_2 with one genus 1 vertex, one marked leg, and one κ-leg, and the graph Γ_3 with one genus 0 vertex, a leg, and a loop.



Γ_1 Γ_2 Γ_3

- Γ_1 contributes $\phi^{-r/2}\,\boldsymbol{b}_{r,a}\,\omega^{r,\tau}_{1,1}(\widehat{\partial}_{r-2-a})\;\psi_1\,,$
- Γ_2 contributes $-\phi^{-r/2}\,\boldsymbol{b}_{r,0}\,\omega^{r,\tau}_{1,2}(\widehat{\partial}_a \otimes \widehat{\partial}_{r-2})\;\kappa_1\,,$
- Γ_3 contributes $-\phi^{-r/2}\;\sum_{a'=0}^{r-2} \boldsymbol{b}_{r,r-2-a'}\,\omega^{r,\tau}_{0,3}(\widehat{\partial}_a \otimes \widehat{\partial}_{a'} \otimes \widehat{\partial}_{a'})\;\delta\,.$

Here, δ is the class Poincaré dual to the boundary. The factor $\phi^{-r/2}$ comes from the series $\boldsymbol{B}_{r,a}(\phi^{-r/2}z)$ in the R-matrix.

Using Proposition 2.4, we obtain the evaluations:

$$\begin{aligned}
\omega^{r,\tau}_{1,1}(\widehat{\partial}_{r-2-a}) &= (a+1)\phi^{(r-2)/4}\delta^{\mathrm{even}}_{r-a}\,,\\
\omega^{r,\tau}_{1,2}(\widehat{\partial}_a \otimes \widehat{\partial}_{r-2}) &= (a+1)\phi^{(r-2)/4}\delta^{\mathrm{even}}_{r-a}\,,
\end{aligned}$$

[12] We refer the reader to [25, Section 2] for a review of Givental's action of R on $\omega^{r,\tau}$.

$$\omega_{0,3}^{r,\tau}(\widehat{\partial}_a \otimes \widehat{\partial}_{a'} \otimes \widehat{\partial}_{a'}) = \left| \begin{array}{ll} \phi^{(r-2)/4} & \text{if } r-a \text{ is even and } \frac{r-2-a}{2} \leq a' \leq \frac{r-2+a}{2}, \\ 0 & \text{otherwise.} \end{array} \right.$$

Hence, we have

$$\begin{aligned} \phi^{-\frac{r}{2}}\, \boldsymbol{b}_{r,a}\, \omega_{1,1}^{r,\tau}(\widehat{\partial}_{r-2-a}) &= \phi^{-\frac{r+2}{4}}\, \frac{(2a+2+r)(2a+2-r)(a+1)}{16r^2} \delta_{r-a}^{\text{even}}\,, \\ -\phi^{-\frac{r}{2}}\, \boldsymbol{b}_{r,0}\, \omega_{1,2}^{r,\tau}(\widehat{\partial}_a \otimes \widehat{\partial}_{r-2}) &= \phi^{-\frac{r+2}{4}}\, \frac{(r+2)(r-2)(a+1)}{16r^2} \delta_{r-a}^{\text{even}}\,, \end{aligned}$$

$$\begin{aligned} &-\phi^{-\frac{r}{2}} \sum_{a'=0}^{r-2} \boldsymbol{b}_{r,r-2-a'} \omega_{0,3}^{r,\tau}(\widehat{\partial}_a \otimes \widehat{\partial}_{a'} \otimes \widehat{\partial}_{a'}) \\ &\qquad = -\phi^{-\frac{r+2}{4}}\, \delta_{r-a}^{\text{even}} \sum_{a'=(r-2-a)/2}^{(r-2+a)/2} \frac{(2a+2-r)(2a+2-3r)}{16r^2} \\ &\qquad\qquad = -\phi^{-\frac{r+2}{4}}\, \frac{a(a+1)(a+2)}{48r^2} \delta_{r-a}^{\text{even}}\,. \end{aligned}$$

After dividing by the common factor $-\phi^{-(r+2)/4}\, \dfrac{(a+1)}{16r^2}$, we obtain the following statement.

Proposition 3.1. *For $a \in \{0, \ldots, r-2\}$ of the same parity as r, we have*

$$(r-2a-2)(r+2a+2) \cdot \psi_1 - (r-2)(r+2) \cdot \kappa_1 + \frac{a(a+2)}{3} \cdot \delta = 0 \in H^2(\overline{\mathcal{M}}_{1,1}).$$

After regrouping the terms, we write the relation of Proposition 3.1 as

$$(r^2-4)(\psi_1 - \kappa_1) + (2a + a^2)\left(\frac{\delta}{3} - 4\psi_1\right) = 0\,,$$

which can only be satisfied for all the possible choices of a and r if

$$\frac{\delta}{12} = \kappa_1 = \psi_1.$$

3.3 Restriction to $\mathcal{M}_{g,n}$

The tautological relations of Theorem 4 become much simpler when restricted to the interior $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ as the graph sum is then reduced to a single term. In order to prove Theorem 5, we will alter the relations slightly before restricting to the interior.

Let $\sigma, a_1, \ldots, a_n$ and d be as in the statement of Theorem 5, so

$$rd > (r-2)(g-1) + |\sigma| + \sum_{i=1}^{n} a_i \,.$$

Since a_i and σ_j avoid $r-1$ mod r, we can write

$$a_i = b_i + rc_i \,, \qquad \sigma_j = b_{n+j} + rc_{n+j}$$

with $0 \leq b_i \leq r-2$ for all $1 \leq i \leq n+\ell(\sigma)$. For $C = \sum_{i=1}^{n+\ell(\sigma)} c_i$, we have

$$r(d-C) > (r-2)(g-1) + \sum_{i=1}^{n+\ell(\sigma)} b_i \,.$$

By Theorem 4 the degree $d-C$ part of

$$\Omega^{r,\tau}_{g,n+\ell(\sigma)}(\widehat{\partial}_{b_1} \otimes \cdots \otimes \widehat{\partial}_{b_{n+\ell(\sigma)}})$$

yields a tautological relation on $\overline{\mathcal{M}}_{g,n+\ell(\sigma)}$ which we write as

$$X = 0 \in H^{2(d-C)}(\overline{\mathcal{M}}_{g,n+\ell(\sigma)}) \,. \tag{24}$$

Push-forward yields a tautological relation of degree d on $\overline{\mathcal{M}}_{g,n}$,

$$p_{\ell(\sigma)*}\left(\prod_{j=1}^{\ell(\sigma)} \psi_{n+j}^{c_{n+j}+1} \cdot X\right) \;=\; 0 \;\in\; H^{2d}(\overline{\mathcal{M}}_{g,n}) \,. \tag{25}$$

We restrict (25) to the interior to obtain a tautological relation of degree d on $\mathcal{M}_{g,n}$.

The only stable graph for $\overline{\mathcal{M}}_{g,n+\ell(\sigma)}$ which contributes to the relation (24) is the principal graph $\Gamma_\bullet$ with a single vertex and no edges. All other strata classes are either annihilated by multiplying by $\prod_j \psi_{n+j}$ or remain supported on the boundary after push-forward by the forgetful map to $\overline{\mathcal{M}}_{g,n}$.

All the factors in the contribution of $\Gamma_\bullet$ match up exactly with the formula given in Theorem 5 except for the values of the topological field theory $\omega^{r,\tau}$ which are absent. A leg (or κ-leg) assigned vector $\widehat{\partial}_a$ produces the series $\boldsymbol{B}_{r,a}$

by combining the two nonzero entries in column a of the matrix R^{-1}. The topological field theory value which appears is

$$\omega^{r,\tau}_{g,n+\ell(\sigma)+m}(\widehat{\partial}_{\widehat{b}_1},\ldots,\widehat{\partial}_{\widehat{b}_{n+\ell(\sigma)}},\widehat{\partial}_{\widehat{0}},\ldots,\widehat{\partial}_{\widehat{0}}),$$

where $\widehat{x}$ represents either x or $r-2-x$ depending on whether the diagonal or antidiagonal entry in R^{-1} was chosen. The number of times an antidiagonal[13] entry is chosen is congruent mod 2 to the degree $d-C$. After specialization to $\phi=1$, the easily checked identity

$$\omega^{r,\tau}_{g,s+1}(\widehat{\partial}_{x_1},\ldots,\widehat{\partial}_{x_s},\widehat{\partial}_{r-2}) = \omega^{r,\tau}_{g,s}(\widehat{\partial}_{x_1},\ldots,\widehat{\partial}_{x_{s-1}},\widehat{\partial}_{r-2-x_s}),$$

then implies that the topological field theory value does not depend on any of the choices made (and can be divided out by the nonvanishing result below).

The parity condition in the statement of Theorem 5,

$$rd \equiv (r-2)(g-1) + |\sigma| + \sum_{i=1}^{n} a_i \mod 2\,,$$

together with parity constraint on the number of times an antidiagonal is chosen, implies

$$\sum_{i=1}^{n+\ell(\sigma)} \widehat{b}_j \equiv r(g-1) \mod 2\,.$$

Lemma 3.2. *Let $g \geq 1$, $s \geq 0$ with $2g-2+s>0$. Let $x_1,\ldots,x_s \in \{0,\ldots,r-2\}$ satisfy*

$$\sum_{i=1}^{s} x_i \equiv r(g-1) \mod 2.$$

Then, $\omega^{r,\tau}_{g,s}(x_1,\ldots,x_s) \neq 0$.

Proof. We will use the formula for $\omega^{r,\tau}_{0,3}$ given by Proposition 2.1 to induct on g and n. In fact, the argument will prove

$$\omega^{r,\tau}_{g,s}(\widehat{\partial}_{x_1},\ldots,\widehat{\partial}_{x_s}) > 0$$

[13] The parity condition here is because the antidiagonal entries are constructed from *odd* functions.

for $\phi > 0$.

Our base case is $\omega^{r,\tau}_{1,1}(x_1)$ with x_1 even. After applying the splitting axiom of the CohFT, we obtain

$$\omega^{r,\tau}_{1,1}(\widehat{\partial}_{x_1}) = \sum_{x=0}^{r-2} \omega^{r,\tau}_{0,3}(\widehat{\partial}_{x_1}, \widehat{\partial}_y, \widehat{\partial}_{r-2-y})$$

and all of the terms are non-negative by Proposition 2.1. For $y = \frac{x_1}{2}$, the point (x_1, y, y) lies in the tetrahedron described in Proposition 2.1,

$$\omega^{r,\tau}_{0,3}\left(\widehat{\partial}_{x_1}, \widehat{\partial}_{\frac{x_1}{2}}, \widehat{\partial}_{r-2-\frac{x_1}{2}}\right) > 0\,.$$

Next, we prove the case $g = 1$, $n > 1$ by induction on n. We have

$$\omega^{r,\tau}_{1,n}(\widehat{\partial}_{a_1}, \ldots, \widehat{\partial}_{a_{n-1}}, \widehat{\partial}_{a_n}) = \sum_{x=0}^{r-2} \omega^{r,\tau}_{1,n-1}(\widehat{\partial}_{a_1}, \ldots, \widehat{\partial}_{a_{n-2}}, \widehat{\partial}_x) \cdot \omega_{0,3}(\widehat{\partial}_{a_{n-1}}, \widehat{\partial}_{a_n}, \widehat{\partial}_{r-2-x}),$$

and we may assume $a_{n-1} \equiv a_n \mod 2$. As before, non-negativity means we need only find a single value of x such that $\omega_{0,3}(a_{n-1}, a_n, r-2-x)$ is nonzero. The tetrahedron constraints are satisfied for $x = |a_n - a_{n-1}|$.

Finally, we treat the case $g > 1$ by induction on g. We have

$$\omega^{r,\tau}_{g,n}(\widehat{\partial}_{a_1}, \ldots, \widehat{\partial}_{a_n}) = \sum_{x=0}^{r-2} \omega^{r,\tau}_{g-1,n+2}(\widehat{\partial}_{a_1}, \ldots, \widehat{\partial}_{a_n}, \widehat{\partial}_x, \widehat{\partial}_{r-2-x})$$

and all of the terms on the right are positive by the inductive hypothesis (since the parity condition is preserved). $\square$

3.4 Proof of Theorem 6

We will now use the relations of Theorem 5 with $r = 4$ to bound the Betti numbers of the tautological ring of $\mathcal{M}_g$.[14]

[14]The analysis here was completed by A. Pixton before our study of r-spin relations started and appears in [27]. Since there is no published reference (and for the convenience of the reader) we have included the short argument here. Several aspects are parallel to the linear algebra required in [22].

In the case $r = 4$ and $n = 0$, the relations of Corollary 5 are parameterized by partitions σ with no parts congruent to 3 mod 4 and positive integers d satisfying

$$4d > 2(g-1) + |\sigma| \quad \text{and} \quad |\sigma| \equiv 0 \pmod 2.$$

We discard the relations coming from partitions σ containing an odd part and then halve all parts of σ. The remaining relations are then simply indexed by partitions σ and positive integers d satisfying

$$2d \geq g + |\sigma|.$$

For $D_s(T) = \boldsymbol{B}_{4,2s}(T)$, the relations of Theorem 5 are obtained by taking the degree d part of

$$\sum_{m\geq 0} \frac{1}{m!}\, p_{m+\ell(\sigma)*} \prod_{j=1}^{\ell(\sigma)} (TD_{\sigma_j})(\psi_j) \prod_{k=1}^{m} (T - TD_0)(\psi_{\ell(\sigma)+k})\,. \tag{26}$$

We will alter the definition of D_1 to kill the constant term:

$$D_1(T) = \boldsymbol{B}_{4,2}(T) - \boldsymbol{B}_{4,0}(T).$$

A straightforward check shows the span of the relations (26) is unchanged by the new definition of D_1.

Proof of Theorem 6. The push-forward kappa polynomials

$$p_{\ell(\tau)*} \prod_{i=1}^{\ell(\tau)} \psi_i^{\tau_i+1}\,, \qquad p_{\ell(\tau)} : \overline{\mathcal{M}}_{g,\ell(\tau)} \to \overline{\mathcal{M}}_g\,, \tag{27}$$

where τ is a partition of d, form a basis for the vector space of (formal) kappa polynomials of degree d. We will use the push-forward basis (27) to obtain a lower bound for the rank of the relations given in (26).

Given any two partitions σ and τ, let $\mathsf{K}(\sigma, \tau)$ be the coefficient of the push-forward kappa polynomial corresponding to τ in (26). Define a matrix M with rows and columns indexed by partitions of d by

$$\mathsf{M}_{\sigma\tau} = \mathsf{K}(\sigma_-, \tau)\,,$$

where σ_- is the partition formed by reducing each part of σ by 1 and discarding the parts of size 0. By Proposition 3.3 below, M is invertible.

The invertibility of M implies Theorem 6 by the following argument. If $|\sigma| = d$ and $\ell(\sigma) \geq g - d$, then

$$|\sigma_-| \leq d - (g-d) = 2d - g\,,$$

so the row corresponding to σ actually contains the coefficients of a relation obtained from Theorem 5. Since M is invertible, all such relations are linearly independent, so the quotient of the space of degree d kappa polynomials by such relations has dimension at most the number of partitions of d of length at most $g - 1 - d$, as desired. $\square$

Proposition 3.3. *The matrix* M *is invertible.*

Proof. We will show the invertibility of M by constructing another matrix A of the same size and checking that the product MA is upper-triangular (with nonvanishing diagonal entries) with respect to any ordering of the partitions of d which places partitions containing more parts of size 1 after partitions containing fewer parts of size 1.

First, we compute the coefficient $\mathsf{K}(\sigma,\tau)$ as a sum over injections from the set of parts of σ to the set of parts of τ describing which factors in (26) produce which psi powers. We write such an injection as

$$\phi : \sigma \hookrightarrow \tau\,.$$

The parts of τ which are not in the image of ϕ are produced by the factors involving D_0. The result is

$$\mathsf{K}(\sigma,\tau) = \frac{(-1)^{\ell(\tau)-\ell(\sigma)}}{|\mathrm{Aut}(\tau)|} \sum_{\phi\,:\,\sigma\hookrightarrow\tau} \prod_{i \overset{\phi}{\mapsto} j} [D_i]_{T^j} \prod_{j\in(\tau\setminus\phi(\sigma))} [D_0]_{T^j}\,. \tag{28}$$

We define the matrix A as follows. For any partitions τ and μ of the same size,

$$\mathsf{A}_{\tau,\mu} = \sum_{\substack{\psi\,:\,\tau\to\mu \\ \text{refinement}}} \frac{|\mathrm{Aut}(\tau)|}{\prod_{k\in\mu}|\mathrm{Aut}(\psi^{-1}(k))|} \prod_{k\in\mu}\big(\ell(\psi^{-1}(k)) + 2k + 1\big)! \prod_{j\in\tau}\frac{1}{(2j+1)!!}\,,$$

where the sum runs over all partition refinements

$$\psi : \tau \to \mu\,,$$

functions from the set of parts of τ to the set of parts of μ such that the preimage of each part k of μ is a partition of k.

We factor the sums appearing in the entries of the product matrix MA:

$$\sum_{\tau} \mathsf{K}(\sigma_-,\tau)\mathsf{A}_{\tau,\mu} = \sum_{\xi:\sigma_-\to\mu} \prod_{\substack{k\in\mu \\ \sigma'=\xi^{-1}(k)}} \left(\sum_{\tau'} \mathsf{K}(\sigma',\tau')\mathsf{A}_{\tau',(k)} \right) , \tag{29}$$

where $\xi : \sigma_- \to \mu$ is a function from the set of parts of σ_- to the set of parts of μ.

In order to understand (29), we must study the sum

$$\sum_{\tau} \mathsf{K}(\sigma_-,\tau)\mathsf{A}_{\tau,(k)} . \tag{30}$$

After expanding (30) via formula (28) for $\mathsf{K}(\sigma_-,\tau)$ and the definition of $\mathsf{A}_{\tau,(k)}$, the result is

$$(-1)^{\ell(\sigma)} \sum_{\tau\vdash k} \sum_{\phi\,:\,\sigma_-\hookrightarrow\tau} \frac{(-1)^{\ell(\tau)}(\ell(\tau)+2k+1)!}{|\mathrm{Aut}(\tau)|} \cdot \prod_{j\in\tau} \frac{1}{(2j+1)!!} \prod_{i\overset{\phi}{\mapsto}j} [D_i]_{T^j} \prod_{j\in(\tau\setminus\phi(\sigma_-))} [D_0]_{T^j} , \tag{31}$$

where the first sum is over all partitions τ of k. Next, we include formal variable t to keep track of the size of τ and factor based on the values of the images of the parts of σ_- under ϕ via the series

$$\widehat{D}_i(t) = \sum_{j\geq 1} [D_i]_{T^j} \frac{t^{j+\frac{1}{2}}}{(2j+1)!!} .$$

After removing nonzero scaling factors, we rewrite (31) as

$$\left[\widehat{D}_0^{-2k-2} \prod_{i\in\sigma_-} \frac{\widehat{D}_i}{\widehat{D}_0} \right]_{t^{-1}} . \tag{32}$$

Up to a triangular change of basis in the $\widehat{D}_i$, we have

$$\widehat{D}_i = \sin\left(\frac{2i+1}{2} \sin^{-1}(\sqrt{t}) \right) .$$

We define

$$\theta = \frac{1}{2}\sin^{-1}(\sqrt{t})$$

and check the following two properties:

$$\left[\frac{1}{\sin^4\theta}\right]_{t^{-1}} \neq 0\,,$$

but

$$\left[\frac{1}{\sin^e\theta}\right]_{t^{-1}} = 0$$

for every even $e \geq 6$.

Using these facts to compute (32), we conclude

$$\sum_{\tau} \mathsf{K}(\sigma_-,\tau)\mathsf{A}_{\tau,(k)} = 0$$

whenever $|\sigma_-| < k-1$, and

$$\sum_{\tau} \mathsf{K}((k-1),\tau)\mathsf{A}_{\tau,(k)} \neq 0\,. \tag{33}$$

We now return to the matrix MA. Suppose that σ and τ are partitions of d containing S and T parts equal to 1 respectively and

$$\mathsf{MA}_{\sigma\tau} \neq 0\,.$$

By the identity (29) and the analysis above, there exists a function

$$\xi : \sigma_- \to \tau$$

for which the preimage of each part $k \in \tau$ is a partition of size at least $k-1$. Thus, we have

$$|\sigma| - \ell(\sigma) = |\sigma_-| \geq |\tau| - \ell(\tau)\,. \tag{34}$$

Since $|\sigma| = |\tau| = d$, we see $\ell(\tau) \geq \ell(\sigma)$. Moreover, comparing lengths of the partitions, we obtain

$$\ell(\sigma) - S = \ell(\sigma_-) \geq \ell(\tau) - T\,. \tag{35}$$

Adding the inequalities (34) and (35), we conclude $S \leq T$.

If $S = T$, then the ξ-preimage of each part $k \in \tau$ must be of size $k-1$ and have length one if $k > 1$, which implies $\tau = \sigma$. Thus, the matrix MA is triangular. The nonvanishing of the diagonal entries follows from the nonvanishing (33). $\square$

4 The semisimple point: $\widetilde{\tau} = (0, r\phi, 0, \ldots, 0)$

4.1 Another shift

The shift along the second basis vector $e_1 \in V_r$ also yields a semisimple CohFT with attractive properties. The associated topological field theory $\omega^{r,\widetilde{\tau}}$ is very simple, much simpler than $\omega^{r,\tau}$, but the R-matrix is not as explicit. A basic polynomiality property of Witten's r-spin class will be proven using $\widetilde{\tau}$.

4.2 The quantum product

Recall the notation for genus 0 correlators,

$$\big\langle a_1, \ldots, a_n \big\rangle^r = \int_{\overline{\mathcal{M}}_{0,n}} W^r_{0,n}(a_1, \ldots, a_n) \in \mathbb{Q}\,,$$

discussed in Section 1. As before, we will often drop the superscript r.

Proposition 4.1. *We have the evaluations*

$$\big\langle a, b, c \big\rangle = 1 \quad \text{for } a + b + c = r - 2\,,$$
$$\big\langle a, b, c, 1 \big\rangle = \frac{1}{r} \quad \text{for } a + b + c = 2r - 3\,.$$

All correlators involving an $a_i = 1$ vanish whenever $n \geq 5$.

Proof. The values of 3- and 4-point correlators are well-known (and were stated in (4) of Section 0.3). The only nontrivial claim here is the vanishing for $n \geq 5$ which is a direct consequence of Proposition 1.4. $\square$

By Proposition 4.1, the *quantum product* at $\widetilde{\tau}$ is given by

$$\partial_a \bullet_{\widetilde{\tau}} \partial_b = \left| \begin{array}{ll} \partial_{a+b} & \text{if } \; a+b \leq r-2\,, \\ \phi\, \partial_{a+b-r+1} & \text{if } \; a+b \geq r-1\,. \end{array} \right.$$

To simplify computations, we introduce a new frame[15]

$$\widetilde{\partial}_a = \phi^{-a/(r-1)} \partial_a.$$

[15]The metric is $\eta(\widetilde{\partial}_a, \widetilde{\partial}_b) = \phi^{-\frac{r-2}{r-1}} \delta_{a+b,r-2}$.

The quantum multiplication then takes the form

$$\widetilde{\partial}_a \bullet_{\widetilde{\tau}} \widetilde{\partial}_b = \left| \begin{array}{ll} \widetilde{\partial}_{a+b} & \text{if} \quad a+b \leq r-2\,, \\ \widetilde{\partial}_{a+b-r+1} & \text{if} \quad a+b \geq r-1\,. \end{array} \right.$$

4.3 The topological field theory

Proposition 4.2. *We have*

$$\omega_{g,n}^{r,\widetilde{\tau}}(\widetilde{\partial}_{a_1} \otimes \cdots \otimes \widetilde{\partial}_{a_n}) = \phi^{(g-1)\frac{r-2}{r-1}}(r-1)^g \cdot \delta\,,$$

where δ *equals* 1 *if* $g-1-\sum_{i=1}^n a_i$ *is divisible by* $r-1$ *and* 0 *otherwise.*

Proof. From Proposition 4.1 and the definition of $\widetilde{\partial}_a$ we get

$$\omega_{0,3}^{r,\widetilde{\tau}}(\widetilde{\partial}_a \otimes \widetilde{\partial}_b \otimes \widetilde{\partial}_c) = \left| \begin{array}{ll} \phi^{-\frac{r-2}{r-1}} & \text{if} \;\; a+b+c=-1 \mod r-1, \\ 0 & \text{else.} \end{array} \right.$$

The topological field theory $\omega_{g,n}^{r,\widetilde{\tau}}$ for general g and n can be computed by restricting the $\widetilde{\tau}$-shifted r-spin theory $\mathsf{W}_{g,n}^{r,\widetilde{\tau}}$ to

$$[C] \in \overline{\mathcal{M}}_{g,n}\,,$$

where C is a completely degenerate curve with $2g-2+n$ rational components and $3g-3+n$ nodes.

The $3g-3+n$ nodes divide the C into genus 0 components with 3 special points each. By the splitting axiom, we must place insertions $\{0,\ldots,r-2\}$ on every branch of every node in a manner such that the following conditions are satisfied:

(i) the sum of the two insertions at each node equals $r-2$,

(ii) the sum of the three insertions on each rational component of the curve plus 1 is divisible by $r-1$.

Conditions (i) and (ii) are impossible to satisfy if $g-1-\sum_{i=1}^n a_i$ is not divisible by $r-1$.

If the divisibility condition is satisfied, we can first place an arbitrary insertion on a single branch of a node of every independent cycle of the dual graph of the curve. Then, the other insertions are uniquely determined. We find exactly $(r-1)^g$ possibilities.

Now each rational component contributes a factor of $\phi^{-\frac{r-2}{r-1}}$ and each node a factor $\phi^{\frac{r-2}{r-1}}$ (the inverse of the metric). Collecting all the factors we get

$$\phi^{(g-1)\frac{r-2}{r-1}}(r-1)^g.$$

□

4.4 Euler field and shifted degree

The operator of quantum multiplication by the Euler field at $\widetilde{\tau}$,

$$E = (r-1)\phi^{\frac{r}{r-1}}\,\widetilde{\partial}_1\,,$$

is given in the frame $\{\widetilde{\partial}_a\}$ by the matrix

$$\xi = (r-1)\phi^{\frac{r}{r-1}} \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

or $\xi^0_{r-2} = \xi^{a+1}_a = (r-1)\phi^{\frac{r}{r-1}}$. In the same frame, the shifted degree operator is

$$\mu = \frac{1}{2r} \begin{pmatrix} -(r-2) & 0 & \cdots & \cdots & 0 \\ 0 & -(r-4) & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & r-4 & 0 \\ 0 & \cdots & \cdots & 0 & r-2 \end{pmatrix}$$

or $\mu^a_a = \frac{2a-r+2}{2r}$.

4.5 The R-matrix

Define the polynomials $P_m(r,a)$ by the following recursive procedure. Let $P_0(r,a)=1$. For $m \geq 1$, let

$$\begin{aligned} P_m(r,a) = {} & \frac{1}{2}\sum_{b=1}^{a}(2mr-r-2b)P_{m-1}(r,b-1) \qquad (36)\\ & -\frac{1}{4mr(r-1)}\sum_{b=1}^{r-2}(r-1-b)(2mr-b)(2mr-r-2b)P_{m-1}(r,b-1)\,. \end{aligned}$$

The second summation can be extended to $r-1$ instead of $r-2$ because of the presence of the factor $r-1-b$. The second sum is then easily seen to be divisible both by r and by $r-1$, so P_m is indeed a polynomial. The first few values are:

$$\begin{aligned} P_0 &= 1,\\ P_1 &= \frac{1}{2}a(r-1-a) - \frac{1}{24}(2r-1)(r-2),\\ P_2 &= \frac{1}{8}a^4 - \frac{1}{12}a^3(5r-1) + \frac{1}{48}a^2(20r^2-5r-4) - \frac{1}{48}a(r-1)(6r^2+7r-2)\\ &\quad + \frac{1}{1152}(2r-1)(r-2)(2r^2+19r+2)\,. \end{aligned}$$

There appears to be no closed formula for the polynomials P_m. However, we will present a closed expression for $P_m(0,a)$ in Proposition A.2 in terms of Bernoulli polynomials.

Lemma 4.3. *The polynomials P_m satisfy the relations*

$$P_m(r,a) - P_m(r,a-1) = \frac{1}{2}(2mr-r-2a)P_{m-1}(r,a-1) \qquad (37)$$

$$P_m(r,0) = P_m(r,r-1) \qquad (38)$$

and are the unique solutions to these equations with initial condition $P_0 = 1$.

Proof. Given P_{m-1}, equation (37) determines P_m uniquely up to a polynomial in r independent of a. Equation (38) for P_{m+1} then determines this polynomial in r. The uniqueness statement is therefore established.

Equation (37) follows directly from the definition of P_m. To show that the second equation is satisfied by P_{m+1}, a calculation is required. The definition of P_m implies

$$P_{m+1}(r, r-1) - P_{m+1}(r, 0) = \frac{1}{2}\sum_{a=0}^{r-2}(2mr + r - 2a - 2)P_m(r, a)\,,$$

where we have substituted $a = b - 1$ for the summation variable. From the definition of P_m, we obtain

$$\begin{aligned}
P_{m+1}&(r, r-1) - P_{m+1}(r, 0) = \\
&\frac{1}{4}\sum_{a=0}^{r-2}\sum_{b=1}^{a}(2mr + r - 2a - 2)(2mr - r - 2b)P_{m-1}(r, b-1) \qquad (39)\\
&- \frac{1}{8mr(r-1)}\sum_{a=0}^{r-2}(2mr + r - 2a - 2)\\
&\qquad \times \sum_{b=1}^{r-2}(r - 1 - b)(2mr - b)(2mr - r - 2b)P_{m-1}(r, b-1)\,.
\end{aligned}$$

Using the evaluation

$$\sum_{a=b}^{r-2}(2mr + r - 2a - 2) = (r - 1 - b)(2mr - b),$$

we obtain

$$\sum_{a=0}^{r-2}(2mr + r - 2a - 2) = 2mr(r-1)\,.$$

In equation (39), we exchange the summation order in the first term and use the identities above. We obtain

$$\begin{aligned}
P_{m+1}&(r, r-1) - P_{m+1}(r, 0) = \\
&\frac{1}{4}\sum_{b=1}^{r-2}(r - 1 - b)(2mr - b)(2mr - r - 2b)P_{m-1}(r, b-1)\\
&- \frac{1}{8mr(r-1)}2mr(r-1)\\
&\qquad \times \sum_{b=1}^{r-2}(r - 1 - b)(2mr - b)(2mr - r - 2b)P_{m-1}(r, b-1)\,,
\end{aligned}$$

which clearly vanishes. $\square$

Proposition 4.4. *The unique solution* $R(z) = \sum_{m=0}^{\infty} R_m z^m \in \mathrm{End}(V_r)[[z]]$ *of the equations*

$$[R_{m+1}, \xi] = (m+\mu) R_m$$

with the initial condition $R_0 = 1$ *has coefficients*

$$(R_m)_a^b = \left[-r(r-1)\phi^{\frac{r}{r-1}}\right]^{-m} P_m(r, r-2-b)\,, \quad \textit{if} \quad b+m = a \mod r-1$$

and 0 otherwise. The inverse matrix $R^{-1}(z)$ *has coefficients*

$$(R_m^{-1})_a^b = \left[r(r-1)\phi^{\frac{r}{r-1}}\right]^{-m} P_m(r, a)\,, \quad \textit{if} \quad b+m = a \mod r-1$$

and 0 otherwise.

Proof. The *uniqueness* of the solution follows from the semisimplicity of the Frobenius manifold V_r at τ (proven in Section 4.2). Since $P_0 = 1$, the formula for R_0 yields the identity matrix. We must check that the formula for R is indeed a solution of the recursion

$$[R_{m+1}, \xi] = (m+\mu) R_m.$$

Explicitly, we must show

$$(R_{m+1})_{a+1}^b \xi_a^{a+1} - \xi_{b-1}^b (R_{m+1})_a^{b-1} = (m + \mu_b^b)(R_m)_a^b,$$

or, equivalently,

$$(r-1)\phi^{\frac{r}{r-1}} \left[(R_{m+1})_{a+1}^b - (R_{m+1})_a^{b-1}\right] = \frac{2mr - r + 2b + 2}{2r} (R_m)_a^b,$$

where both $a+1$ and $b-1$ are understood modulo $r-1$.

The nonvanishing condition $b + m = a \pmod{r-1}$ is simultaneously satisfied or not satisfied in all three terms of the equality. The formula for R_m contains the factor $\left[-r(r-1)\phi^{\frac{r}{r-1}}\right]^{-m}$. After using these two observations, we obtain the final form of the equality to be checked:

$$P_{m+1}(r, r-1-b) - P_{m+1}(r, r-2-b) = \frac{1}{2}(2mr - r + 2b + 2) P_m(r, r-2-b)\,. \quad (40)$$

Here, the argument $r-1-b$ of the polynomials should be taken modulo $r-1$. In other words, when $b=0$ the equality reads

$$P_{m+1}(r,0) - P_{m+1}(r,r-2) = \frac{1}{2}(2mr - r + 2)P_m(r,r-2)\,. \tag{41}$$

We first prove (40). After replacing m with $m+1$ in (37), we obtain

$$P_{m+1}(r,a) - P_{m+1}(r,a-1) = \frac{1}{2}(2mr + r - 2a)P_m(r,a-1)\,.$$

After substituting $a = r-1-b$, we have

$$\begin{aligned} P_{m+1}(r,r-1-b) - P_{m+1}(r,r-2-b) = \\ &= \frac{1}{2}\left(2mr + r - 2(r-1-b)\right) P_m(r,r-2-b) \\ &= \frac{1}{2}(2mr - r + 2b + 2)P_m(r,r-2-b), \end{aligned}$$

which is exactly (40). In particular, for $b=0$, we find

$$P_{m+1}(r,r-1) - P_{m+1}(r,r-2) = \frac{1}{2}(2mr - r + 2)P_m(r,r-2).$$

Equation (41) now follows from the equality $P_{m+1}(r,r-1) = P_{m+1}(r,0)$ of equation (38). $\square$

Let $\mathsf{W}^{r,\widetilde{\tau}}$ be the cohomological field theory given by the shift of Witten's r-spin class by the vector $\widetilde{\tau} = (0, r\phi, 0, \dots, 0)$. Define

$$\Omega^{r,\widetilde{\tau}} = R.\omega^{r,\widetilde{\tau}}$$

by the action of the R-matrix of Proposition 4.4 on the topological field theory $\omega^{r,\widetilde{\tau}}_{g,n}$ of Proposition 4.2. Using Teleman's classification and the dimension analysis of Section 3.1, we obtain the following result parallel to Theorem 4 and Theorem 8 for the shift by $\tau = (0, \dots, 0, r\phi)$. As an outcome, we obtain a second formula for Witten's r-spin class.

Theorem 9. *$W^r_{g,n}(a_1, \dots, a_n)$ equals the part of $\Omega^{r,\widetilde{\tau}}_{g,n}$ of degree*

$$\mathsf{D}^r_{g,n}(a_1, \dots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}$$

in $H^*(\overline{\mathcal{M}}_{g,n})$. The parts of $\Omega^{r,\widetilde{\tau}}_{g,n}$ of degree higher than $\mathsf{D}^r_{g,n}$ vanish.

A more explicit formula for the cohomological field theory $\Omega^{r,\widetilde{\tau}}$ is provided in Proposition 4.6 of Section 4.6.

4.6 Proof of Theorem 7

We prove here Theorem 7: for $\sum_{i=1}^n a_i = 2g-2$, the rescaled Witten class

$$\mathcal{W}_{g;a_1,\dots,a_n}(r) = r^{g-1} W^r_{g,n}(a_1,\dots,a_n) \ \in RH^{g-1}(\overline{\mathcal{M}}_{g,n}) \tag{42}$$

is polynomial for sufficiently large values of r. Our proof will show the polynomial (42) is always divisible by $r-1$.

Theorems 8 and 9 provide two formulas for Witten's r-spin class obtained by shifting by e_{r-2} and e_1 respectively. The formula of Theorem 8 has an explicit R-matrix, but the topological field theory is not explicitly polynomial in r. We will therefore use the formula of Theorem 9 which is explicitly polynomial. From now on, we will always assume that r is sufficiently large.

Denote by $\mathsf{G}_{g,n+k}$ the set of stable graphs with $n+k$ legs. The legs marked by $n+1$ to $n+k$, termed the *κ-legs*, will correspond to marked points forgotten by a forgetful map. The ψ classes corresponding to the κ-legs push-forward to κ classes.

Theorem 9 applied when $\sum_{i=1}^n a_i = 2g-2$ expresses the rescaled Witten class $\mathcal{W}_{g;a_1,\dots,a_n}(r)$ as a sum over stable graphs with weightings.

Definition 4.5. Consider a stable graph Γ of genus g with $n+k$ marked legs. A *weighting* $\mathbf{a}$ of Γ is a function on the set of half-edges

$$\mathsf{H}(\Gamma) \to \{0,\dots,r-2\}\,, \quad h \mapsto a_h$$

satisfying the following properties:

- if h and h' are the two half-edges of a single edge, then $a_h + a_{h'} = r-2$,
- If h corresponds to the leg i for $1 \le i \le n$, then[16] $a_h = a_i$,
- If h is a κ-leg, then $a_h = 0$.

To every vertex v of a stable graph, we assign a formal variable x_v satisfying $x_v^{r-1} = 1$. For a polynomial Π in variables x_v, we will denote by $\{\Pi\}_x$ the term of degree 0 in all variables x_v.

[16]If $r-1$ were smaller than a_i, the weighting of the leg would be $a_i \mod r-1$, but here we assume r is large.

Given a stable graph Γ with weighting, we assign to each edge $e \in \mathsf{E}(\Gamma)$ the *edge factor*

$$\Delta(e) = \frac{1}{x^a y^b} \, \frac{1 - \sum\limits_{m,\ell \geq 0} P_m(r,a) P_\ell(r,b) (x\psi')^m (y\psi'')^\ell}{\psi' + \psi''}.$$

Here, a and b are the weightings of the half-edges of e, ψ' and ψ'' are the corresponding cotangent line classes, and x, y are the vertex variables corresponding to the vertices adjacent to the edge (if the edge is a loop, then $x = y$).

To each leg i for $1 \leq i \leq n$ we assign the *leg factor*

$$L(i) = \frac{1}{x_v^{a_i}} \sum_{m \geq 0} P_m(r, a_i)(x_v \psi_i)^m,$$

where a_i is the weighting of the leg, ψ_i is the cotangent line associated to the leg, and x_v is the vertex variable of the vertex to which the leg is attached.

Finally, to each κ-leg i for $n+1 \leq i \leq n+k$, we assign the *κ-factor*

$$K(i) = -\psi_i \sum_{m \geq 1} P_m(r, 0)(x_v \psi_i)^m,$$

where ψ_i is the cotangent line class of the leg, and x_v is the vertex variable of the vertex to which the leg is attached.

Proposition 4.6. *The class $\mathcal{W}_{g;a_1,\ldots,a_n}(r)$ is given by the degree $g-1$ part of the mixed degree cohomology class*

$$\sum_{k \geq 0} \sum_{\substack{\Gamma \in \mathsf{G}_{g,n+k} \\ \text{weightings } \mathbf{a}}} \frac{(r-1)^{1-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} p_* \left\{ \prod_v x_v^{g_v - 1} \prod_e \Delta(e) \prod_{i=1}^{n} L(i) \prod_{i=n+1}^{n+k} K(i) \right\}_x,$$

where $p : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ is the natural boundary map forgetting the last k marked points.

Proof. The formula is essentially a reformulation of the R-matrix action described in Theorem 9. To simplify the computations, we set $\phi = 1$.

The powers of x_v keep track of the remainders modulo $r-1$. More precisely, the base vector e_a corresponds to x^{-a}. The bi-vector η^{ab} is then encoded by the expression $\frac{1}{x^a y^b}$ with $a+b=r-2$. The matrix R_m^{-1} takes e_a to a multiple of e_b with $b = a-m \mod r-1$. Therefore the coefficients of ψ^m in the formulas come with an mth power of the corresponding vertex variable. Finally, putting a factor x_v^{g-1} on each vertex and taking the coefficient of x_v^0 allows one to encode the condition

$$g - 1 - \sum a_i = 0 \mod r - 1$$

which appears in topological field theory $\omega_{g,n}^{r,\widetilde{\tau}}$.

In order to take into account the rescaling by r^{g-1}, we have removed the factor r^m from the denominator of the mth term R_m of the r-matrix. Thus the degree d part of the resulting mixed degree class is multiplied by r^d. In particular, the degree $g-1$ part, which corresponds to Witten's r-spin class, is multiplied by r^{g-1} giving the rescaled class.

Finally, we account for all the occurrences of $r-1$. There is a factor of $(r-1)^m$ in the denominator or R_m, leading, as above, to a global factor of $(r-1)^{1-g}$. There is also a factor $(r-1)^{g_v}$ in the topological field theory at the vertex v. The latter yield $(r-1)^{g-h^1(\Gamma)}$. After multiplying the two factors, we obtain $(r-1)^{1-h^1(\Gamma)}$. $\square$

We will use Proposition 4.6 to prove the polynomiality assertion of Theorem 7. However, we will temporarily remove the division by $\psi' + \psi''$ from the edge factor Δ. We will study the polynomiality in r of the formula of Proposition 4.6 *without* the division by $\prod_{e\in \mathsf{E}(\Gamma)}(\psi'_e + \psi''_e)$. For each stable graph Γ, we will prove the polynomiality in r of the degree

$$g - 1 + |\mathsf{E}(\Gamma)|$$

part of the formula of Proposition 4.6 without denominators, where $|\mathsf{E}(\Gamma)|$ is the number of edges.

The division by $\prod_{e\in \mathsf{E}(\Gamma)}(\psi'_e+\psi''_e)$ will be taken afterwards by the following argument. Consider the expression of Proposition 4.6 as an element of the strata algebra *not* quotiented by any tautological relations, not even the relations due to the degree of the cohomology class supported by $\overline{\mathcal{M}}_v$ being higher than the dimension of $\overline{\mathcal{M}}_v$ for some vertex v. Then, $\psi'_e + \psi''_e$ is not a zero divisor in the strata algebra. Division by $\prod_{e\in \mathsf{E}(\Gamma)} \psi'_e+\psi''_e$, when possible at all, is therefore uniquely defined and preserves the property of being a polynomial in r.

Let Γ be a stable graph with $n+k$ legs. Let $\mathbf{m}$ be a function

$$\mathbf{m} : \mathsf{H}(\Gamma) \to \mathbb{Z}_{\geq 0} \ , \qquad h \mapsto m_h$$

satisfying the constraint $\sum_{h \in \mathsf{H}(\Gamma)} m_h = g-1$ and the condition

- if h and h' are the two half-edges of a single edge, then $(m_h, m_{h'}) \neq (0,0)$.

Define the sum

$$S_{\Gamma,\mathbf{m}} = \sum_{\text{weightings } \mathbf{a}} p_* \left\{ \prod_v x_v^{g_v - 1} \prod_h P_{m_h}(r, a_h) x_{v(h)}^{m_h - a_h} \right\}_x .$$

By Lemmas 4.7 and 4.8 below, for r large enough, $S_{\Gamma,\mathbf{m}}$ *is a polynomial in* r *divisible by* $(r-1)^{h^1(\Gamma)}$. By writing the coefficients of the formula of Proposition 4.6 without denominators in terms of the $S_{\Gamma,\mathbf{m}}$, we obtain the polynomiality required for Theorem 7. Moreover, since the prefactor in Proposition 4.6 for the rescaled Witten class is $(r-1)^{1-h^1(\Gamma)}$, we also conclude that the rescaled Witten class is a polynomial in r *divisible by* $r-1$.

Lemma 4.7. *The sum* $S_{\Gamma,\mathbf{m}}$ *is a polynomial in* r *for* r *large enough.*

Proof. The proof here follows closely Pixton's proof of polynomiality in [17, Appendix]. We will use [17, Proposition A1], but with Pixton's r replaced with $r-1$ (which we assume to be large enough). Let Γ' be the graph obtained from Γ by adding a vertex at the end of each leg and in the middle of each edge. Let M be the edge-vertex adjacency matrix of Γ'. The matrix M satisfies the assumptions of [17, Proposition A1]. The vector x of [17, Proposition A1] assigns an integer x_h to each edge of Γ' or, in other words, to each half-edge h of Γ. The vectors a and b of [17, Proposition A1] assign an integer to each vertex of Γ'. The following table summarizes what these integers are for each vertex and what conditions the equation

$$\mathsf{M}\mathsf{x} = \mathsf{a} + (r-1)\mathsf{b}$$

imposes. In fact, the latter conditions are equivalent to x being a weighting.

type of vertex of Γ'	a	b	effect on x
midpoint of edge h–h' in Γ	$r-2$	0	$x_h + x_{h'} = r-2$
endpoint of leg h in Γ	a_h	0	$x_h = a_h$
vertex v of Γ	$g_v - 1 + \sum_{h \mapsto v} m_h$	b_v	TopFT condition mod $r-1$ at v

The numbers b_v in the table can take different values for different weightings. However, for a given graph Γ and a given choice of integers m_h, there are only *finitely* many possible values b_v. Thus, the sum $S_{\Gamma,\mathbf{m}}$ over all weightings can be decomposed into a finite number of sums of the form of [17, Proposition A1]. Hence, by [17, Proposition A1], $S_{\Gamma,\mathbf{m}}$ is a polynomial in r. □

Lemma 4.8. *The polynomial $S_{\Gamma,\mathbf{m}}$ is divisible by $(r-1)^{h^1(\Gamma)}$.*

Proof. Once again we follow Pixton's proof in [17, Appendix]. Let Q be a polynomial in N variables with (p-integral) $\mathbb{Q}$-coefficients. According to [17, Equation 33], the sum

$$\sum_{0 \le w_1, \ldots, w_N \le p} \mathsf{Q}(w_1, \ldots, w_N) \tag{43}$$

over the N-tuples satisfying D integral linear equations mod p is divisible by p^{N-D} for every large enough prime p. In our case, the sum $S_{\Gamma,\mathbf{m}}$ has exactly the form (43) if we take $p = r-1$ to be prime.

The number N of variables is equal to the number $|\mathsf{E}(\Gamma)|$ of edges of the graph Γ. The number of mod p linear equations is equal to $|\mathsf{V}(\Gamma)| - 1$, where $V(\Gamma)$ is vertex set of Γ. Indeed, there is one mod p condition per vertex, but one condition is redundant, since the sum of the conditions is equal to

$$2g - 2 - \sum a_i = 0 \mod p\,, \tag{44}$$

a condition that is automatically satisfied.

We check the assertion that the sum of the vertex condition yields (44) as follows. First, we sum the vertex conditions

$$\sum_{e\in\mathsf{E}(\Gamma)} w_e + (r-2-w_e) + \sum_{i=1}^{n} a_i - \sum_{h\in\mathsf{H}(\Gamma)} m_h = \sum_{v\in\mathsf{V}(\Gamma)} (g_v - 1) \mod r-1\,.$$

We rewrite the above as

$$(r-2)|\mathsf{E}(\Gamma)| + \sum_{i=1}^{n} a_i - (g-1) = g-1-|\mathsf{E}(\Gamma)| \mod r-1\,,$$

or equivalently,

$$(r-1)|\mathsf{E}(\Gamma)| + \sum_{i=1}^{n} a_i = 2g-2 \mod r-1\,,$$

which is exactly (44).

Thus, by [17, Appendix A.3], $S_{\Gamma,\mathbf{m}}$ is divisible by

$$(r-1)^{N-D} = (r-1)^{|\mathsf{E}(\Gamma)|-|\mathsf{V}(\Gamma)|+1} = (r-1)^{h^1(\Gamma)}$$

for $r-1$ prime and large. Since we already know that $S_{\Gamma,\mathbf{m}}$ is a polynomial in r for r large enough, we conclude the polynomial is divisible by $(r-1)^{h^1(\Gamma)}$. $\square$

A Holomorphic differentials

by F. Janda, R. Pandharipande, A. Pixton, D. Zvonkine

A.1 **Moduli space.** Let g and n be in the stable range $2g-2+n>0$, and let

$$(a_1,\ldots,a_n)\,, \qquad \sum_{i=1}^n a_i = 2g-2$$

be a partition with $a_i \geq 0$ for all i.

We define the *moduli space of holomorphic differentials* as the closed substack

$$\mathcal{H}_g(a_1,\ldots,a_n) = \left\{ [C,\mathsf{p}_1,\ldots,\mathsf{p}_n] \in \mathcal{M}_{g,n} \;\middle|\; \mathcal{O}_C\Big(\sum_{i=1}^n a_i \mathsf{p}_i\Big) = \omega_C \right\} \subset \mathcal{M}_{g,n}\,.$$

Since $\mathcal{H}_g(a_1,\ldots,a_n)$ is the locus of points

$$[C,\mathsf{p}_1,\ldots,\mathsf{p}_n] \in \mathcal{M}_{g,n}$$

for which the evaluation map

$$H^0(C,\omega_C) \to H^0\big(C, \omega_{C|a_1\mathsf{p}_1+\cdots+a_n\mathsf{p}_n}\big)$$

is not injective, every component of $\mathcal{H}_g(a_1,\ldots,a_n)$ has dimension at least $2g-2+n$ in $\mathcal{M}_{g,n}$ by degeneracy loci considerations [9]. Polishchuk [30] has shown that $\mathcal{H}_g(a_1,\ldots,a_n)$ is a *nonsingular* substack of $\mathcal{M}_{g,n}$ of pure dimension $2g-2+n$. Hence, the Zariski closure

$$\mathcal{H}_g(a_1,\ldots,a_n) \subset \overline{\mathcal{H}}_g(a_1,\ldots,a_n) \subset \overline{\mathcal{M}}_{g,n}$$

defines a cycle class

$$[\overline{\mathcal{H}}_g(a_1,\ldots,a_n)] \in H^{2(g-1)}(\overline{\mathcal{M}}_{g,n})\,.$$

Our goal in the Appendix is to relate $[\overline{\mathcal{H}}_g(a_1,\ldots,a_n)]$ to a certain limit of Witten's r-spin classes.

A compact moduli space of twisted canonical divisors which includes the moduli of holomorphic differentials $\mathcal{H}_g(a_1,\ldots,a_n)$ is defined in [8]. A detailed study of the points of the closure $\overline{\mathcal{H}}_{g,n}(a_1, \ldots, a_n)$ can be found in [1]. In [8, Appendix], a conjecture determining

$$[\overline{\mathcal{H}}_g(a_1, \ldots, a_n)] \in A^{g-1}(\overline{\mathcal{M}}_{g,n})$$

in terms the fundamental classes of the moduli spaces of twisted canonical divisors and a formula of Pixton is presented.

The relationship of the conjecture of [8, Appendix] to our conjecture here is a direction for future study.

A.2 **The limit** $r = 0$**.** Let $(a_1, \ldots, a_n)$ be a partition of $2g-2$ with non-negative parts (as in A.1). For

$$r - 2 \geq \max\{a_1, \ldots, a_n\}\,,$$

Witten's r-spin class $W^r_{g,n}(a_1, \ldots, a_n)$ is well-defined and of degree *independent* of r,

$$\mathsf{D}^r_{g,n}(a_1, \ldots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r} = g - 1\,.$$

By Theorem 7, after scaling by r^{g-1},

$$\mathcal{W}_{g;a_1,\ldots,a_n}(r) = r^{g-1} \cdot W^r_{g,n}(a_1, \ldots, a_n) \in RH^{g-1}(\overline{\mathcal{M}}_{g,n})$$

is a *polynomial* in r for all sufficiently large r.

Conjecture A.1. *We have*

$$(-1)^g \mathcal{W}_{g;a_1,\ldots,a_n}(0) = [\overline{\mathcal{H}}_g(a_1, \ldots, a_n)] \in H^{2(g-1)}(\overline{\mathcal{M}}_{g,n}).$$

If the polynomiality of Theorem 7 were to hold in Chow (which we expect), then Conjecture A.1 could also be formulated in $A^{g-1}(\overline{\mathcal{M}}_{g,n})$. By Conjecture A.1, $[\overline{\mathcal{H}}_g(a_1, \ldots, a_n)]$ is a tautological class – a claim which has been proven[17] by A. Sauvaget [31].

[17] In fact, Sauvaget proves $[\overline{\mathcal{H}}_g(a_1, \ldots, a_n)]$ is tautological in Chow.

A.3 **Evidence.**

Genus 1. Witten's class $W^r_{1,n}(0,\dots,0)$ has degree 0. To evaluate the class, the topological field theory is enough (we do not need the R-matrix). A simple computation shows that Witten's class here is equal to $r-1$. Indeed,

$$\begin{aligned} W^r_{1,n}(0,\dots,0) &= \omega_{1,n}(0,\dots,0) \\ &= \sum_{a=0}^{r-2} \omega_{0,n+2}(0,\dots,0,a,r-2-a) \\ &= \sum_{a=0}^{r-2} 1 \;=\; r-1\,. \end{aligned}$$

Thus $\mathcal{W}_{1;0,\dots,0}(r) = r-1$, $\mathcal{W}_{1;0,\dots,0}(0) = -1$. According to the conjecture we have

$$(-1)^g \cdot (-1) = 1 = [\overline{\mathcal{H}}_1(0,\dots,0)] \in RH^0(\overline{\mathcal{M}}_{1,n}),$$

which is, indeed, true.

Genus 2, $n=1$**,** $a_1=2$**.** The tautological space $RH^1(\overline{\mathcal{M}}_{2,1})$ is spanned by the classes ψ_1, δ_{sep}, and δ_{nonsep}, where the indices sep and nonsep refer to the boundary divisors with a separating or a nonseparating node. The class κ_1 may be expressed as

$$\kappa_1 = \psi_1 + \frac{7}{5}\delta_{\mathrm{sep}} + \frac{1}{5}\delta_{\mathrm{nonsep}}. \tag{45}$$

Equation (45) is obtained by pulling back the relation on $\overline{\mathcal{M}}_2$ expressing κ_1 in terms of boundary divisors.

Theorem 9 gives an expression for Witten's class as a linear combination of classes κ_1, ψ_1, δ_{sep} and δ_{nonsep} with the following coefficients:

Class	R-matrix	TopFT	Coefficient
κ_1	$-\frac{P_1(r,0)}{r(r-1)}$	$(r-1)^2$	$\frac{(r-1)(r-2)(2r-1)}{24r}$
ψ_1	$\frac{P_1(r,2)}{r(r-1)}$	$(r-1)^2$	$-\frac{(r-1)(2r^2-29r+74)}{24r}$
δ_{sep}	$-\frac{P_1(r,1)}{r(r-1)}$	$(r-1)^2$	$\frac{(r-1)(r-2)(2r-13)}{24r}$
δ_{nonsep}	$-\sum_{a=0}^{r-2}\frac{P_1(r,a)}{r(r-1)}$	$r-1$	$-\frac{(r-1)(r-2)}{24r}$

After multiplying Witten's class by $(-1)^g r^{g-1} = r$ and extracting the constant term in r, we obtain

$$\frac{1}{12}(-\kappa_1 + 37\psi_1 - 13\delta_{\mathrm{sep}} - \delta_{\mathrm{nonsep}})\,.$$

We remove κ_1 using equation (45). Conjecture A.1 predicts

$$[\overline{\mathcal{H}}_2(2)] = 3\psi_1 - \frac{6}{5}\delta_{\mathrm{sep}} - \frac{1}{10}\delta_{\mathrm{nonsep}} \; \in RH^1(\overline{\mathcal{M}}_{2,1})\,.$$

The result coincides with the well-known formula for the locus of the Weierstrass points, see [2, Lemma 5].

Genus 2, $n = 2$, $a_1 = a_2 = 1$. The tautological space $RH^1(\overline{\mathcal{M}}_{2,2})$ is spanned by six classes ψ_1, ψ_2, α, β, γ and δ_{nonsep} where

- α is the locus of curves with a rational component carrying both markings and a genus 2 component,
- β is the locus of curves with two elliptic components carrying one marking each,
- γ is the locus of curves with two elliptic components one of which carries both markings and the other one no markings,
- δ_{nonsep} is the locus of curves with a nonseparating node.

The class κ_1 is expressed as

$$\kappa_1 = \psi_1 + \psi_2 + \alpha + \frac{7}{5}\beta + \frac{7}{5}\gamma + \frac{1}{5}\delta_{\mathrm{nonsep}}$$

by pulling back the boundary relation on $\overline{\mathcal{M}}_2$.

Theorem 9 gives an expression for Witten's class as a linear combination of classes κ_1, ψ_1, ψ_2, α, β, γ and δ_{nonsep} with the following coefficients:

Class	R-matrix	TopFT	Coefficient
κ_1	$-\frac{P_1(r,0)}{r(r-1)}$	$(r-1)^2$	$\frac{(r-1)(r-2)(2r-1)}{24r}$
ψ_1	$\frac{P_1(r,1)}{r(r-1)}$	$(r-1)^2$	$-\frac{(r-1)(r-2)(2r-13)}{24r}$
ψ_2	$\frac{P_1(r,1)}{r(r-1)}$	$(r-1)^2$	$-\frac{(r-1)(r-2)(2r-13)}{24r}$
α	$-\frac{P_1(r,2)}{r(r-1)}$	$(r-1)^2$	$\frac{(r-1)(2r^2-29r+74)}{24r}$
β	$-\frac{P_1(r,0)}{r(r-1)}$	$(r-1)^2$	$\frac{(r-1)(r-2)(2r-1)}{24r}$
γ	$-\frac{P_1(r,1)}{r(r-1)}$	$(r-1)^2$	$\frac{(r-1)(r-2)(2r-13)}{24r}$
δ_{nonsep}	$-\sum_{a=0}^{r-2}\frac{P_1(r,a)}{r(r-1)}$	$r-1$	$-\frac{(r-1)(r-2)}{24r}$

After multiplying Witten's class by $(-1)^g r^{g-1} = r$ and extracting the constant term in r, we obtain

$$\frac{1}{12}(-\kappa_1 + 13\psi_1 + 13\psi_2 - 37\alpha - \beta - 13\gamma - \delta_{\mathrm{nonsep}})\,.$$

We remove κ_1 using equation (45). Conjecture A.1 predicts

$$[\overline{\mathcal{H}}_2(1,1)] = \psi_1 + \psi_2 - 3\alpha - \frac{1}{5}\beta - \frac{6}{5}\gamma - \frac{1}{10}\delta_{\mathrm{nonsep}} \ \in RH^1(\overline{\mathcal{M}}_{2,2})\,.$$

The result coincides with the well-known formula for the locus of genus 2 curves with a pair of conjugate points, see [2, Lemma 6].

A.4 **The constant term.** We will now present a more explicit approach to the constant term

$$(-1)^g \mathcal{W}_{g;a_1,\ldots,a_n}(0) \in RH^{g-1}(\overline{\mathcal{M}}_{g,n})$$

which, according to Conjecture A.1, equals $[\overline{\mathcal{H}}_g(a_1,\ldots,a_n)]$.

We will use the shift along e_1 studied in Section 4. The corresponding R-matrix involves a sequence of polynomials $P_m(r,a)$ for which we know no closed formula. However, in Proposition A.2 below, we obtain a closed formula for the polynomials $P_m(0,a)$.

Let $B_m(x)$ be the Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \,.$$

Let $P_m(r,a)$ be the polynomials defined by equation (36).

Proposition A.2. *We have*

$$\sum_{m\geq 0} z^m P_m(0,a) = \exp\left(-\sum_{m\geq 1} z^m \frac{B_{m+1}(a+1)}{m(m+1)}\right) .$$

Proof. Define polynomials $Q_m(a)$ by

$$\sum_{m\geq 0} z^m Q_m(a) = \exp\left(-\sum_{m\geq 1} z^m \frac{B_{m+1}(a+1)}{m(m+1)}\right) . \tag{46}$$

We will show that the polynomials $Q_m(a)$ satisfy the mod r reduction of equations (37) and (38) of Lemma 4.3,

$$Q_m(a) - Q_m(a-1) = -aQ_{m-1}(a-1) \,, \tag{47}$$

$$Q_m(-1) = Q_m(0) \,. \tag{48}$$

Together with the initial condition $Q_0 = 1$, these properties determine the polynomials Q_m uniquely and immediately imply

$$Q_m(a) = P_m(0,a) \,.$$

• Property (48) is implied by the simple equality

$$B_{m+1}(0) = B_{m+1}(1)$$

for every $m \geq 1$ and definition (46).

• Property (47) follows from a generating function calculation:

$$\sum_{m\geq 0} z^m Q_m(a) - \sum_{m\geq 0} z^m Q_m(a-1) = -az \sum_{m\geq 0} z^m Q_m(a-1)$$

which is equivalent to

$$\sum_{m\geq 0} z^m Q_m(a) = (1-az) \sum_{m\geq 0} z^m Q_m(a-1)\,.$$

After taking the logarithm

$$\log \sum_{m\geq 0} z^m Q_m(a) = \log(1-az) + \log \sum_{m\geq 0} z^m Q_m(a-1)\,,$$

we must show

$$\sum_{m\geq 1} z^m \frac{B_{m+1}(a+1)}{m(m+1)} = \sum_{m\geq 1} z^m \frac{a^m}{m} + \sum_{m\geq 1} z^m \frac{B_{m+1}(a)}{m(m+1)}\,.$$

The latter follows from

$$B_{m+1}(a+1) = (m+1)z^m + B_{m+1}(a)$$

which is a well-known property of Bernoulli polynomials. $\square$

Using Proposition A.2, we can derive a more explicit formula for the constant term

$$(-1)^g \mathcal{W}_{g,(a_1,\ldots,a_n)}(0)$$

as a sum over stable graphs. We will require here only stable graphs with exactly n legs. We will replace the push-forward of the ψ classes on the κ-legs with an equivalent vertex factor involving κ classes. The equivalence requires the following well-known equality. If

$$f = \sum_{m\geq 1} c_m z^m$$

is a power series without constant term and $F = \exp(f)$ is the exponential, then

$$\sum_{k\geq 0} p_{k*}\left[\prod_{i=n+1}^{n+k} \psi_i(1-F(\psi_i))\right] = \exp\left(-\sum_{m\geq 1} c_m\kappa_m\right), \tag{49}$$

where $p_k : \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map.

As before, to every vertex v of a stable graph we assign a formal variable x_v satisfying $x_v^{r-1} = 1$. For a polynomial Π in variables x_v, we will denote by $\{\Pi\}_x$ the term of degree 0 in all variables x_v.

Given a stable graph, to each edge e we assign the *edge factor*[18]

$$\widetilde{\Delta}(e) = \sum_{a+b=r-2} \frac{1}{x^a y^b} \frac{1-\exp\left[-\sum_{m\geq 1} \frac{B_{m+1}(a+1)\cdot(x\psi')^m + B_{m+1}(b+1)\cdot(y\psi'')^m}{m(m+1)}\right]}{\psi'+\psi''}.$$

Here, a and b are non-negative integers, ψ' and ψ'' are the cotangent line classes corresponding to the half-edges of e, and x, y are the vertex variables corresponding to the vertices adjacent to the edge (if the edge is a loop, then $x = y$).

Furthermore, to each leg i we assign the *leg factor*

$$L(i) = \frac{1}{x_v^{a_i}} \exp\left(-\sum_{m\geq 1} \frac{B_{m+1}(a_i+1)}{m(m+1)}(x_v\psi_i)^m\right),$$

where ψ_i is the cotangent line class associated to the leg, and x_v is the vertex variable of the vertex to which the leg is attached.

Finally, we assign to each vertex v the *vertex factor*

$$\kappa(v) = x_v^{g-1} \exp\left[\sum_{m\geq 1} \frac{B_{m+1}(1)}{m(m+1)} x_v^m \kappa_m\right].$$

Here, x_v is the variable of the vertex and κ_m is the κ class on the moduli space $\overline{\mathcal{M}}_v$ corresponding to v.

Proposition A.3. *Let $a_1, \ldots a_n \in \mathbb{Z}_{\geq 0}$ satisfy*

$$\sum_{i=1}^{n} a_n = 2g-2\,.$$

[18]Division by $\psi' + \psi''$ is only possible for the constant term in r of the edge factor.

The coefficient $(-1)^g \mathcal{W}_{g;a_1,\ldots,a_n}(0)$ *of the rescaled Witten class*

$$(-1)^g r^{g-1} \cdot W_g^r(a_1,\ldots,a_n)$$

is given by the r^0 *coefficient of the degree* $g-1$ *part of*

$$\sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{(-1)^{g-1+h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} p_* \left\{ \prod_e \widetilde{\Delta}(e) \prod_{i=1}^n L(i) \prod_v \kappa(v) \right\}_x ,$$

where $p : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ *is the natural boundary map.*

Proof. The result is obtained by a mod r reduction of the formula of Proposition 4.6 for the rescaled Witten class. The polynomials $P_m(r,a)$ are replaced by the expression for

$$Q_m(a) = P_m(0,a)$$

given in Proposition A.2. The sum over weightings $\mathbf{a}$ has been moved to the sum in the definition of the edge term $\widetilde{\Delta}(e)$. The push-forward of the κ-factors associated to the legs marked by $n+1 \leq i \leq n+k$ adjacent to a vertex v is collected in the vertex factor $\kappa(v)$. The factor $r-1$ becomes -1. Finally, we multiply the resulting expression by the global factor of $(-1)^g$. $\square$

The advantage of Proposition A.3 is that all the inputs are explicit. Proposition 4.6 is more general (and determines the full r dependence of $\mathcal{W}_{g;a_1,\ldots,a_n}(r)$), but involves the polynomials $P(r,a)$ for which we know no closed formula.

Remark A.4. The class $[\mathcal{H}_g(a_1,\ldots,a_n)] \in H^{2(g-1)}(\mathcal{M}_{g,n})$ on the moduli of nonsingular curves is easily determined by the classical Thom-Porteous formula[19] to be

$$c_{g-1}\left(R^1\pi_*\omega\left(-\sum_{i=1}^n a_i \mathsf{p}_i\right) - R^0\pi_*\omega\left(-\sum_{i=1}^n a_i \mathsf{p}_i\right)\right) , \qquad (50)$$

where $\pi : \mathcal{C} \to \mathcal{M}_{g,n}$ is the universal curve. We have checked that the restriction of Proposition A.3 to $\mathcal{M}_{g,n}$ agrees with (50) calculated by Grothendieck-Riemann-Roch. As a result, the restriction of Conjecture A.1 to $\mathcal{M}_{g,n}$ is correct.

[19] See, for example, [3, Section 2] for the Thom-Porteous approach.

References

[1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Moeller, *Compactification of strata of abelian differentials*, `arXiv:1604.08834`.

[2] P. Belorousski and R. Pandharipande, *A descendent relation in genus 2*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **29** (2000), 171-191.

[3] D. Chen, *Strata of abelian differentials and the Teichmller dynamics*, J. Mod. Dyn. **7** (2013), no. 1, 135–152.

[4] A. Chiodo, *The Witten top Chern class via K-theory*, J. Algebraic Geom. **15** (2006), 681–707.

[5] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and abelian varieties, 109–129, Aspects Math., Vieweg, Braunschweig, 1999.

[6] C. Faber and R. Pandharipande (with an appendix by D. Zagier), *Logarithmic series and Hodge integrals in the tautological ring*, Michigan Math. J. **48** (2000), 215–252.

[7] C. Faber and R. Pandharipande, *Tautological and non-tautological cohomology of the moduli space of curves* in *Handbook of moduli, Vol. I*, 293–330, Adv. Lect. Math. **24**, Int. Press: Somerville, MA, 2013.

[8] G. Farkas and R. Pandharipande, *The moduli space of twisted canonical divisors*, J. Institute Math. Jussieu **17** (2018), 615–672.

[9] W. Fulton, *Intersection theory*, Springer-Verlag: Berlin, 1984.

[10] W. Fulton and J. Harris, *Representation theory*, Springer-Verlag: Berlin, 1991.

[11] A. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), 551–568.

[12] T. Graber and R. Pandharipande, *Constructions of nontautological classes on moduli spaces of curves*, Michigan Math. J. **51** (2003), 93–109.

[13] H. Fan, T. Jarvis, and Y. Ruan, *The Witten equation, mirror symmetry and quantum singularity theory*, Ann. of Math. (2) **178** (2013), no. 1, 1–106.

[14] F. Janda, *Comparing tautological relations from the equivariant Gromov-Witten theory of projective spaces and spin structures*, arXiv:1407.4778.

[15] F. Janda, *Frobenius manifolds near the discriminant and relations in the tautological ring*, Lett. Math. Phys **108** (2018), 1649–1675.

[16] F. Janda, *Relations on $\overline{\mathcal{M}}_{g,n}$ via equivariant Gromov-Witten theory of* $\mathbb{P}^1$, Algebr. Geom. **4** (2017), 311–336.

[17] F. Janda, R. Pandharipande, A. Pixton, D. Zvonkine. *Double ramification cycles on the moduli spaces of curves.*, Publ. Math. IHES **125** (2017), 221–266.

[18] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525–562.

[19] E. Looijenga, *On the tautological ring of M_g*. Invent. Math. **121** (1995), 411–419.

[20] T. Mochizuki, *The virtual class of the moduli stack of stable r-spin curves*, Comm. Math. Phys. **264** (2006), no. 1, 1–40.

[21] D. Mumford, *Towards an enumerative geometry of the moduli space of curves* in *Arithmetic and Geometry* (M. Artin and J. Tate, eds.), Part II, Birkhäuser, 1983, 271–328.

[22] R. Pandharipande, *The κ ring of the moduli of curves of compact type*, Acta Math. **208** (2012), 335–388.

[23] R. Pandharipande, *A calculus for the moduli space of curves*, Proceedings of Algebraic geometry – Salt Lake City 2015, Proc. Sympos. Pure Math. **97**, Part 1, 459–488.

[24] R. Pandharipande, *Cohomological field theory calculations*, Proceedings of the ICM (Rio de Janeiro 2018), Vol. 1, 869–898.

[25] R. Pandharipande, A. Pixton, and D. Zvonkine, *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures*, J. Amer. Math. Soc. **28** (2015), 279–309.

[26] A. Pixton, *Conjectural relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$*, `arXiv:1207.1918`.

[27] A. Pixton, *The tautological ring of the moduli space of curves*, Princeton Ph.D. 2013.

[28] A. Polishchuk and A. Vaintrob, *Algebraic construction of Witten's top Chern class* in *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, 229-249, Contemp. Math. **276**, AMS: Providence, RI, 2001.

[29] A. Polishchuk, *Witten's top Chern class on the moduli space of higher spin curves* in *Frobenius manifolds*, 253-264, Aspects Math. **E36**, Vieweg: Wiesbaden, 2004.

[30] A. Polishchuk, *Moduli spaces of curves with effective r-spin structures* in *Gromov-Witten theory of spin curves and orbifolds*, 120, Contemporary Mathematics **403**, AMS: Providence, RI, 2006.

[31] A. Sauvaget, *Cohomology classes of strata of differentials*, arXiv:1701.07867.

[32] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. **188** (2012), 525–588.

[33] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory,* Nuclear Physics B300 [FS22] (1988), 360–376.

[34] E. Witten, *Algebraic geometry associated with matrix models of two-dimensional gravity* in *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, 235–269, Publish or Perish: Houston, TX, 1993.

Departement Mathematik, ETH Zürich
rahul@math.ethz.ch

Department of Mathematics, MIT
apixton@mit.edu

CNRS, Institut Mathématique de Jussieu
dimitri.zvonkine@ump-prg.fr