VP160 Honors Physics I Recitation Class

W.Peng

UM-SJTU JI

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Scientific Notations

Definition

Scientific notation expresses numerical values in **powers of 10.** It is used to represent very large numbers or very small numbers, giving the correct number of *significant figures*.

Example

The distance from the earth to the moon is denoted as

$$3.84 \times 10^8 \text{ m}$$

Unit Prefixes

Definition

SI (*Système International*) units are used to keep measurements consistent around the world. By adding a **prefix** to the fundamental units, additional units are derived.

Example

$$1\ \mathrm{nm} = 10^{-9}\ \mathrm{m} \quad 1\ \mu\mathrm{m} = 10^{-6}\ \mathrm{m} \quad 1\ \mathrm{mm} = 10^{-3}\ \mathrm{m}$$

$$1\ \mathrm{cm} = 10^{-2}\ \mathrm{m} \quad 1\ \mathrm{km} = 10^{3}\ \mathrm{m}$$

Unit Conversions

Definition

Expressing the *same physical quantity* in two *different* units forms a unit **conversion factor**.

Example

$$3\ \mathrm{min} = (3\ \mathrm{min})\left(\frac{60\ \mathrm{s}}{1\ \mathrm{min}}\right) = 180\ \mathrm{s}$$

Uncertainty

Definition

Uncertainties exist in all measurements. They are the maximum possible deviation (to some confidence level) of the **true value** of the quantity from the **measured value**. The **significant figures** are composed of one or two uncertain digit with all the digits preceding it being **certain**.

Example

In my Vp 141 lab report for Exercise 1, I wrote:
The moment of inertia for cylinder B in hole 2 is calculated as

$$\begin{aligned} I_{\text{B,2,math}} &= I_{\text{B,principal,math}} + m_B d_2^2 \\ &= 1.860 \times 10^{-5} + 0.1656 \times (45.09 \times 10^{-3})^2 \\ &= 3.5528 \times 10^{-4}\mathrm{kg} \cdot \mathrm{m}^2 \pm 0.0025 \times 10^{-4}\mathrm{kg} \cdot \mathrm{m}^2 \end{aligned}$$

Back-of-the-Envelope Calculations

Definition

Order-of-magnitude estimates are calculations where we make some rough approximations to carry them out quickly. Since they are carried out so quickly that they can be calculated at the back of an envelope, they are also called back-of-the-envelope calculations.

Example

How many gallons of gasoline are used in the United States in one day? Assume that there are two cars for every three people, that each car is driven an average of $10,000$ mi per year, and that the average car gets 20 miles per gallon.
The US Population on 05/06/2016 is around $323,496$ thousand, which we approximate to 323 million.

$$323 \times 10^6 \times (2/3) \times 10,000/20 \approx 10^{11} \text{ gallons}$$

- Notions of Units
- Uncertainty and Significant Figures
- Estimates and Orders of Magnitude
- **Vectors and vector operations**
- 3D Curvilinear Coordinate Systems
- 1D Kinematics
- Exercises

Vectors

Definition

Vectors are quantities that have both **magnitude** and **direction.** A vector in an *n*-dimensional real vector space is denoted as

$$X \in \mathbb{R}^n: \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T = (x_1, x_2, \ldots, x_n)$$

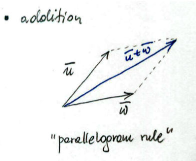
Example

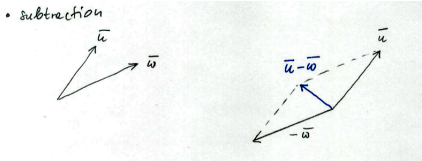
Displacement $\overline{s}$, velocity $\overline{v}$, acceleration $\overline{a}$, force $\overline{v}$, momentum $\overline{P}$, angular velocity $\overline{\omega}$ are vectors in $\mathbb{R}^3$.

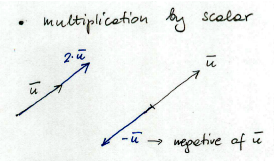
Vector Addition and Scalar Multiplication

Definition

The **addition** and **subtraction** of vectors follows the “parallelogram rule”. The **scalar multiplication** changes the magnitude (perhaps reserve the direction) of the vector.







Dot Product in $\mathbb{R}^n$ and Cross Product in $\mathbb{R}^3$

Definition

The **dot product** of two vectors $\overline{u}$, $\overline{v}$ in $\mathbb{R}^n$ is denoted as $\overline{u} \circ \overline{v}$.

$$\overline{u} \circ \overline{v} = \sum_{i=1}^{n} u_i v_i = |\overline{u}||\overline{v}| \cos \angle(\overline{u}, \overline{v})$$

Definition

The **cross product** of two vectors $\overline{u}$, $\overline{v}$ in $\mathbb{R}^3$ is denoted as $\overline{u} \times \overline{v}$.

$$\overline{u} \times \overline{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Dot Product: Perpendicular (Orthogonal) Projections

Unit Vector

The unit vector in the direction of $\overline{\omega}$ is given by $\frac{\overline{\omega}}{|\overline{\omega}|}$.

Magnitude of Projection

The magnitude of the projection of vector $\overline{v}$ on vector $\overline{\omega}$ is

$$|\overline{u}| \cdot \cos\angle(\overline{u}, \overline{v}) = \frac{\overline{u} \circ \overline{\omega}}{|\overline{\omega}|}$$

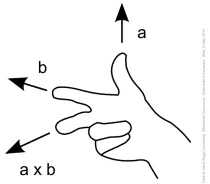
Orthogonal Projections

The **orthogonal projection** of vector $\overline{v}$ on vector $\overline{\omega}$ is

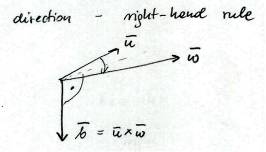
$$\frac{\overline{u} \circ \overline{\omega}}{|\overline{\omega}|} \cdot \frac{\overline{\omega}}{|\overline{\omega}|}$$

The direction of the cross product follows the right-hand rule. The

The Right Hand Rule



length of the cross product $|\overline{b}| = |\overline{u}||\overline{\omega}| \sin \angle(\overline{u}, \overline{v})$



Properties

The Cross Product has the following properties:

1. $\overline{\omega} \times \overline{u} = -\overline{u} \times \overline{\omega}$
2. $\overline{u} \times \overline{\omega} \perp \overline{u}$; $\overline{u} \times \overline{\omega} \perp \overline{\omega}$
3. $\overline{u} \times \overline{\omega} = 0 \Leftrightarrow \overline{u} \parallel \overline{\omega}$

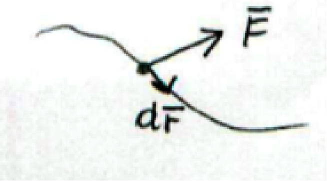
Examples for Dot Product and Cross Product

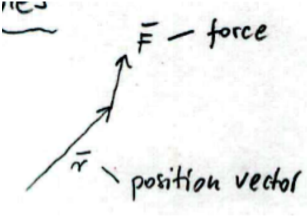
Example

The elementary work δw is defined as the **dot** product of force $\overline{F}$ and infinitesimal displacement $\mathrm{d}\overline{r}$: $\delta w = \overline{F} \circ \mathrm{d}\overline{r}$

Example

Torque $\overline{\tau}$ is defined as the **cross** product of position vector $\overline{r}$ and force $\overline{F}$: $\overline{\tau} = \overline{r} \times \overline{F}$





- Notions of Units
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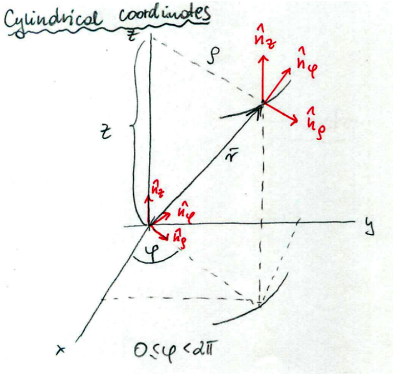
Cylindrical Coordinates



Coordinates: ρ, φ, z

Unit vectors: $\hat{n}_\rho$, $\hat{n}_\varphi$, $\hat{n}_z$

Versors are NOT Fixed:

Careful with derivatives

$\rho = \sqrt{x^2 + y^2}$,

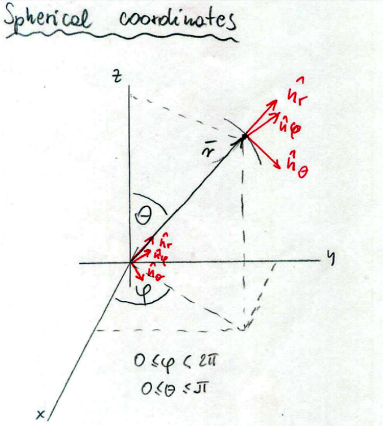
$\varphi = \arctan(y/x)$, $z = z$

$x = \rho\cos\varphi$, $y = \rho\sin\varphi$

$\bar{r} = \rho\hat{n}_\rho + z\hat{n}_z$, where $\hat{n}_\rho$ carries information about φ

Polar coordinates is the special case $z = 0$.

Spherical Coordinates



Coordinates: r, θ, φ

Unit vectors: $\hat{n}_r$, $\hat{n}_\theta$, $\hat{n}_\varphi$

$r = \sqrt{x^2 + y^2 + z^2}$,

$\theta = \arctan \frac{\sqrt{x^2+y^2}}{z}$,

$\varphi = \arctan(y/x)$

$x = r \sin\theta \cos\varphi$,

$y = r \sin\theta \sin\varphi$, $z = r\cos\theta$

$\bar{r} = r\hat{n}_r$, where $\hat{n}_r$ carries information for θ and φ.

Polar coordinates is the special case $\theta = \pi/2$.

Gradient, Divergence, and Curl

$$\nabla U = \hat{n}_x \frac{\partial u}{\partial x} + \hat{n}_y \frac{\partial u}{\partial y} + \hat{n}_z \frac{\partial u}{\partial z}$$

$$\nabla U = \hat{n}_r \frac{\partial u}{\partial r} + \hat{n}_\varphi \frac{1}{r}\frac{\partial u}{\partial \varphi} + \hat{n}_z \frac{\partial u}{\partial z}$$

$$\nabla U = \hat{n}_r \frac{\partial u}{\partial r} + \hat{n}_\theta \frac{1}{r}\frac{\partial u}{\partial \theta} + \hat{n}_\varphi \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}$$

$$\nabla\cdot\bar{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

$$\nabla\cdot\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

Total flux $\oint_S \bar{A}\cdot d\bar{S}$ Divergence $\text{div}\,\bar{A} = \lim \frac{\oint \bar{A}\cdot d\bar{S}}{\Delta V}$

Divergence Theorem $\int_V \nabla\cdot\bar{A}\,dV = \oint_S \bar{A}\cdot d\bar{S}$

Circulation $\oint_C \bar{A}\cdot d\bar{l}$ Curl $\nabla\times\bar{A} = \lim \frac{1}{\Delta S}\left(\hat{n}\oint_C \bar{A}\cdot d\bar{l}\right)$

Stoke's Theorem $\int_S (\nabla\times\bar{A})\cdot d\bar{S} = \oint_C \bar{A}\cdot d\bar{l}$

$$\nabla\times\bar{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{n}_{u_1}h_1 & \hat{n}_{u_2}h_2 & \hat{n}_{u_3}h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

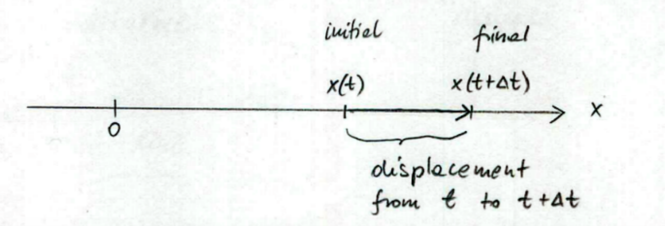
Cartesian $h_1 = h_2 = h_3 = 1$

Cylindrical $h_1 = h_3 = 1$ $h_2 = r$

Spherical $h_1 = 1$ $h_2 = R$ $h_3 = R\sin\theta$

Motion Along a Straight Line

Define positive direction first. As a convention, the vectors *x*, *v* and *a* are written as positive if they have the same direction as the positive direction of the axis, and are written as negative if their direction is oppositve to the positive direction of the axis.

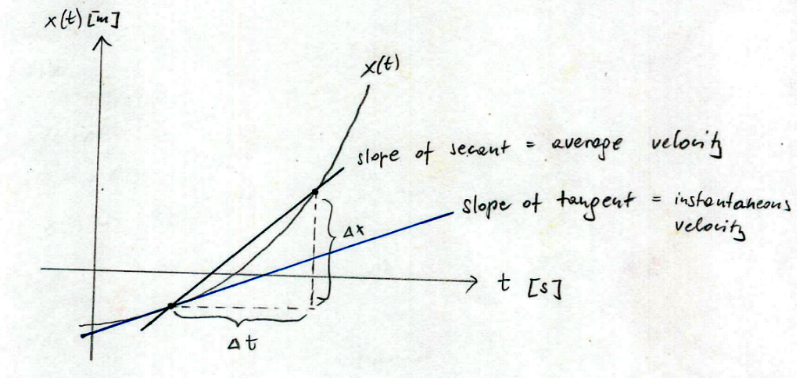


Here we assume that *x* is twice differentiable if there are no impulses. The reasons will be clear when we study the Newton's laws.

Average and Instantaneous Velocity

Average Velocity over $(t, t+\Delta t)$: $v_{av,x} = \frac{x(t+\Delta t)-x(t)}{\Delta t}$

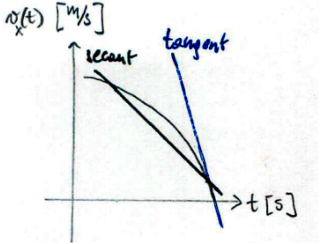
Instantaneous Velocity at t: $v_x(t) = \left.\frac{\mathrm{d}x(\cdot)}{\mathrm{d}t}\right|_t$



Average and Instantaneous Acceleration

Average Acceleration over $(t, t+\Delta t)$: $a_{av,x} = \frac{v(t+\Delta t)-v(t)}{\Delta t}$

Instantaneous Acceleration at t: $a_x(t) = \left.\frac{\mathrm{d}v(\cdot)}{\mathrm{d}t}\right|_t$



Newton's notation for derivatives W.R.T time: $v_x = \dot{x}$, $a_x = \dot{v}_x = \ddot{x}$

Average Speed vs. Average Velocity

Average speed=(distance traveled)/(time interval)
Average velocity=(displacement)/(time interval)

Obtain Displacement from Acceleration

Obtain Velocity from Acceleration

$v(t) = v(0) + \int_0^t a(\tau)\mathrm{d}\tau$ $v(0)$: Initial (t=0) Condition

Obtain Displacement from Velocity

$x(t) = x(0) + \int_0^t v(\tau)\mathrm{d}\tau$

Special Case: Constant Acceleration a

$x(t) = x(0) + v(0)t + \frac{1}{2}at^2$

General Case: Varying Acceleration a

$$x(t) = x(0) + \int_0^t v(\tau)\mathrm{d}\tau = x(0) + v(0)t + \int_0^t \mathrm{d}\tau \int_0^\tau a(s)\mathrm{d}s$$

Relative Motion

Relative Velocity

Velocity of Particle in FoR A
=Velocity of Origin of FoR A'+Velocity of Particle in FoR A'

$$v_x = v_{O'x} + v'_x$$

Analogously, $a_x = a_{O'x} + a'_x$ for acceleration.

Galilean Transformation ($V_{O'x} = const, x_{O'}(0) = 0$)

$$\begin{cases} a_x &= a'_x \\ v_x &= v_{O'x} + v'_x \\ x &= v_{O'x}t + x' \end{cases}$$

where $v_{O'x}t = x'_o$

Planck's Units

Given the Dirac's constant $\hbar = h/(2\pi)$, gravitational constant G, and the speed of light in vacuum c, use dimensional analysis to construct the so called *natural units* of time, length, and mass. These are also called *Planck's units*: Planck's time t_p, Planck's length l_p, and Planck's mass m_P. Find their values in the SI units. How do they compare to the time, distance, and mass that we are able to measure nowadays?

Hints

From Chapter 6 in Vc 210, we learnt the uncertainty principle
$\Delta x \cdot \Delta(mv) \geq h/(4\pi)$, so $\hbar$ has dimension
$[\mathrm{m}] \cdot [\mathrm{kg}] \cdot [\mathrm{m/s}] = [\mathrm{m}^2 \cdot \mathrm{kg} \cdot \mathrm{s}^{-1}]$
c is the speed of light, so it has dimension $[m/s]$
The gravitational force $F = GMm/r^2$, so G has dimension
$[\mathrm{kg} \cdot \mathrm{m/s}^2] \cdot [\mathrm{m}^2] \cdot [\mathrm{kg}^{-2}] = [\mathrm{m}^3 \cdot \mathrm{s}^{-2} \cdot \mathrm{kg}^{-1}]$

Constants

$\hbar = 1.054 \times 10^{-34}\ \mathrm{m}^2 \cdot \mathrm{kg} \cdot \mathrm{s}^{-1}$
$G = 6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$
$c = 2.998 \times 10^8\ \mathrm{m/s}$

Solution

Express m_P as $m_p = \hbar^\alpha G^\beta c^\gamma$, so the power for m, kg, and s shall match.

$$\begin{cases} 2\alpha + 3\beta + 1\gamma &= 0 \\ \alpha - \beta &= 1 \\ -\alpha - 2\beta - \gamma &= 0 \end{cases} \implies \begin{cases} a &= \frac{1}{2} \\ b &= -\frac{1}{2} \\ c &= \frac{1}{2} \end{cases} \implies$$

$$\implies m_P = \sqrt{\frac{c \cdot \hbar}{G}} = 2.176 \times 10^{-8}\ \mathrm{kg}$$

Similarly, $t_P = c^{-5/2} G^{1/2} \hbar^{1/2} = 5.391 \times 10^{-44}\ \mathrm{s}$, and
$l_P = c^{3/2} G^{1/2} \hbar^{1/2} = 1.616 \times 10^{-35}\ \mathrm{m}$

Dimension Analysis on a Simple Pendulum

Question

A simple pendulum consists of a light inextensible string AB with length L, with the end A fixed, and a point mass M attached to B. The pendulum oscillates with a small amplitude, and the period of oscillation is T. It is suggested that T is proportional to the product of powers of M, L, and g, where g is the acceleration due to gravity. Use dimensional analysis to find this relationship.

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Solution

$$T = M^{\alpha} L^{\beta} g^{\gamma} \implies [s] = [kg]^{\alpha}[m]^{\beta}[m/s^2]^{\gamma}$$

$$\implies \alpha = 0, \beta = 1/2, \gamma = -1/2 \quad T = k\sqrt{L/g}$$

Chain Rules in v-x Relations

Suppose a particle in 1 dimensional motion has the following v-x (SI) relation:

$$v = \sqrt{x+1}$$

Determine $v(t)$.

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Determine $v(t)$.

Solution

By the chain rule of differentiation,

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2\sqrt{x+1}}\sqrt{x+1} = \frac{1}{2}\ \mathrm{m/s^2}$$

$$v(t) = \frac{1}{2}t + v(0)$$

Now $v(t)^2 - v(0)^2 = 2a(t)x(t)$, we obtain $v(0) = 1\ \mathrm{m/s}$

Dot Product in Cartesian Coordinates

Check that in the Cartesian coordinates, the dot product of two vectors $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{w} = (w_x, w_y, w_z)$ can be equivalently found either as $\mathbf{u} \circ \mathbf{w} = u_x w_x + u_y w_y + u_z w_z$, or as $\mathbf{u} \cdot \mathbf{w} = uw \cos \alpha$, where α is the smaller angle between $\mathbf{u}$ and $\mathbf{w}$.

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Solution

$$|\mathbf{u} - \mathbf{w}|^2 = u^2 + w^2 - 2uw\cos\alpha$$

$$uw\cos\alpha = \frac{u^2 + w^2 - |\mathbf{u} - \mathbf{w}|^2}{2}$$

$$= \frac{2(u_x w_x + u_y w_y + u_z w_z)}{2}$$

where $\mathbf{u} - \mathbf{w} = (u_x - w_x)\hat{n}_x + (u_y - w_y)\hat{n}_y + (u_z - w_z)\hat{n}_z$

Inverse Cross Product

Question

Is it possible to find a vector $\mathbf{u}$, such that $(2, -3, 4) \times \mathbf{u} = (4, 3, -1)$? What is a quick way to check it?

Inverse Cross Product

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Is it possible to find a vector $\mathbf{u}$, such that $(2, -3, 4) \times \mathbf{u} = (4, 3, -1)$? What is a quick way to check it?

Solution

Suppose $\mathbf{u} = (u_x, u_y, u_z)$ satisfies this relation.

$$\begin{cases} -1 & = 2u_y + 3u_x \\ 4 & = -3u_z - 4u_y \\ 3 & = 4u_x - 2u_z \end{cases} \implies \begin{cases} -\frac{1}{3} & = \frac{2}{3}u_y + u_x \\ 1 & = -\frac{3}{4}u_z - u_y \\ \frac{3}{4} & = u_x - \frac{1}{2}u_z \end{cases} \implies$$

$\frac{13}{8} = -u_y - \frac{3}{4}u_z$ and $1 = -\frac{3}{4}u_z - u_y \implies \frac{5}{8} = 0$, i.e., not possible.

Quick way: $(2, -3, 4) \circ (4, 3, -1) = -5 \neq 0$

Pulling a Boat at Constant Speed

Question

Suppose a person convolves a rope at constant speed v_0 on the left riverbank that is h above the water. The other end of the rope is fixed on a small boat floating on the surface of the water. Find the speed and the acceleration of the boat when it is x from the person (assuming the rope is weightless).

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Solution

The fact that the motion of the boat is constrained on a straight line allows us to use the magnitude of position vector, velocity, and acceleration directly. Let r denote the length of the rope, then:

$$\begin{cases} \frac{\mathrm{d}r}{\mathrm{d}t} &= -v_0 \\ r &= \sqrt{x^2 + h^2} \end{cases}$$

Pulling a Boat at Constant Speed

Solution (continued)

Our goal is to express $\dot{x}$ and $\ddot{x}$ using x, v_0 and h.
Taking the derivative w.r.t t on both sides of $r = \sqrt{x^2 + h^2}$ using the chain rule,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2x}{2\sqrt{x^2 + h^2}} \frac{\mathrm{d}x}{\mathrm{d}t}$$

so $v = \dot{x} = -\sqrt{x^2 + h^2} v_0 / x$, where the $-$ sign indicates that the boat is moving toward the left.

$$\dot{v} = -v_0 \left[\frac{\frac{2x}{2\sqrt{x^2+h^2}} x - \sqrt{x^2 + h^2}}{x^2} \right] \dot{x} = -v_0 \left[\frac{\frac{x^2 - x^2 - h^2}{\sqrt{x^2+h^2}}}{x^2} \right] \dot{x}$$

$$= [v_0^2 \sqrt{x^2 + h^2}/x][-h^2/(x^2\sqrt{x^2 + h^2})] = -\frac{v_0^2 h^2}{x^3}$$

Parallel and Perpendicular Components of Vectors

Question

Consider two vectors $\mathbf{u} = 3\hat{n}_x + 4\hat{n}_y$ and $\mathbf{w} = 6\hat{n}_x + 16\hat{n}_y$. Find (a) the components of the vector **w** that are parallel and perpendicular to the vector **u**, (b) the angle between **w** and **u**.

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Solution

(a) The parallel component of **w** to **u** is given by the orthogonal projection $w_{||} = \frac{\mathbf{u} \circ \mathbf{w}}{|w|} \cdot \frac{\mathbf{u}}{|u|} = \frac{3\times6+4\times16}{\sqrt{3^2+4^2}} \frac{(3,4)}{\sqrt{3^2+4^2}} = (9.84, 13.12)$. The orthogonal component is given by $\mathbf{w}_\perp = \mathbf{w} - \mathbf{w}_{||} = (-3.84, 2.88)$
(b)

$$\angle(\mathbf{w}, \mathbf{u}) = \arccos \frac{\mathbf{u} \circ \mathbf{w}}{uw} = \arccos[\frac{3 \times 6 + 4 \times 16}{5 \times 17.088}] = 0.285 \text{ rad}$$

Harmonic Oscillation Drifting in One Direction

Question

A particle moves along a straight line with non-constant acceleration $a_x(t) = -A\omega^2 \cos\omega t$, where A and ω are positive constants with proper units. At the instant of time $t = 0$ its velocity $v_x(0) = 3\ [\mathrm{m/s}]$ and position $x(0) = 4\ [\mathrm{m}]$. Find $v_x(t)$ and $x(t)$ at any instant of time. Sketch the graphs of $x(t)$, $v_x(t)$, and $a_x(t)$. What kind of motion may these results describe?

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Solution

$$v_x(t) = v_x(0) + \int_0^t a(\tau)\mathrm{d}\tau = 3 - A\omega \sin\omega t$$

$$x(t) = x(0) + \int_0^t v_x(\tau)\mathrm{d}\tau = 4 + 3t + A(\cos\omega t - 1)$$

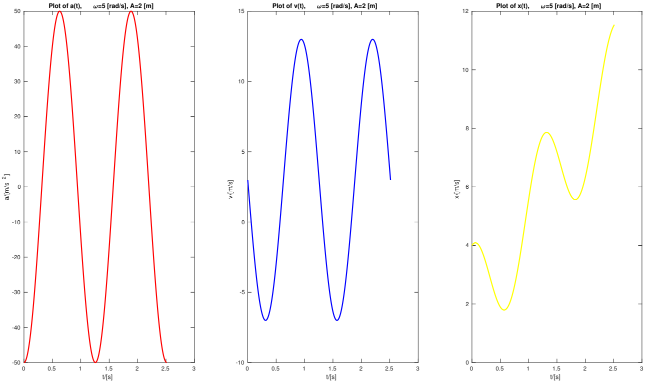
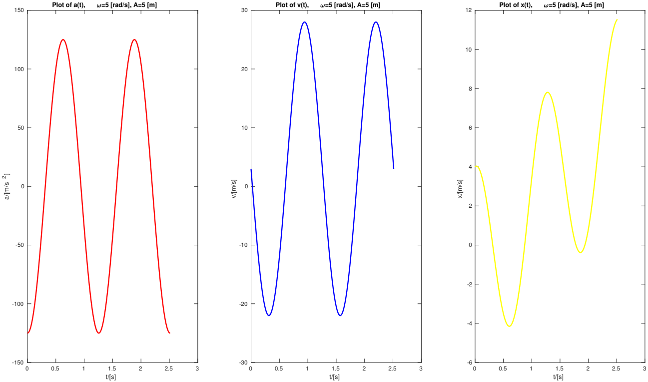


Figure: Plot for x, v, and a given $\omega = 5$ [rad/s], $A = 2$ [m]



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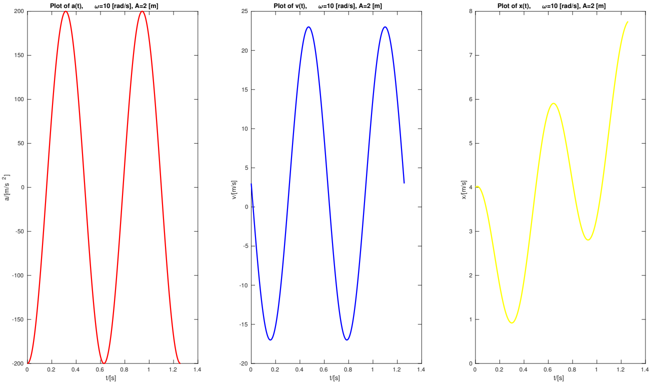


Figure: Plot for x, v, and a given $\omega = 10$ [rad/s], $A = 2$ [m]

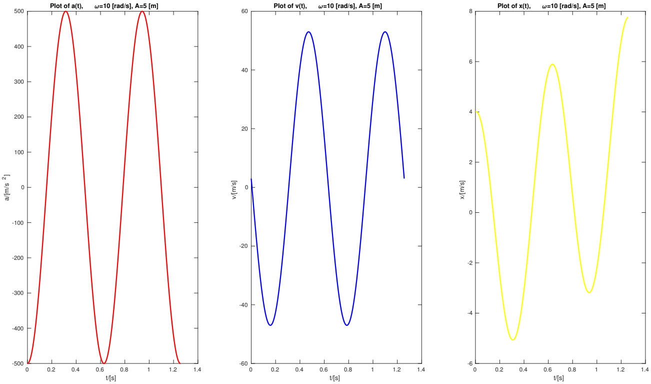


Figure: Plot for x, v, and a given $\omega = 10$ $[\mathrm{rad/s}]$, $A = 5$ $[\mathrm{m}]$

MATLAB Scripts

```
omega=10;t=0:pi/2000:2*2*pi/omega;A=2;

figure
subplot(1,3,1)
plot(t,-A.*omega.^2.*cos(omega.*t),'r-','LineWidth',2);
xlabel('t/[s]');ylabel('a/[m/s^2]');title('Plot of a(t), \omega=10 [rad/s], A=2 [m]');
subplot(1,3,2)
plot(t,3-A.*omega.*sin(omega.*t),'b-','LineWidth',2);
xlabel('t/[s]');ylabel('v/[m/s]');title('Plot of v(t), \omega=10 [rad/s], A=2 [m]');
subplot(1,3,3)
plot(t,4+3.*t+A.*(cos(omega.*t)-1),'y-','LineWidth',2);
xlabel('t/[s]');ylabel('x/[m/s]');title('Plot of x(t), \omega=10 [rad/s], A=2 [m]');
```

An Under-Damped Oscillation

A particle is moving along a straight line with velocity $v_x(t) = -\beta A\omega e^{-\beta t}\cos\omega t$, where A, ω, β are positive constants.

1. What are the units of these constants?
2. Find acceleration $a_x(t)$ and position $x(t)$ of the particle, assuming that $x(0) = 5\ [\mathrm{m}]$.
3. Sketch $x(t)$, $v_x(t)$, and $a_x(t)$
4. What kind of motion could these results refer to (qualitatively)?

An Under-Damped Oscillation (Solution)

βt is dimensionless, so β has unit $[s^{-1}]$. The same holds for ω. $\beta A\omega$ has unit $[m/s]$, so A has unit $[m \cdot s]$

$a_x(t) = \dot{v}_x(t) = \beta^2 A\omega e^{-\beta t}\cos\omega t + \beta A\omega^2 e^{-\beta t}\sin\omega t$

$x(t) = x(0) + \int_0^t v_x(\tau)\mathrm{d}\tau$, where we need to integrate by part.

$$\int_0^t e^{-\beta\tau}\cos\omega\tau\mathrm{d}\tau = -\frac{1}{\beta}e^{-\beta\tau}\cos\omega\tau\Big|_0^t - \int_0^t \frac{\omega}{\beta}e^{-\beta\tau}\sin\omega\tau\mathrm{d}\tau$$

$$\int_0^t e^{-\beta\tau}\sin\omega\tau\mathrm{d}\tau = -\frac{1}{\beta}e^{-\beta\tau}\sin\omega\tau\Big|_0^t + \int_0^t \frac{\omega}{\beta}e^{-\beta\tau}\cos\omega\tau\mathrm{d}\tau$$

so denoting $C = \int_0^t e^{-\beta\tau}\cos\omega\tau\mathrm{d}\tau$, we have

$C = -\frac{1}{\beta}e^{-\beta\tau}\cos\omega\tau\Big|_0^t - \frac{\omega}{\beta}[-\frac{1}{\beta}e^{-\beta\tau}\sin\omega\tau\Big|_0^t + \frac{\omega}{\beta}C]$, i.e.,

$(1 + \frac{\omega^2}{\beta^2})C = -\frac{1}{\beta}e^{-\beta t}\cos\omega t + \frac{1}{\beta} + \frac{\omega}{\beta^2}[e^{-\beta t}\sin\omega t]$

$$C = \frac{\beta^2}{\beta^2+\omega^2}\left[\frac{1}{\beta}(1-e^{-\beta t}\cos\omega t) + \frac{\omega}{\beta^2}e^{-\beta t}\sin\omega t\right]$$

$$x(t) = x(0) - \beta A\omega C = 5 - \tfrac{\beta A\omega}{\beta^2+\omega^2}\left[\beta((1-e^{-\beta t}\cos\omega t)) + \omega e^{-\beta t}\sin\omega t\right]$$

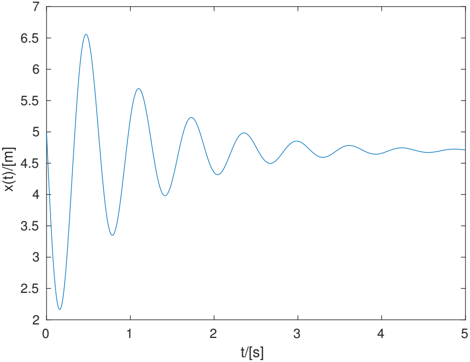
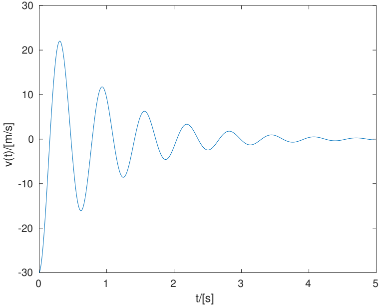


Figure: $x(t)$ given $A = 3\ \mathrm{m}\cdot\mathrm{s}$, $\beta = 1\ \mathrm{s}^{-1}$, $\omega = 10\ \mathrm{rad/s}$

Sketch of $v_x(t)$ and $a_x(t)$



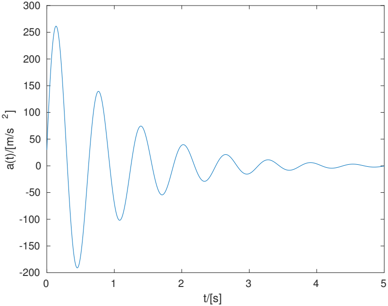


Figure: $v(t)$ and $a(t)$ given $A = 3\ \mathrm{m \cdot s}$, $\beta = 1\ \mathrm{s}^{-1}$, $\omega = 10\ \mathrm{rad/s}$

This represents an underdamped oscillation.

A Moving Car

Question

A car is moving in one direction along a straight line. Find the average velocity of the car if: (a) it travels *half of the journey* with velocity v_1 and the other half with velocity v_2, (b) it covers *half the distance* with velocity v_1 and the other with velocity v_2. Both v_1 and v_2 are constants.

Solution

The formula we use is the definition: $v_{avg,x} = \frac{x}{t}$.
(a) $x = v_1 t/2 + v_2 t/2$, so $v_{avg,x} = \frac{v_1 + v_2}{2}$
(b) $t = x/(2v_1) + x/(2v_2)$, so $v_{avg,x} = \frac{1}{1/(2v_1) + 1/(2v_2)}$

Kinematics in Cartesian Coordinates

The velocity and acceleration are just the Derivatives of the position vector.

Position Vector

$$\bar{r}(t) = x(t)\hat{n}_x + y(t)\hat{n}_y + z(t)\hat{n}_z$$

Velocity

$$\bar{v}(t) = \dot{\bar{r}}(t) = \dot{x}(t)\hat{n}_x + \dot{y}(t)\hat{n}_y + \dot{z}(t)\hat{n}_z$$

Instantaneous Speed $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

Acceleration

$$\bar{a}(t) = \dot{\bar{v}}(t) = \ddot{x}(t)\hat{n}_x + \ddot{y}(t)\hat{n}_y + \ddot{z}(t)\hat{n}_z$$

$a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$

Derivatives of Versors w.r.t. Time

Based on the position vector, we find the velocity and acceleration.

Position Vector in Cylindrical Coordinates

$$\bar{r}(t) = \rho(t)\hat{n}_\rho + z(t)\hat{n}_z$$

Relation Between Versors

$$\hat{n}_\rho = \hat{n}_x \cos\varphi + \hat{n}_y \sin\varphi \quad \hat{n}_\varphi = -\hat{n}_x \sin\varphi + \hat{n}_y \cos\varphi \quad \hat{n}_z = \hat{n}_z$$

Derivatives of Versors w.r.t. Time

Based on the position vector, we find the velocity and acceleration.

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Derivatives of Versors

$$\dot{\hat{n}}_\rho = -\hat{n}_x\dot{\varphi}\sin\varphi + \hat{n}_y\dot{\varphi}\cos\varphi = \dot{\varphi}\hat{n}_\varphi$$

$$\dot{\hat{n}}_\varphi = -\hat{n}_x\dot{\varphi}\cos\varphi - \hat{n}_y\dot{\varphi}\sin\varphi = -\dot{\varphi}\hat{n}\rho$$

Then using the product rule of differentiation, we calculate velocity and acceleration.

Velocity and Acceleration in Cylindrical Coordinates

$\dot{\hat{n}}_\rho = \dot{\varphi}\hat{n}_\varphi$ and $\dot{\hat{n}}_\varphi = -\dot{\varphi}\hat{n}\rho$ are used in the following derivation.

Velocity

$$\overline{v} = \dot{\rho}\hat{n}_\rho + \rho\dot{\hat{n}}_\rho + \dot{z}\hat{n}_z = \dot{\rho}\hat{n}_\rho + \rho\dot{\varphi}\hat{n}_\varphi + \dot{z}\hat{n}_z$$

Velocity and Acceleration in Cylindrical Coordinates

$\dot{\hat{n}}_\rho = \dot{\varphi}\hat{n}_\varphi$ and $\dot{\hat{n}}_\varphi = -\dot{\varphi}\hat{n}\rho$ are used in the following derivation.

Velocity

$$\overline{v} = \dot{\rho}\hat{n}_\rho + \rho\dot{\hat{n}}_\rho + \dot{z}\hat{n}_z = \dot{\rho}\hat{n}_\rho + \rho\dot{\varphi}\hat{n}_\varphi + \dot{z}\hat{n}_z$$

Acceleration

$$\begin{aligned}\overline{a} &= \ddot{\rho}\hat{n}_\rho + \dot{\rho}\dot{\hat{n}}_\rho + \dot{\rho}\dot{\varphi}\hat{n}_\varphi + \rho\ddot{\varphi}\hat{n}_\varphi + \rho\dot{\varphi}\dot{\hat{n}}_\varphi + \ddot{z}\hat{n}_z \\ &= \ddot{\rho}\hat{n}_\rho + \dot{\rho}\dot{\varphi}\hat{n}_\varphi + \dot{\rho}\dot{\varphi}\hat{n}_\varphi + \rho\ddot{\varphi}\hat{n}_\varphi + \rho\dot{\varphi}(-\dot{\varphi}\hat{n}\rho) + \ddot{z}\hat{n}_z \\ &= \underbrace{(\ddot{\rho} - \rho\dot{\varphi}^2)\hat{n}_\rho}_{\text{radial component}} + \underbrace{(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{n}_\varphi}_{\text{transversal component}} + \ddot{z}\hat{n}_z\end{aligned}$$

Setting $z \equiv 0$ in the preceding formulas yields the formulas for the polar coordinates.

- Kinematics in Cartesian Coordinates
- Kinematics in Cylindrical Coordinates
- Kinematics in Spherical Coordinates
- Kinematics in Natural Coordinates
- Discussion
- Exercises

Position Vector in Spherical Coordinates

$$\bar{r}(t) = r(t)\hat{n}_r$$

Relation Between Versors

$$\begin{aligned}
\hat{n}_r &= \sin\theta(\hat{n}_x\cos\varphi + \hat{n}_y\sin\varphi) + \hat{n}_z\cos\theta \\
\hat{n}_\varphi &= -\hat{n}_x\sin\varphi + \hat{n}_y\cos\varphi \\
\hat{n}_\theta &= \cos\theta(\hat{n}_x\cos\varphi + \hat{n}_y\sin\varphi)
\end{aligned}$$

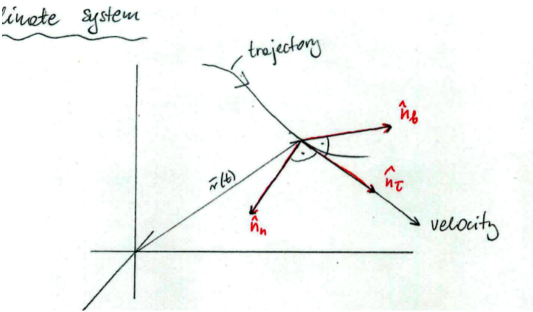
Derivatives of Versors w.r.t. Time

$$\dot{\hat{n}}_r = \dot{\theta}\hat{n}_\theta + \dot{\varphi}\sin\theta\hat{n}_\varphi$$

$$\dot{\hat{n}}_\varphi = -\dot{\varphi}\sin\theta\hat{n}_r - \dot{\varphi}\cos\theta\hat{n}_\theta$$

$$\dot{\hat{n}}_\theta = -\dot{\theta}\hat{n}_r + \dot{\varphi}\cos\theta\hat{n}_\varphi$$

Natural Coordinates



Versors:

$\hat{n}_\tau$: tangent (along $\overline{v}$)

$\hat{n}_n$: normal

$\hat{n}_b$: binormal

Velocity:

$\overline{v}(t) = v\hat{n}_\tau$

$\hat{n}_\tau = \frac{\overline{v}}{v} = \frac{\dot{\overline{r}}}{|\dot{\overline{r}}|}$

Assumption: the trajectory is not straight;

the particle moves in one direction.

$$\hat{n}_n = \frac{\dot{\hat{n}}_\tau}{|\dot{\hat{n}}_\tau|} \quad \hat{n}_b = \hat{n}_\tau \times \hat{n}_n$$

Acceleration and Curvature

Acceleration

$$\bar{a} = \underbrace{\dot{v}\hat{n}_\tau}_{\text{tangent component}} + \underbrace{v|\dot{\hat{n}}_\tau|\hat{n}_n}_{\text{normal component}}$$

Radius of Curvature

$$R_c = \frac{v}{|\dot{\hat{n}}_\tau|}$$

$$\bar{a} = \underbrace{\dot{v}\hat{n}_\tau}_{\text{tangential component } \bar{a}_\tau} + \underbrace{(v^2/R_c)\hat{n}_n}_{\text{normal component } \bar{a}_n}$$

- Kinematics in Cartesian Coordinates
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The Difference Between $\dot{\mathbf{v}}$ and $\dot{v}$

The derivative of a vector **v** is the vector whose components are derivatives of the components in the original vector. It is exactly the **acceleration** of the particle. The derivative of a scalar v is the rate of change of the magnitude of velocity. It is precisely the magnitude of the **tangential component** of acceleration.

Example

Consider a particle moving with velocity $\overline{v}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$, so $\dot{\overline{v}} = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$.

Now $v = \sqrt{t^2 + t^4 + t^6}$, so $\dot{v} = \frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}}$.

Now the unit tangent vector $\hat{n}_\tau = \frac{v}{|v|} = \frac{1}{\sqrt{t^2+t^4+t^6}} \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$

$\dot{v}$ as Magnitude of Tangential Component of **a**

Example

Now the tangential component and normal component of the acceleration can be calculated using the inner product of these unit vectors and acceleration. The magnitude $a_\tau = \langle \overline{a}, \hat{n}_\tau \rangle$ and $a_n = \langle \overline{a}, \hat{n}_n \rangle$.

$$a_\tau = \frac{1}{\sqrt{t^2 + t^4 + t^6}} \left\langle \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}, \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \right\rangle = \frac{t + 2t^3 + 3t^5}{\sqrt{t^2 + t^4 + t^6}} = \dot{v}$$

This results conforms with the assertion that $\dot{v}$ is just the magnitude of the tangential component $\overline{a}_\tau$ of the acceleration $\overline{a} = \dot{\overline{v}}$.

Then, applying the quotient rule for the derivative, we find the unit normal vector:

Calculating $\hat{n}_n$

Example

$$\dot{\hat{n}}_\tau = \frac{1}{t^2+t^4+t^6}\begin{pmatrix} \sqrt{t^2+t^4+t^6} - t\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \\ 2t\sqrt{t^2+t^4+t^6} - t^2\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \\ 3t^2\sqrt{t^2+t^4+t^6} - t^3\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \end{pmatrix}$$

$$|\dot{\hat{n}}_\tau| = \frac{\sqrt{t^4+4t^6+t^8}}{t^2+t^4+t^6}$$

$$\hat{n}_n = \frac{\dot{\hat{n}}_\tau}{|\dot{\hat{n}}_\tau|} = \frac{1}{\sqrt{t^4+4t^6+t^8}}\begin{pmatrix} \sqrt{t^2+t^4+t^6} - t\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \\ 2t\sqrt{t^2+t^4+t^6} - t^2\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \\ 3t^2\sqrt{t^2+t^4+t^6} - t^3\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \end{pmatrix}$$

Calculating a_n

Example

$$a_n = \frac{1}{\sqrt{t^4 + 4t^6 + t^8}} \left\langle \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}, \begin{pmatrix} \sqrt{t^2 + t^4 + t^6} - t\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \\ 2t\sqrt{t^2 + t^4 + t^6} - t^2\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \\ 3t^2\sqrt{t^2 + t^4 + t^6} - t^3\frac{2t+4t^3+6t^5}{2\sqrt{t^2+t^4+t^6}} \end{pmatrix} \right\rangle$$

$$= \frac{\sqrt{t^8 + 4t^6 + t^4}}{\sqrt{t^6 + t^4 + t^2}}$$

and we can check that $a_n^2 + a_\tau^2 = a^2$

Role of the normal component $\mathbf{a}_n$

Remarks

$$\hat{n}_\tau \circ \hat{n}_\tau = 1 \implies \frac{\mathrm{d}}{\mathrm{d}t}[\hat{n}_\tau \circ \hat{n}_\tau] = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\hat{n}_\tau \circ \hat{n}_\tau + \hat{n}_\tau \circ \frac{\mathrm{d}}{\mathrm{d}t}\hat{n}_\tau = 0$$

Notice that $\dot{\hat{n}}_\tau$ is perpendicular to $\hat{n}_\tau$ because $\hat{n}_\tau$ has unit length. The normal component of acceleration, therefore, only changes the direction of velocity, and has no effect on the magnitude of velocity.

Differential Geometry in Polar Coordinates

Changing r ,keeping φ constant, results in displacement along $\bar{r}$, while changing φ, keeping r constant, results in displacement perpendicular to $\bar{r}$. Putting these two kinds of changes in the form of infinitesimal displacement vector: $\hat{n}_r \mathrm{d}r$ and $\hat{n}_\varphi r \mathrm{d}\varphi$, we note that in fact,

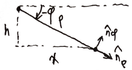
$$\underbrace{\mathrm{d}\bar{r}}_{\text{Infinitesimal displacement}} = \underbrace{\hat{n}_r \mathrm{d}r}_{\text{Radial Component}} + \underbrace{\hat{n}_\varphi r \mathrm{d}\varphi}_{\text{Transversal Component}}$$

Therefore, by the Pythagoras' theorem,

$$|\mathrm{d}\bar{r}|^2 = (\mathrm{d}r)^2 + (r\mathrm{d}\varphi)^2$$

In fact, this is exactly the case for velocity: we can decompose the velocity into radial and transversal components, and exploit the fact that they are mutually perpendicular to each other.

In polar coordinates



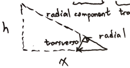
$$\dot\rho = -v_0$$

$$-\sin\varphi = \frac{h}{\rho} \qquad \cos\varphi = \frac{x}{\rho}$$

$$-\cos\varphi\,\dot\varphi = -\frac{h}{\rho^2}\dot\rho = \frac{h}{\rho^2}v_0$$

$$\dot\varphi = -\frac{hv_0}{\rho^2\frac{x}{\rho}} = -\frac{hv_0}{x\rho}$$

$$\vec v = \dot\rho\,\hat n_\rho + \rho\dot\varphi\,\hat n_\varphi = \underbrace{-v_0\hat n_\rho}_{\text{radial component}} - \underbrace{\frac{hv_0}{x}\hat n_\varphi}_{\text{transversal component}}$$



(also fine to derive the transversal velocity using differential geometry)

$$\ddot\rho = 0 \qquad \ddot\varphi = \frac{d}{dt}\left[-\frac{hv_0}{\rho\sqrt{\rho^2-h^2}}\right]$$

$$\ddot\varphi = \frac{d}{d\rho}\left[-\frac{hv_0}{\rho\sqrt{\rho^2-h^2}}\right]\frac{d\rho}{dt}$$

$$= \frac{hv_0\left[\sqrt{\rho^2-h^2}+\frac{2\rho^2}{2\sqrt{\rho^2-h^2}}\right]}{\rho^2(\rho^2-h^2)}(-v_0)$$

$$= \frac{hv_0[\rho^2-h^2+\rho^2](-v_0)}{\rho^2(\rho^2-h^2)^{3/2}}$$

$$= \frac{-hv_0^2[2\rho^2-h^2]}{\rho^2(\rho^2-h^2)^{3/2}}$$

$$|\vec v(t)| = \sqrt{1+\left(\frac{h}{x}\right)^2}\,v_0$$

$$|\vec a(t)| = \frac{h^2v_0^2}{\rho x^3}\sqrt{h^2+x^2} = \frac{h^2v_0^2}{x^3}$$

$$\vec a(t) = (\ddot\rho-\rho\dot\varphi^2)\hat n_\rho + (\rho\ddot\varphi+2\dot\rho\dot\varphi)\hat n_\varphi$$

$$= -\rho\left[\frac{hv_0}{x\rho}\right]^2\hat n_\rho + \left(\frac{-hv_0^2[2\rho^2-h^2]}{\rho(\rho^2-h^2)^{\frac{3}{2}}} + 2(-v_0)\left(-\frac{hv_0}{x\rho}\right)\right)\hat n_\varphi$$

$$= -\rho\frac{h^2v_0^2}{x^2\rho^2}\hat n_\rho + \left(\frac{-hv_0^2[\rho^2+x^2]}{\rho x^3} + \frac{2hv_0^2}{x\rho}\right)\hat n_\varphi$$

$$= -\frac{h^2v_0^2}{x^2\rho}\hat n_\rho + \frac{hv_0^2}{\rho x^3}\left[2x^2-x^2-\rho^2\right]\hat n_\varphi$$

$$= \underbrace{-\frac{h^2v_0^2}{x^2\rho}\hat n_\rho}_{\text{radial component}} \underbrace{- \frac{h^3v_0^2}{\rho x^3}\hat n_\varphi}_{\text{transversal component}}$$

A Parabolic Motion

A particle moves in the $x - y$ plane so that

$$x(t) = at, \quad y(t) = bt^2$$

where a, b are positive constants. Find its trajectory, velocity, and acceleration (its tangential and normal components).

Solution

The trajectory is $y = b(x/a)^2$. The position vector $\bar{r} = \begin{pmatrix} at \\ bt^2 \end{pmatrix}$, so the velocity is $\dot{\bar{r}} = \begin{pmatrix} a \\ 2bt \end{pmatrix}$. The acceleration is $\ddot{\bar{r}} = \begin{pmatrix} 0 \\ 2b \end{pmatrix}$.

The unit tangent vector $\hat{n}_\tau = \frac{\bar{v}}{v} = \frac{1}{\sqrt{a^2+4b^2t^2}} \begin{pmatrix} a \\ 2bt \end{pmatrix}$.

A Parabolic Motion

Solution (Continued)

The tangential component of acceleration $\mathbf{a}_\tau = \langle \bar{a}, \hat{n}_\tau \rangle \, \hat{n}_\tau$

$$\mathbf{a}_\tau = \frac{4b^2 t}{\sqrt{a^2 + 4b^2 t^2}} \frac{1}{\sqrt{a^2 + 4b^2 t^2}} \begin{pmatrix} a \\ 2bt \end{pmatrix} = \frac{1}{a^2 + 4b^2 t^2} \begin{pmatrix} 4ab^2 t \\ 8b^3 t^2 \end{pmatrix}$$

The normal component of acceleration $\mathbf{a}_n = \mathbf{a} - \mathbf{a}_\tau$

$$\mathbf{a}_n = \frac{1}{a^2 + 4b^2 t^2} \begin{pmatrix} -4ab^2 t \\ 2b(a^2 + 4b^2 t^2) - 8b^3 t^2 \end{pmatrix} = \frac{1}{a^2 + 4b^2 t^2} \begin{pmatrix} -4ab^2 t \\ 2ba^2 \end{pmatrix}$$

Relative Motion of Two Particles

Question

The velocities of two particles observe from a fixed frame of reference are given in the Cartesian coordinates by vectors $\mathbf{v}_1(t) = (0, 2, 0) + (3, 1, 2)t^2$ and $\mathbf{v}_2(t) = (1, 0, 1)$. At the initial instant of time $t = 0$, the positions of these particles are $\mathbf{r}_1(0) = (1, 0, 0)$, and $\mathbf{r}_2(0) = (0, 1, 1)$.
Find the positions of both particles and the acceleration of particle 1 (and its tangential and normal components), relative position, and relative acceleration of particle 1 with respect to particle 2 at any instant of time t.

Relative Motion of Two Particles (Solution)

The positions are found as follows:

$\mathbf{r}_1(t) = \mathbf{r}_1(0) + \int_0^t \mathbf{v}_1(\tau)\mathrm{d}\tau = (1,0,0) + (0,2,0)t + (1,1/3,2/3)t^3$

$\mathbf{r}_2(t) = \mathbf{r}_2(0) + \int_0^t \mathbf{v}_2(\tau)\mathrm{d}\tau = (0,1,1) + (1,0,1)t$

The acceleration of particle 1 and 2 are found as follows:

$\mathbf{a}_1(t) = \dot{\mathbf{v}}_1(t) = (6,2,4)t \quad \mathbf{a}_2(t) = \overline{0}$

The unit tangent vector for particle 1 is found as

$\hat{n}_{\tau,1} = \frac{\mathbf{v}_1(t)}{|\mathbf{v}_1(t)|} = \frac{1}{\sqrt{9t^4+(2+t^2)^2+4t^4}}[(0,2,0)+(3,1,2)t^2]$

so the tangential component of acceleration is found as

$\mathbf{a}_{\tau,1} = \left\langle \mathbf{a}_1(t), \hat{n}_{\tau,1} \right\rangle \hat{n}_{\tau,1} = \frac{t(18t^2+4+2t^2+8t^2)}{9t^4+(2+t^2)^2+4t^4}[(0,2,0)+(3,1,2)t^2]$

Relative Motion of Two Particles (Continued Solution)

$\mathbf{a}_{\tau,1} = \frac{2\left(7t^3+t\right)}{7t^4+2t^2+2}\begin{pmatrix}3t^2\\2+t^2\\2t^2\end{pmatrix}$, so the normal component of accelration

$\mathbf{a}_{n,1} = \mathbf{a}_1 - \mathbf{a}_{\tau,1} = \frac{1}{2+2t^2+7t^4}\begin{pmatrix}6t(2+t^2)\\-26t^3\\4t(2+t^2)\end{pmatrix}$. Check they are orthogornal!

The relative position of particle 1 w.r.t. particle 2 is

$$\mathbf{r}_1(t) - \mathbf{r}_2(t) = (1,-1,-1) + (-1,2,-1)t + (1,1/3,2/3)t^3$$

The relative acceleration of particle 1 w.r.t. particle 2 is

$$\mathbf{a}_1(t) - \mathbf{a}_2(t) = (6,2,4)t$$

Beetle on the Wheel

A disc of radius R rotates about its axis of symmetry (perpendicular to the disk surface) with constant angular velocity $\dot{\varphi} = \omega = const$. At the instant of time $t = 0$ a beetle starts to walk with constant speed v_0 along a radius of the disk, from its center to the edge. Find

1. the position of the beetle and its trajectory in the Cartesian and polar coordinate systems,
2. its velocity in both systems,
3. its acceleration in both systems (Cartesian components, polar components, as well as tangential and normal components).

Beetle on the Wheel (Solution)

Position and Trajectory

In the Polar Coordinate system, $r = v_0 t$, $\varphi = \omega t$. Hence in the Cartesian Coordinate system, $x(t) = v_0 t \cos \omega t$, $y(t) = v_0 t \sin \omega t$. The trajectory in the Polar coordinates is $r = v_0 \varphi / \omega$. The trajectory in the Cartesian coordinates is found by

$$\begin{cases} \tan \omega t & = y/x \\ x^2 + y^2 & = v_0^2 t^2 \end{cases}$$

so the trajectory is $y/x = \tan(\omega \sqrt{x^2 + y^2}/v_0)$, known as Archimedes' spiral.

Velocity

In the Polar Coordinate system, $\dot{r} = v_0$, $\dot{\varphi} = \omega$. Therefore,

$$v_r = \dot{r} = v_0, \text{ and } v_\varphi = r\dot{\varphi} = v_0 \omega t.$$

In the Cartesian Coordinate system, $v_x = \dot{x}(t) = v_0 \cos\omega t - \omega v_0 t \sin\omega t$, $v_y = \dot{y}(t) = v_0 \sin\omega t + \omega v_0 t \cos\omega t$.

Acceleration

In the Polar Coordinate system, $\ddot{r} = 0$, and $\ddot{\varphi} = 0$. Therefore,

$$a_r = \ddot{r} - r\dot{\varphi}^2 = -v_0 t\omega^2 \text{ and } a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 2v_0\omega.$$

In the Cartesian Coordinate system,
$a_x = \ddot{x}(t) = -\omega v_0(2\sin\omega t + \omega t \cos\omega t)$, and
$a_y = \ddot{y}(t) = \omega v_0(2\cos\omega t - \omega t \sin\omega t)$.

CAUTION: Tangential component is not radial component in this case.

Tangential Component and Normal Component

Based on the previous results, we calculate v, with which we find the magnitude of the tangential component of acceleration.

$$v = \sqrt{v_r^2 + v_\varphi^2} = v_0\sqrt{1 + (\omega t)^2}$$

$$a_\tau = \dot{v} = v_0 \frac{\omega^2 t}{\sqrt{1 + (\omega t)^2}}$$

Then we exploit the fact that the tangential and the normal components are perpendicular to each other to find the magnitude of the normal component from a: $a = \sqrt{a_r^2 + a_\varphi^2} = v_0\omega\sqrt{(\omega t)^2 + 4}$

$$a_n = \sqrt{a^2 - a_\tau^2} = \frac{v_0\omega(2 + (\omega t)^2)}{\sqrt{1 + (\omega t)^2}}$$

More on the Beetle

1. What is the distance covered by the beetle?

$$s = \int_0^T v\mathrm{d}t = \int_0^T v_0\sqrt{1+(\omega t)^2}\mathrm{d}t$$
$$= v_0\left(\frac{1}{2}T\sqrt{\omega^2 T^2+1}+\frac{\sinh^{-1}(\omega T)}{2\omega}\right)$$

2. What is the radius of curvature of the trajectory?

$$R_c = \frac{v^2}{a_n} = \frac{v_0(1+\omega^2 t^2)^{3/2}}{\omega(2+\omega^2 t^2)}$$

Hyperbolic Spiral Motion

Question

A particle moves along a hyperbolic spiral (i.e. a curve $r = c/\varphi$, where c is a positive constant), so that $\varphi(t) = \varphi_0 + \omega t$, where φ_0 and ω are positive constants. Fint its velocity and acceleration (all components and magnitudes of both vectors).

Hyperbolic Spiral Motion

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Solution

$\dot{\varphi} = \omega \quad \dot{r} = -c/\varphi^2 \cdot \omega$, so $v_r = -c\omega/(\varphi_0 + \omega t)^2$, and $v_\varphi = \omega c/(\varphi_0 + \omega t)$

$v = \sqrt{v_r^2 + v_\tau^2} = [\omega c/(\varphi_0 + \omega t)^2]\sqrt{1 + (\varphi_0 + \omega t)^2}$

$\ddot{\varphi} = 0 \quad \ddot{r} = (2\omega^2 c)/\varphi^3$, so $a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = -2c\omega^2/(\varphi_0 + \omega t)^2$

$a_r = \ddot{r} - r\dot{\varphi}^2 = (2\omega^2 c)/(\varphi_0 + \omega t)^3 - \omega^2 c/(\varphi_0 + \omega t)$

$a = \sqrt{a_\varphi^2 + a_r^2} = \sqrt{\frac{c^2\omega^4(4+(\varphi_0+\omega t)^4)}{(\varphi_0+\omega t)^6}}$

Four Crawling Spiders

Four spiders are initially placed at the four corners of a square with side length l. The spiders crawl counter-clockwise at the same speed v and each spider crawls directly toward the next spider at all times. They approach the center of the square along spiral paths. Find

1. polar coordinates of a spider at any instant of time, assuming the origin is at the center of the square.
2. the time after which all spiders meet.
3. the trajectory of a spider in polar coordinates.
4. the acceleration of a spider, and the radius of curvature at any instant of time.

CAUTION: The transversal component is not the tangential component in this case.

Four Crawling Spiders (Solution)

Due to the symmetry of the problem, we study the spider starting at $r(0) = l/\sqrt{2}$ and $\varphi(0) = 0$. Notice that the four spiders always lie on the four corners of a square due to symmetry. Now the fact that one spider always aims directly at the next spider is interpreted as each spider having a radial velocity $v_r = -v/\sqrt{2}$ and a transversal velocity $v_\varphi = v/\sqrt{2}$. Therefore, $\dot{r} = -v/\sqrt{2}$ and $\dot{\varphi} = (v/\sqrt{2})/r(t)$. Now $r(t) = r(0) + \int_0^t \dot{r}(\tau)\mathrm{d}\tau = l/\sqrt{2} - vt/\sqrt{2}$, and $\varphi(t) = \varphi(0) + \int_0^t \dot{\varphi}(\tau)\mathrm{d}\tau$.

$$\varphi(t) = \int_0^t \frac{v}{(l - v\tau)}\mathrm{d}\tau = \int_0^{vt} \frac{\mathrm{d}s}{l - s} = -\int_0^{vt} \frac{\mathrm{d}s}{s - l} = -\int_{-l}^{vt-l} \frac{\mathrm{d}w}{w}$$

so the polar coordinates are given by

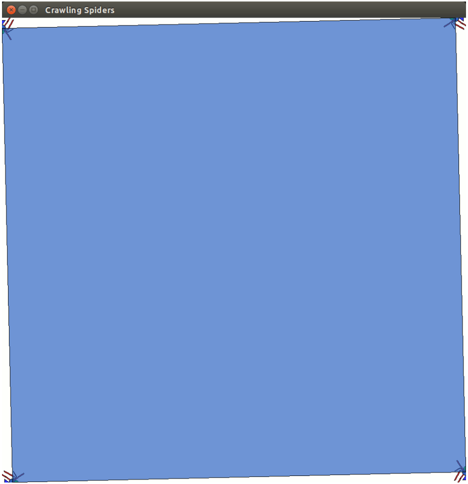
$$r(t) = \frac{l - vt}{\sqrt{2}} \quad \varphi(t) = -\ln\left(\frac{vt - l}{-l}\right)$$

The time t_f the spiders meet is the time when $r(t_f) = 0$, so $t_f = l/v$
The trajectory of the spider is given by

$$\varphi = -\ln\left(\frac{\sqrt{2}r}{l}\right)$$

The acceleration of the spider is given by
$\mathbf{a}(t) = (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi$, where $\ddot{r} = 0$ and
$\ddot{\varphi} = -[(v/\sqrt{2})/r(t)^2](-v/\sqrt{2}) = v^2/(l - vt)^2$. Hence,
$a_r = -v^2/[\sqrt{2}(l - vt)]$,
$a_\varphi = v^2/[\sqrt{2}(l - vt)] + \sqrt{2}(-v)(v/\sqrt{2})/[(l - vt)/\sqrt{2}] =$
$-(\sqrt{2} - 1/\sqrt{2})v^2/(l - vt)$
$a = \sqrt{a_r^2 + a_\varphi^2} = v^2/(l - vt)$
Since there is no tangential acceleration, this is the normal acceleration, so the radius of curvature is $(l - vt)$.

A Numerical Animation



The animation works with Adobe Reader XI or Adobe Acrobat Reader DC. Equivalent GIF is uploaded to CANVAS.

```
void SpiderChase(Point* spiders,double* angle, int size){
        double step=0.00004,newx,newy;
        double distance=sqrt(pow((spiders[0].x-spiders[1].x),2)
            +pow((spiders[0].y-spiders[1].y),2));
        if (distance<=step*5.0){
                spiders[0]={-1.0,1.0};spiders[1]={-1.0,-1.0};
                    spiders[2]={1.0,-1.0};spiders[3]={1.0,1.0};
                distance=sqrt(pow((spiders[0].x-spiders[1].x)
                    ,2)+pow((spiders[0].y-spiders[1].y),2));
        }
        for (int i=0;i<size;i++){
                newx=spiders[i].x+step/distance*(spiders[(i+1)%
                    size].x-spiders[i].x);
                newy=spiders[i].y+step/distance*(spiders[(i+1)%
                    size].y-spiders[i].y);
                spiders[i]={newx,newy};
                angle[i]=atan2(spiders[(i+1)%size].y-spiders[i
                    ].y,spiders[(i+1)%size].x-spiders[i].x)-PI
                    *0.5;
        }
}
```

- Force
- Newton's Laws
- Application of Newton's Laws
- Motion with Air/Fluid Drag
- Simple and linearly damped Oscillator
- Exercises

Force

Definition

Force is interaction between two objects or an object and its environment. The interactions are of material origin. Force is a vector quantity with SI unit Newton. 1 $\mathrm{N} =$ 1 $\mathrm{kg} \cdot \mathrm{m/s^2}$

Several Forces

Normal Force When an object pushes on a surface, the surface pushes back on the object in the direction perpendicular to the surface.
Friction When an object slides on a surface, the surface resists such sliding parallel to the surface.
Tension A pulling force exerted on an object by rope/cord.
Weight Pull of gravity on an object.

Newton's First Law

Essence

An Inertial frame of reference exists.

Inertial frame of reference

A special class of frames of reference is inertial frames of reference, where a particle acted upon by zero net force moves with constant velocity.

$$\sum \overline{F} = 0 \Leftrightarrow \overline{a} = 0$$

Newton's Second Law

In an inertial frame of reference (identified by the first law), acceleration of a particle is directly proportional to the net force, and is inversely proportional to the mass.

1. $\overline{F} \neq 0 \Leftrightarrow \overline{a} \neq 0$
2. $\overline{a} \propto \overline{F}$
3. $\overline{a} \propto 1/m$

Equivalence of all Inertial FoRs (Galilean Invariation)

$r(t) = r_{O'}(t) + r'(t)$
$v(t) = v_{O'}(t) + v'(t)$
$a(t) = a'(t)$

Conclusion: Enough to have one inertial FoR.

Free Body Diagram

Definition

A free-body diagram is a sketch showing all forces acting upon an object. When kinematics and dynamics are both involved, we sketch two diagrams, with one diagram is sketched for alertkinematics, and the other for dynamics.

Remarks

Newton's Second Law bridges kinematics and dynamics.

Newton's Third Law

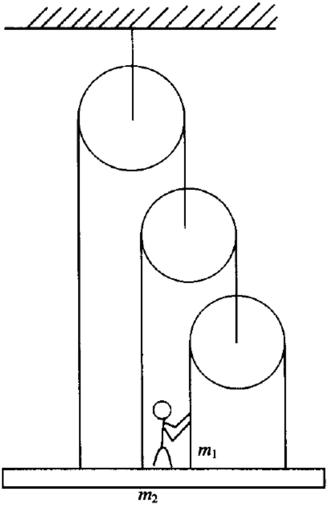
Statement

The mutual forces of action and reaction between two bodies are equal in magnitude and opposite in direction.

Remarks

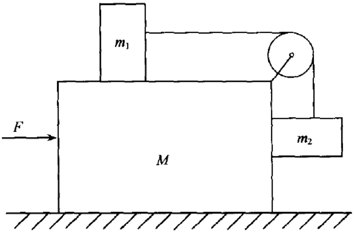
Newton's third law allows us to consider several objects as a system and ignore the internal forces of the system when we study the kinematics and dynamics of the system as a whole.

Particles in Static Balance



Now consider a person with mass $m_1 = 60\ \mathrm{kg}$ standing on a board with mass $m_2 = 20\ \mathrm{kg}$. Ignoring the friction between the rope and the wheels and the mass of them. How much force does the person need to exert on the rope to keep himself and the board static?

Particles in Motion



Now consider the situation shown in the figure. All the surfaces are frictionless, and the weight of the wheel and the ropes can be ignored. Find the horizontal force F and the stress N block M exerts on the horizontal surface in the following two cases:

1. There is no relative motion among block m_1, m_2, and M
2. M is static

Friction

Consider a brick sliding upward an inclined surface 30° to the horizontal plane. Its initial speed is 1.5 $\mathrm{m/s}$, and the coefficient of kinetic friction $\mu = \sqrt{3}/12$. How far is the brick from its initial position after 0.5 s?

Projectile Motion with Linear Drag

Question

Consider a particle launched with horizontal speed $v_x(0)$ and vertical speed $v_y(0)$ from the origin. The drag is linear, i.e., $\bar{f} = -\alpha\bar{v}$. Find its position at time t.

ODE Solution as IVP

$$\begin{cases} m\dot{v}_x &= -\alpha v_x \\ m\dot{v}_y &= -mg - \alpha v_y \end{cases} \implies \begin{cases} \frac{\mathrm{d}v_x}{v_x} &= -(\alpha/m)\mathrm{d}t \\ \frac{\mathrm{d}(v_y+mg/\alpha)}{v_y+mg/\alpha} &= -(\alpha/m)\mathrm{d}t \end{cases} \implies$$

$$\begin{cases} \ln(v_x(t)) - \ln(v_x(0)) &= -(\alpha/m)t \\ \ln(v_y(t) + mg/\alpha) - \ln(v_y(0) + mg/\alpha) &= -(\alpha/m)t \end{cases}$$

$v_x(t) = v_x(0)e^{-(\alpha/m)t}$ $v_y(t) = (v_y(0) + mg/\alpha)e^{-(\alpha/m)t} - mg/\alpha$
$x(t) = v_x(0)(1 - e^{-(\alpha/m)t})m/\alpha$
$y(t) = (v_y(0) + mg/\alpha)(1 - e^{-(\alpha/m)t})m/\alpha - mgt/\alpha$

Free Fall with Quadratic Drag $f = -kv^2$

Taking the vertically downward direction as the positive direction,

$$m\dot{v} = mg - kv^2 \implies (k/m)v^2 + \dot{v} = g$$

This is a Ricatti's equation with one trivial solution being $v = \sqrt{mg/k}$. $v = \sqrt{mg/k} + 1/z$, where z is the solution to $z' - (2\sqrt{mg/k})(k/m)z = (k/m)$. Now $z' - 2\sqrt{g(k/m)}z = 0$ is the homogeneous equation, $z^{hom} = Ce^{2\sqrt{g(k/m)}t}$, and a particular solution is given by $z^{part} = -\sqrt{k/(mg)}/2$, so the general solution for v is

$$v = \sqrt{mg/k} + \frac{1}{Ce^{2\sqrt{g(k/m)}t} - \sqrt{\frac{k}{4mg}}}$$

Now the initial condition is $v(0) = 0$, so $C = -\sqrt{k/(4mg)}$, the solution

$$v(t) = \sqrt{mg/k} - \frac{1}{\sqrt{\frac{k}{4mg}}e^{2\sqrt{g(k/m)}t} + \sqrt{\frac{k}{4mg}}}$$

Simple Harmonic Oscillator

Definition

A **simple harmonic oscillator** is a particle under a net external force proportional in magnitude to its displacement from equilibrium, and towards equilibrium in direction. Such an external force is called the restoring force.

In the case of 1 dimension,

$$\sum F = -kx \implies \ddot{x} = -\frac{k}{m}x \implies \ddot{x} + \frac{k}{m}x = 0$$

Characteristic equation $s^2 + \frac{k}{m} = 0$, Characteristic roots $s_{1,2} = \pm j\sqrt{\frac{k}{m}}$
General solution given by

$$x = C_1 e^{s_1 t} + C_2 e^{s_2 t} = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

where natural frequency$\omega_0 = \sqrt{k/m}$, so period $T = 2\pi/\omega = 2\pi\sqrt{m/k}$

Harmonic Oscillator with Linear Damping

x is displacement from equilibrium, $b > 0$ is constant.

$$m\ddot{x} = \underbrace{-b\dot{x}}_{\text{Linear Drag}} - kx$$

A linear, second order, homogeneous ODE with constant coefficients is obtained:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

Characteristic Equation $s^2 + \frac{b}{m}s + \frac{k}{m} = 0$, so Characteristic Roots

$$s_{1,2} = \begin{cases} \frac{-b \pm \sqrt{b^2 - 4km}}{2m} & \text{if } b^2 > 4km \\ -\frac{b}{2m} & \text{if } b^2 = 4km \\ \frac{-b \pm j\sqrt{-b^2 + 4km}}{2m} & \text{if } b^2 < 4km \end{cases}$$

Three Regimes: b^2 vs. $4km$

General solution

$$x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \text{ if } s_1 \neq s_2 \quad x = C_1 e^{s_1 t} + C_2 t e^{s_2 t} \text{ if } s_1 = s_2$$

Overdamped Regime: $b^2 > 4km$

$$x(t) = C_1 e^{-\left(\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t} + C_2 e^{-\left(\frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t}$$

Critically Damped Regime: $b^2 = 4km$

$$x(t) = C_1 e^{-\frac{b}{2m}t} + C_2 t e^{-\frac{b}{2m}t}$$

Under Damped Regime: $b^2 < 4km$

$$x(t) = e^{-\frac{b}{2m}t}\left[A\cos\left(\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}\,t\right) + B\sin\left(\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}\,t\right)\right]$$

Mass on a Car

Question

Mass m hangs on a massless rope in a car moving with (a) constant velocity $\mathbf{v}$, (b) constant acceleration $\mathbf{a}$ on a horizontal surface. What is the angle the rope forms with the vertical direction?

Solution

Recall: tension on a massless rope is along the rope. (a) The mass is moving with constant velocity, i.e., zero net force. Now gravity and tension are the only two forces on this mass, so they are equal in magnitude and opposite in direction. Hence the rope is parallel to the vertical direction. (b) Now the net force on the mass is $m\mathbf{a}$, horizontal, so the horizontal component of the tension is ma, and the vertical component of the tension is mg. The rope forms $\arctan(a/g)$ with the vertical direction.

Sliding car on an Inclined Plane

Question

Mass m hangs on a massless rope in a car sliding down an inclined plane (frictionless) at an angle α. What is the angle the rope forms with the vertical direction?

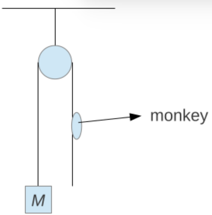
Solution

Consider the mass sliding down the same inclined plane. It slides in an identical fashion as the car. Apart from gravity, a normal force is exerted on the mass perpendicular to the surface of the plane. The parallel component of net force is completely due to gravity. Therefore, when the mass is attached to the rope, to follow a same motion, the parallel component of net force is also due to gravity. The tension shall only contribute to the normal component. Therefore, the rope forms α with the vertical direction.

Monkey and Pulley

A monkey with mass m holds a rope hanging over a frictionless pulley attached to mass M. Discuss the motion of the system if the monkey

1. does not move with respect to the rope,
2. climbs up the rope with constant velocity v_0 with respect to the rope,
3. climbs up the rope with constant acceleration a_0 with respect to the rope.



Monkey and Pulley (Solution)

In case a and b, the monkey and the mass have accelerations that are equal in magnitude and opposite in direction. The acceleration a must satisfy Newton's second law for both the monkey and the mass. For the monkey,

$$ma = T - mg$$

for the mass,

$$Ma = Mg - T$$

Adding them together, we get $a = \frac{M-m}{M+m}g$. For case c, let a denote the acceleration of the mass.

$$m(a + a_0) = T - mg \quad Ma = Mg - T$$

so we get $a = \frac{Mg-m(g+a_0)}{M+m}$

Free Fall with Quadratic Air Drag (Continued)

Question

Consider fall of an object (mass m) without initial speed. Assuming quadratic air drag. Find the time dependence of the object's velocity and position. Find the terminal speed (Sol. to Velocity on Slide 99).

Solution

Taking downward as positive. $f = -kv^2 \implies a = g - \frac{k}{m}v^2$.

$$v(t) = \sqrt{\frac{mg}{k}} - \sqrt{\frac{4mg}{k}}\frac{1}{e^{2\sqrt{gk/m}t} + 1}$$

$$x(t) = x(0) + \sqrt{\frac{mg}{k}}t - \sqrt{\frac{4mg}{k}}\frac{2\sqrt{gk/m}t + \ln 2 - \ln(1 + e^{2\sqrt{gk/m}t})}{2\sqrt{gk/m}}$$

Separation of Variables Approach

It turns out that the ODE on Slide 99 can be solved using separation of variables!

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{k}{m}v^2 \implies \frac{\mathrm{d}v}{(v+\sqrt{mg/k})(v-\sqrt{mg/k})} = -\frac{k}{m}\mathrm{d}t$$

$$\frac{\mathrm{d}(v-\sqrt{mg/k})}{v-\sqrt{mg/k}} - \frac{\mathrm{d}(v+\sqrt{mg/k})}{v+\sqrt{mg/k}} = -2\sqrt{kg/m}\mathrm{d}t$$

$$v(t) = \sqrt{\frac{mg}{k}}\frac{1-e^{-2\sqrt{kg/m}t}}{1+e^{-2\sqrt{kg/m}t}} = v_{\text{terminal}}\tanh(\sqrt{kg/m}t)$$

$$x(t) = x(0) + \frac{m}{k}\left[\ln(\cosh(\sqrt{kg/m}t))\right]$$

where $\cosh(x) = \frac{e^x+e^{-x}}{2}$

Oscillation at the bottom of a Pot

Question

Discuss motion of a particle that is placed on the inner surface of a spherical pot, close to its bottom, and released from hold (no friction).

Solution

The potential energy of the particle x from the axis of symmetry of the pot is $U = -mg\sqrt{R^2 - x^2}$. Our goal is to find the coefficient for the quadratic term in the analytic expansion of the potential energy, and conclude that it is a simple harmonic oscillation around the bottom of the potential well. The bottom of the potential well is identified at $U'(x_0) = 0$ and $U''(x_0) > 0$.

Coefficients of Series Expansion

Suppose within the radius of convergence around x_0 f is analytic, $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

Coefficients of Series Expansion

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$
$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \dots$$
$$f''(x) = 2a_2 + 6a_3(x - x_0) + 12a_4(x - x_0)^2 + 20a_5(x - x_0)^3 + \dots$$

Our goal is to determine a_n, and in fact we can calculate a_n by differentiating both sides n times and taking the value at x_0.
$f(x_0) = a_0$; $f'(x_0) = a_1$; $f''(x_0) = 2a_2$; $f'''(x_0) = 6a_3$. In general,

$$f^{(n)}(x_0) = n!a_n \implies a_n = \frac{f^{(n)}(x_0)}{n!}$$

Oscillation at the bottom of a Pot (Continued)

Now in our case, $U = -mg\sqrt{R^2 - x^2}$, $U' = -mg\frac{-2x}{2\sqrt{R^2-x^2}}$,
$U'' = mg\frac{\sqrt{R^2-x^2}-x\frac{-2x}{2\sqrt{R^2-x^2}}}{(R^2-x^2)} = mg\frac{(R^2-x^2)+x^2}{(R^2-x^2)^{3/2}} = \frac{mgR^2}{(R^2-x^2)^{3/2}}$ so $x_0 = 0$.
$U = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, $a_1 = 0$, $a_2 = \frac{mgR^2}{2R^3} = \frac{mg}{2R}$ Therefore, the restoring force $F = -U' = -a_1 - 2a_2(x - x_0) + \sum_{n=3}^{\infty} na_n(x - x_0)^{n-1}$.
When x is close to x_0, $F \approx -2a_2(x - x_0)$, so when the amplitude is small, the motion is approximated by a simple harmonic oscillation with natural frequency $\omega_0 = \sqrt{\frac{2a_2}{m}} = \sqrt{\frac{g}{R}}$ and period $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R}{g}}$

A More Difficult Pot

Question

The same pot with cross-section in the shape of a cycloid placed upside-down

$$x = R(\gamma + \sin\gamma), \quad y = R(1 - \cos\gamma) \quad \text{where } -\pi \leq \gamma \leq \pi$$

Solution

We still want to find evidence that the oscillation is simple harmonic, but this time we have to go with the parametrized form. Suppose the particle is in such a position that $\gamma = \theta$ close to 0. We need to exploit the Series expansion of sine and cosine: $\cos\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}$ and $\sin\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}$ The potential energy of the particle is
$U = mgR(1 - \cos\theta) = mgR(1 - (1 - \frac{1}{2}\theta^2) + o(\theta^2)) = \frac{1}{2}mgR\theta^2 + o(\theta^2)$
$x = R(\theta + \theta + o(\theta^2)) = 2R\theta + o(\theta^2)$

A More Difficult Pot (Continued)

Now $U = \frac{1}{2}mgR\theta^2 + o(\theta^2)$ and $x = 2R\theta + o(\theta^2)$. $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2R + o(\theta)$, so by the inverse function theorem (Use series expansion to see $o(\theta)$),

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{2R + \underbrace{o(\theta)}_{\text{A polynomial}}} = \frac{1}{2R} + o(\theta)$$

Restoring force

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{\mathrm{d}U}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = -[mgR\theta + o(\theta)][\frac{1}{2R} + o(\theta)] = -\frac{mg\theta}{2} + o(\theta)$$

$$\frac{F}{x} = \frac{-mg\theta/2 + o(\theta)}{2R\theta + o(\theta^2)} \approx -\frac{mg}{4R}$$

so the natural frequency of the simple harmonic oscillation is $\omega_0 = \sqrt{\frac{g}{4R}}$, and the period is $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{4R}{g}}$

Driven Oscillations

Definition

A driven oscillation in our context is a linearly damped simple harmonic oscillation under a periodic driving force.

Equation of Motion

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{k}{m}x=\frac{F_0}{m}\cos\omega_{dr}t$$

This is an inhomogeneous, second order, linear ODE with constant coefficients.

Applying Laplace Transform on Both Sides

$$s^2X(s)-sx(0^-)-x'(0^-)+\frac{b}{m}(sX(s)-x(0^-))+\frac{k}{m}X(s)=\frac{F_0}{m}\frac{s}{s^2+\omega_{dr}^2}$$

Laplace Transformed Equation

$$(s^2 + \frac{b}{m}s + \frac{k}{m})X(s) = \frac{F_0}{m}\frac{s}{s^2 + \omega_{dr}^2} + (s + \frac{b}{m})x(0^-) + x'(0^-)$$

Suppose there are two distinct roots s_1 and s_2 for $s^2 + \frac{b}{m}s + \frac{k}{m} = 0$, then assuming zero state $x(0^-) = 0$ and $x'(0^-) = 0$, there are four distinct first-order poles.

$$X(s) = \frac{F_0}{m}\frac{s}{(s + j\omega_{dr})(s - j\omega_{dr})(s - s_1)(s - s_2)}$$

$$X(s) = \frac{F_0}{m}\left[\frac{E}{s + j\omega_{dr}} + \frac{B}{s - j\omega_{dr}} + \frac{C}{s - s_1} + \frac{D}{s - s_2}\right]$$

To perform Inverse Laplace Transform, we need to expand $X(s)$ into a sum of first order fractions.

Partial Fraction Expansion, Inverse Laplace Transform

$$E = \left.\frac{s}{(s-j\omega_{dr})(s-s_1)(s-s_2)}\right|_{s=-j\omega_{dr}} = \frac{1}{2(s_1s_2+(s_1+s_2)j\omega_{dr}-\omega_{dr}^2)}$$

$$B = \left.\frac{s}{(s+j\omega_{dr})(s-s_1)(s-s_2)}\right|_{s=j\omega_{dr}} = \frac{1}{2(s_1s_2-(s_1+s_2)j\omega_{dr}-\omega_{dr}^2)}$$

$$C = \left.\frac{s}{(s^2+\omega_{dr}^2)(s-s_2)}\right|_{s=s_1} = \frac{s_1}{(s_1^2+\omega_{dr}^2)(s_1-s_2)}$$

$$D = \left.\frac{s}{(s^2+\omega_{dr}^2)(s-s_1)}\right|_{s=s_2} = \frac{s_2}{(s_2^2+\omega_{dr}^2)(s_2-s_1)}$$

$$X(s) = \frac{F_0}{m}\left[\frac{E}{s+j\omega_{dr}} + \frac{B}{s-j\omega_{dr}} + \frac{C}{s-s_1} + \frac{D}{s-s_2}\right]$$

Applying the Inverse Laplace Transform on both sides, for $t > 0$,

$$x(t) = \frac{F_0}{m}\left[Ee^{-j\omega_{dr}t} + Be^{+j\omega_{dr}t} + Ce^{s_1t} + De^{s_2t}\right]$$

Be aware that $\mathfrak{Re}\{s_1\} = \mathfrak{Re}\{s_2\} = -\frac{b}{2m} < 0$, so $Ce^{s_1t} + De^{s_2t}$ decays.

Sinusoidal Steady-State Response

The other two terms are complex exponentials that are oscillating. Now $s_1 s_2 = \frac{k}{m}$, and $s_1 + s_2 = -\frac{b}{m}$, so $E = \frac{1}{2(\frac{k}{m} - \frac{b}{m} j\omega_{dr} - \omega_{dr}^2)}$, and $B = \frac{1}{2(\frac{k}{m} + \frac{b}{m} j\omega_{dr} - \omega_{dr}^2)}$. $x(t) = 2\frac{F_0}{m}|B|\cos(\omega_{dr} t + \angle B)$, so the amplitude of the sinusoidal steady state response is $A = \frac{F_0}{m\sqrt{(k/m - \omega_{dr}^2)^2 + (b\omega_{dr}/m)^2}}$ and the phase lag φ satisfies $\tan\varphi = \frac{b\omega_{dr}}{m\omega_{dr}^2 - k}$. Therefore, $x(t) = A\cos(\omega_{dr} t + \varphi)$, where the amplitude of the sinusoidal steady state response is

$$A = \frac{F_0}{m\sqrt{(k/m - \omega_{dr}^2)^2 + (b\omega_{dr}/m)^2}}$$

and the phase lag (φ takes value from 0 to $-\pi$) φ satisfies

$$\tan\varphi = \frac{b\omega_{dr}}{m\omega_{dr}^2 - k}$$

Start with Position Vector

Einstein's notation $r_\alpha \hat{n}_\alpha = \sum_{\alpha=x,y,z} r_\alpha \hat{n}_\alpha$.

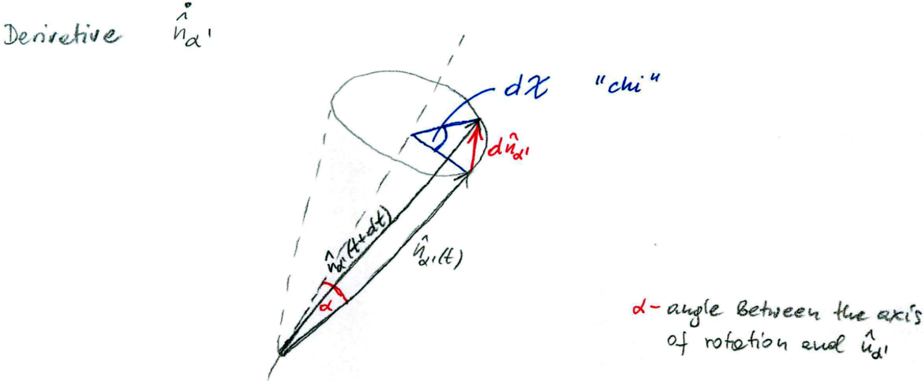
$$\overline{r}(t) = \overline{r}_{O'}(t) + \overline{r}'(t)$$

Differentiate both sides w.r.t. time,

$$\frac{\mathrm{d}\overline{r}}{\mathrm{d}t} = \overline{v} = \frac{\mathrm{d}\overline{r}_{O'}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\overline{r'}(t)}{\mathrm{d}t} = \overline{v}_{O'} + \frac{\mathrm{d}\overline{r'}(t)}{\mathrm{d}t}$$

Now $\frac{\mathrm{d}\overline{r'}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r_{\alpha'}\hat{n}_{\alpha'}) = \dot{r}_{\alpha'}\hat{n}_{\alpha'} + r_{\alpha'}\dot{\hat{n}}_{\alpha'} = \overline{v'} + r_{\alpha'}\dot{\hat{n}}_{\alpha'}$

Derivative $\dot{\hat{n}}_{\alpha'}$



$|\mathrm{d}\hat{n}_{\alpha'}| = \mathrm{d}\chi|\hat{n}_{\alpha'}|\sin\alpha$, so define vector $\mathrm{d}\overline{\chi}$ as the vector along the instantaneous axis of rotation, such that $\mathrm{d}\overline{\chi}$ is the angle that the tips of $\hat{n}_{\alpha'}(t)$, $\hat{n}_{\alpha'}(t+\mathrm{d}t)$ form over time $\mathrm{d}t$. Then ($\overline{\omega} = \frac{\mathrm{d}\overline{\chi}}{\mathrm{d}t}$)

$$\mathrm{d}\hat{n}_{\alpha'} = \mathrm{d}\overline{\chi} \times \hat{n}_{\alpha'} \quad \frac{\mathrm{d}\hat{n}_{\alpha'}}{\mathrm{d}t} = \frac{\mathrm{d}\overline{\chi}}{\mathrm{d}t} \times \hat{n}_{\alpha'} = \overline{\omega} \times \hat{n}_{\alpha'}$$

Velocity and Acceleration in Non Inertial FoR

The upshot of all these calculations is that the motion of a particle observed in one Inertial FoR $OXYZ$ and one Non Inertial FoR $O'X'Y'Z'$ described by the relation $\overline{r}(t) = \overline{r}_{O'}(t) + \overline{r'}(t)$ and that the axes of $O'X'Y'Z'$ rotates with angular velocity $\overline{\omega}$ in $OXYZ$ around O' has velocity relation

$$\overline{v} = \overline{v}_{O'} + \overline{v'} + (\overline{\omega} \times \overline{r'})$$

and acceleration relation

$$\overline{a} = \overline{a}_{O'} + \overline{a'} + 2\overline{\omega} \times \overline{v'} + \frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \times \overline{r'} + \overline{\omega} \times (\overline{\omega} \times \overline{r'})$$

or, multiplying by mass m and noting that $m\overline{a} = \overline{F}$,

$$m\overline{a'} = \overline{F} \underbrace{- m\overline{a}_{O'} - m\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \times \overline{r'} - 2m(\overline{\omega} \times \overline{v'}) - m\overline{\omega} \times (\overline{\omega} \times \overline{r'})}_{\text{Pseudo Forces}}$$

Pseudo Forces

Term	Name	Cause
$-m\overline{a}_{O'}$	d'Alembert "force"	acceleration of O'
$-m\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \times \overline{r'}$	Euler "force"	angular acceleration of O'
$-2m\overline{\omega} \times \overline{v'}$	Coriolis "force"	motion in O' and rotation of O'
$-m\overline{\omega} \times (\overline{\omega} \times \overline{r'})$	Centrifugal "force"	rotation of O'

On Slide 180, a comparison is made among solutions using Non Inertial FoR and Lagrangian Mechanics.

- Driven Oscillations
- Non Inertial FoR
- **The Earth as a Frame of Reference**
- Discussion
- Exercises

The Earth as a Frame of Reference

The Earth is a non-inertial frame of reference that performs orbital motion and rotational motion.

$$m\overline{a'} = \overline{F} - m\overline{a}_0 - m\overline{\omega} \times (\overline{\omega} \times \overline{r'}) - 2m(\overline{\omega} \times \overline{v'})$$

The gravitational attraction of the sun $\overline{F}_{sun}$ provides the mass m with $m\overline{a_0}$, so for objects on the earth under gravity,

$$m\overline{a'} = \overline{F}_{earth} - m\overline{\omega} \times (\overline{\omega} \times \overline{r'}) - 2m(\overline{\omega} \times \overline{v'})$$

In general, the earth can be treated as an inertial frame of reference with a good approximation, but when $\overline{v'}$ is large (such as the velocity of a missile), the Coriolis “force” becomes more significant.

- Driven Oscillations
- Non Inertial FoR
- The Earth as a Frame of Reference
- **Discussion**
- Exercises

Phase Lag of Driven Oscillation

$x(t) = A\cos(\omega_{dr}t + \varphi)$

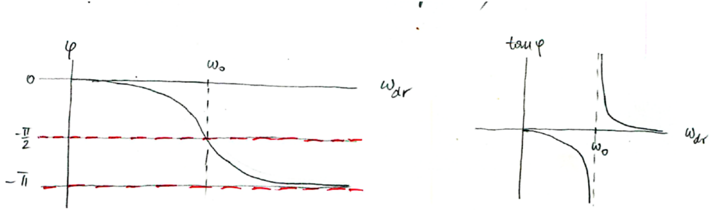


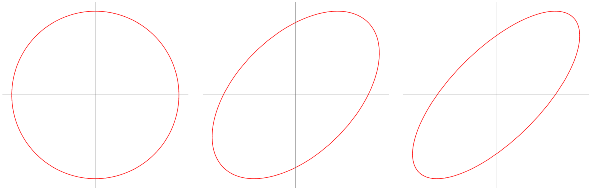
Figure: Relation between Phase Lag φ and Driving Frequency f. Notice how $x(t)$ is defined.

Harmonic Oscillator in 2D: Lissajous Figures

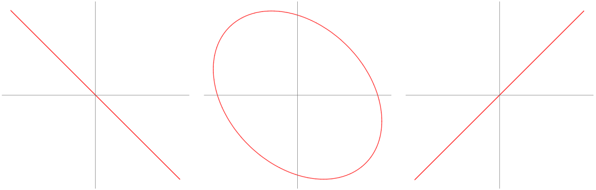
The position coordinates of a 2D Harmonic Oscillator are given by

$$\begin{cases} x(t) = A\cos(\omega_x t - \varphi_x) \\ y(t) = B\cos(\omega_y t - \varphi_y) \end{cases}$$

A special case is $\omega_x = \omega_y$, and $\varphi_x = 0$, in which case we can observe the phase lag using Lissajous Figures.
Weisstein, Eric W. "Lissajous Curve." From MathWorld–A Wolfram Web Resource.
`http://mathworld.wolfram.com/LissajousCurve.html`



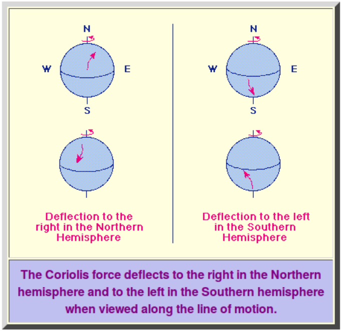
$\varphi_y = \pi/2$, $\pi/3$, $\pi/4$

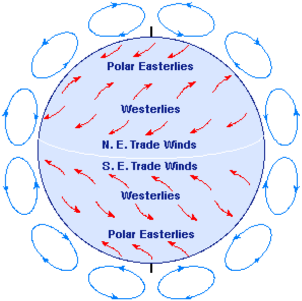


$\varphi_y = \pi$, $3\pi/5$, 2π.

Consequences of Coriolis Force in Nature

http://csep10.phys.utk.edu/astr161/lect/earth/coriolis.html The following diagram on the left illustrates the effect of Coriolis forces in the Northern and Southern hemispheres.

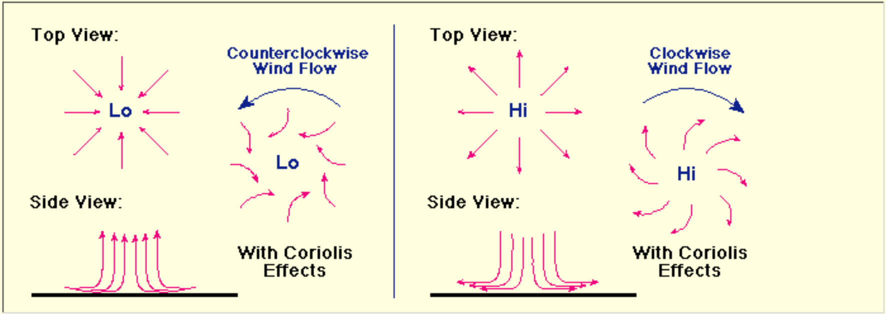




This produces the prevailing surface winds illustrated in the figure on the right.

Cyclones and anticyclones

The wind flow around high pressure (anticyclonic) systems is clockwise in the Northern hemisphere and counterclockwise in the Southern hemisphere. The corresponding flow around low pressure (cyclonic) systems is counterclockwise in the Northern hemisphere and clockwise in the Southern hemisphere.



Low pressure systems (left) and high pressure systems (right) in the Northern hemisphere

Centrifugal force and Centripetal force

We CANNOT say that there is a centrifugal force and a centripetal force acting upon a particle at the same time. When we state a centrifugal force, we are describing the effect of a pseudo force in a non-inertial FoR. When we state a centripetal force, we are describing the effect of some concrete force in an inertial FoR.

Particle Sliding down a fixed Hemisphere: Zero State

Question

A particle with mass m slides with 0 initial speed from the top of a fixed frictionless hemisphere with radius R. Find the place where the particle loses contact with the surface of the ball. What is its speed at this instant?

Solution

The moment the mass loses contact with the surface of the ball, the mass is just able to maintain a circular motion using the normal component of gravity. Suppose it traverses θ from the top, $v = \sqrt{2gR(1 - \cos\theta)}$, and $m\frac{v^2}{R} = mg\cos\theta$. Therefore, $\theta = \arccos\frac{2}{3}$, and $v = \sqrt{2gR/3}$.

Particle Sliding down a fixed Hemisphere

Question

A particle with mass m slides with 0 initial speed from the top of a fixed frictionless hemisphere with radius R. Find the place where the particle loses contact with the surface of the ball. What is its speed at this instant?

Solution

The moment the mass loses contact with the surface of the ball, the mass is just able to maintain a circular motion using the normal component of gravity. Suppose it traverses θ from the top, $v = \sqrt{v_0^2 + 2gR(1 - \cos\theta)}$, and $m\frac{v^2}{R} = mg\cos\theta$. Therefore, $\theta = \arccos\left[\frac{v_0^2+2gR}{3gR}\right]$, and $v = \sqrt{(v_0^2 + 2gR)/3}$.

Angle the Surface of Liquid Forms

Question

A box is filled with a liquid and is placed on a horizontal surface. Find the angle that the surface of the liquid forms with the horizontal surface if we pull the box with acceleration a.

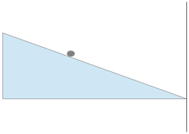
Solution

The surface of the liquid can only exert pressure on the liquid particles at the surface of the liquid, so study the force along the surface. Either an Inertial FoR or an Non-Inertial FoR works. $\alpha = \arctan(a/g)$.

Stay on a Rotating Plane

Question

A plane, inclined at an angle α to the horizontal, rotates with constant angular speed ω about a vertical axis (see the figure). Where on the inclined plane should we place a particle, so that it remains at rest?



The plane is frictionless.

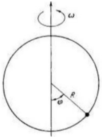
Solution

The plane can only support the particle in the normal direction, so study the force along the plane. $\tan\alpha = \frac{\omega^2 R}{g}$, $R = \frac{g}{\omega^2}\tan\alpha$.

Bead on a Hoop

Question

A small bead can slide without friction on a circular hoop that is in a vertical plane and has a radius R. Find points on the hoop, such that if we place the bead there it will remain at rest. Acceleration due to gravity is g.



Solution

$\tan(\varphi) = \frac{\omega^2 R \sin\varphi}{g}$, so $\cos\varphi = \frac{g}{\omega^2 R}$, $\varphi = \arccos(g/(\omega^2 R))$

Foucault Pendulum on the Equator

Question

Will the oscillation plane of a Foucault pendulum, that is placed on the equator, rotate?

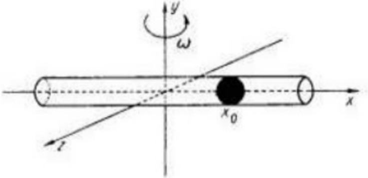
Solution

No. The rotation of the oscillation plane is due to $\overline{\omega} \times \overline{v'}$. Now $\overline{\omega} \times \overline{v'}$ lies in the plane of oscillation.

Mass inside a Rotating Pipe

Question

A particle with mass m is inside a pipe that rotates with constant angular velocity ω about an axis perpendicular to the pipe. The kinetic coefficient of friction is equal to μ_k. Write down (do not solve!) the equation of motion for this particle in the non-inertial frame of reference of the rotating pipe.



There is no gravitational force in this problem.

Mass inside a Rotating Pipe (Solution)

$m\overline{a'} = \overline{F} - m\overline{a}_{O'} - m\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \times \overline{r'} - 2m(\overline{\omega} \times \overline{v'}) - m\overline{\omega} \times (\overline{\omega} \times \overline{r'})$ There are two concrete forces (normal force and friction) and two pseudo forces (Coriolis "force" and Centrifugal "force") in this non inertial FoR. Now set $O'X'$ along the pipe, $O'Z'$ along the axis of rotation. $\overline{F} = \overline{N} + \overline{f}$. Furthermore, there is no acceleration along $O'Y'$ and $O'Z'$. Now

$$\overline{\omega} = \omega \hat{n}_{z'}, \text{ and } \overline{v'} = v' \hat{n}_{x'}, \text{ so } \overline{\omega} \times \overline{v'} = \omega v' \hat{n}_{y'}.$$

Furthermore, $\overline{f} = f\hat{n}_{x'}$, so the balance in $O'Y'$ direction tells $\overline{N} - 2m(\overline{\omega} \times \overline{v'}) = 0$, i.e., $\overline{N} = 2m\omega v' \hat{n}_{y'}$. Centrifugal force is $-m\overline{\omega} \times (\omega \hat{n}_{z'} \times r\hat{n}_{x'}) = -m\overline{\omega} \times \omega r \hat{n}_{y'} = -m\omega^2 r \hat{n}_{z'} \times \hat{n}_{y'} = m\omega^2 r \hat{n}_{x'}$. As long as the mass is sliding (in which case it has to be sliding along the positive direction of the $O'X'$ axis), $\overline{f} = -2\mu_k m\omega v' \hat{n}_{x'}$, so the motion of equation in this non inertial FoR is given by

$$\overline{a'} = (\omega^2 r - 2\mu_k \omega v')\hat{n}_{x'}$$

- **Work and Energy; Power**
- Potential Force Fields
- Potential Energy
- Non-conservative Forces
- Exercises

Work

Definition

Elementary work δW done by $\overline{F}$ when particle moves from $\overline{r}$ to $\overline{r} + \mathrm{d}\overline{r}$

$$\delta W := \overline{F} \circ \mathrm{d}\overline{r}$$

Total work w_{AB} when particle moves from A to B along curve Γ_{AB} is the line integral of the force field

$$w_{AB} = \int_{\Gamma_{AB}} \overline{F} \circ \mathrm{d}\overline{r}$$

Line Integral Along Parametrized Curve (Discussed in Calculus III)

If we calculate the line integral using a concrete parametrization $\gamma : I \to \mathcal{C}$, we obtain $\int_{\mathcal{C}^*} F \mathrm{d}\overline{s} = \int_I \langle F(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t$

Line Integral: Example

Example

Calculate

$$\oint_{\mathcal{C}^+} \begin{pmatrix} y^2 \\ 3xy \end{pmatrix} \mathrm{d}\overline{s}$$

where $\mathcal{C}^+$ is the positively oriented curve

$$\mathcal{C} = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, y > 0\right\} \cup$$

$$\left\{(x, y) \in \mathbb{R}^2 : y = 0, -1 \leq x \leq 1\right\}$$

We choose these two parameterizations:

$$\gamma_1 : [0, \pi] \to \mathbb{R}^3 : t \mapsto \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \quad \gamma_2 : [-1, 1] \to \mathbb{R}^3 : t \mapsto \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
&\oint_{\mathcal{C}^+} y^2 \mathrm{d}x + 3xy\mathrm{d}y \\
=& \int_0^\pi \left\langle \begin{pmatrix} \sin^2 t \\ 3\cos t \sin t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle \mathrm{d}t + \int_{-1}^1 \left\langle \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \mathrm{d}t \\
=& \int_0^\pi (-\sin^3 t + 3\cos^2 t \sin t)\mathrm{d}t + 0 \\
=& \int_0^\pi -(-\sin^2 t + 3\cos^2 t)\mathrm{d}(\cos t) \\
=& \int_0^\pi [(1 - \cos^2 t) - 3\cos^2 t]\mathrm{d}\cos t \\
=& -1 - 1 + (-4/3)(-1 - 1) = 2/3
\end{aligned}$$

Kinetic Energy, Work-Kinetic Energy Theorem

Recall that $\delta w = \overline{F} \circ \mathrm{d}\overline{r}$, and exploiting $v^2 = \overline{v} \circ \overline{v}$,

$$\frac{\delta w}{\mathrm{d}t} = \overline{F} \circ \frac{\mathrm{d}\overline{r}}{\mathrm{d}t} = \overline{F} \circ \overline{v} = m\overline{a} \circ \overline{v} = \mathrm{d}\frac{1}{2}mv^2$$

so kinetic energy is defined as $K = \frac{1}{2}mv^2$

Work-Kinetic Energy Theorem

The work done by the net force on a particle is equal to the change in the particle's kinetic energy.

$$\delta w = \mathrm{d}K$$

or, for finite increments,

$$w = \Delta K$$

Power

Power characterizes how fast work is being done.

Definition

Instantaneous power

$$\underbrace{\frac{\delta w}{\mathrm{d}t}}_{\text{rate of work done}} = \overline{F} \circ \overline{v} = \underbrace{P}_{\text{instantaneous power}}$$

Definition

Average power

$$\frac{\overbrace{w}^{\text{work done in the interval } (t, t+\Delta t)}}{\delta t} = \underbrace{P_{av}}_{\text{average power}}$$

Potential Force Fields

Definition

If there exists a scalar function u of x, y, z such that $\overline{F} = -\nabla u$, then the force field is called **potential** (conservative).
$-\nabla u = \left(-\frac{\partial u}{\partial x}\Big|_{x,y,z}, -\frac{\partial u}{\partial y}\Big|_{x,y,z}, -\frac{\partial u}{\partial z}\Big|_{x,y,z}\right)$

Properties

Work done by $\overline{F}$ depends only on the final position and initial position.

$$w = u(\mathbf{r}_{\text{final}}) - u(\mathbf{r}_{\text{initial}})$$

Criteria

In a simply connected region, $\overline{F}$ is conservative if and only if $\text{rot}\overline{F} = 0$.

Rotation (Curl) of **F**

$$\mathrm{rot}\overline{F} = \nabla \times \overline{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z)$$

$$= \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y, \frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z, \frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)$$

Simply Connected

The concept simply connected can be interpreted as being possible to retract a rubber band within the region to any point in the region.

- Work and Energy; Power
- Potential Force Fields
- **Potential Energy**
- Non-conservative Forces
- Exercises

Potential Energy

To find the potential energy once we have proved that a force field is conservative, we need to find a compatible u for all three integrations $\int F_x \mathrm{d}x + C_x(y, z)$, $\int F_y \mathrm{d}y + C_y(x, z)$, and $\int F_z \mathrm{d}z + C_z(x, y)$.

Example

Consider $\overline{F} = x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$, so $\int F_x \mathrm{d}x = \frac{1}{2}x^2 + C_x(y, z)$, $\int F_y \mathrm{d}y = \frac{1}{2}y^2 + C_y(x, z)$, $\int F_z \mathrm{d}z = \frac{1}{2}z^2 + C_z(x, y)$, we decide $-u(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + C$

Conservation of Mechanical Energy in Potential Fields

Suppose $\overline{F}$ is the net force on a particle and $\overline{F}$ is conservative, then $\delta w = \overline{F} \circ \mathrm{d}\overline{r} = -\mathrm{d}U$. Now by the work-kinetic energy theorem, $\delta w = \mathrm{d}K$, so $\mathrm{d}(K + U) = 0$, $K + U = const$. The constant is the mechanical energy of the particle in this Potential Field.

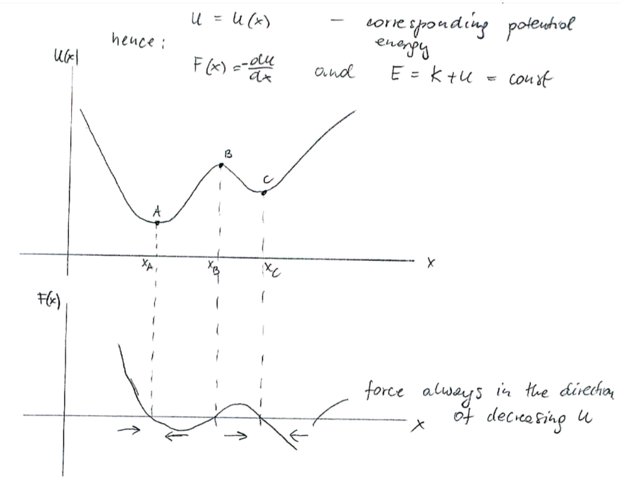
Non-conservative Forces

If non-conservative forces present, then the work done by non-conservative forces is equal to the change in the total mechanical energy. In fact, $w_{n-cons} = -\Delta u_{int}$, i.e., internal energy (other form of energy). The sum of all these energies is constant. In other words,

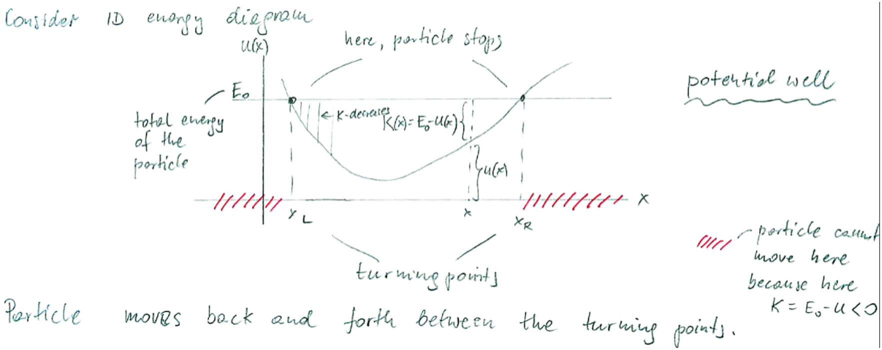
$$\Delta K + \Delta U + \Delta U_{int} = 0$$

This is the law of conservation of total energy.

Energy Diagrams



1D Energy Diagram; Harmonic Approximation



Harmonic approximation of oscillation in the vicinity of a stable equilibrium x_0:

$$U(x) \approx U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2$$

$\omega_0 = \sqrt{\frac{U''(x_0)}{m}}$, $x(t) = x(0) + A\cos(\omega_0 t + \varphi)$.

- Work and Energy; Power
- Potential Force Fields
- Potential Energy
- Non-conservative Forces
- Exercises

Pull a Cylinder out of Liquid

Question

A uniform cylinder of mass m, radius R, and height h is floating vertically in a liquid, so that it is half-immersed in the liquid. Find the density of the liquid and minimum work needed to pull the cylinder completely above the liquid's surface.

Solution

$mg = \frac{1}{2}\rho g \pi R^2 h$, so the density of the liquid $\rho = \frac{2m}{\pi R^2 h}$. The minimum work is attained when we pull the cylinder slowly so that the kinetic energy is always almost 0.

Consider the cylinder has been pulled up by x. The pulling force F is

$F = mg - \frac{h/2-x}{h/2}mg = \frac{x}{h/2}mg$, so by definition,

$w = \int_0^{h/2} F \mathrm{d}x = \frac{2mg}{h}\frac{1}{2}(h/2)^2 = \frac{mg}{4h}$

Find Work

Question

Find work done by the force $\mathbf{F}_1(x, y) = -x\hat{n}_x - y\hat{n}_y$ and by the force $\mathbf{F}_2(x, y) = (2xy + y)\hat{n}_x + (x^2 + 1)\hat{n}_y$ if the particle is being moved from $(-1, 0)$ to $(0, 1)$ along

1. the straight line connecting these points
2. the (shorter) arc of the circle $x^2 + y^2 = 1$
3. the axes of the Cartesian coordinate system: first from $(-1, 0)$ to $(0, 0)$ along the x axis, then from $(0, 0)$ to $(0, 1)$ along the y axis.

Parametrization

1. $\gamma : [0, 1] \to \mathbb{R}^2$, $\gamma(t) = (t - 1, t)$
2. $\gamma : [\pi, \pi/2] \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$
3. $t \in [0, 1]$, $\gamma_1(t) = (t - 1, 0)$, $\gamma_2(t) = (0, t)$

Find Work (Solution)

1. $w_1 = \int_0^1 \left\langle \binom{-t+1}{-t}, \binom{1}{1} \right\rangle \mathrm{d}t = \int_0^1 -2t + 1 \mathrm{d}t = -t^2 + t\big|_0^1 = 0$
   $w_2 = \int_0^1 \left\langle \binom{2(t-1)t+t}{(t-1)^2+1}, \binom{1}{1} \right\rangle \mathrm{d}t = \int_0^1 3t^2 - 3t + 2\mathrm{d}t = 3/2$
2. $w_1 = \int_{\pi}^{\pi/2} \left\langle \binom{-\cos t}{-\sin t}, \binom{-\sin t}{\cos t} \right\rangle \mathrm{d}t = 0$
   $w_2 = \int_{\pi}^{\pi/2} \left\langle \binom{2\sin t\cos t+\sin t}{\cos^2 t+1}, \binom{-\sin t}{\cos t} \right\rangle \mathrm{d}t =$
   $-\frac{t}{2} + \frac{5\sin(t)}{4} + \frac{1}{4}\sin(2t) + \frac{1}{4}\sin(3t)\Big|_{\pi}^{\pi/2} = \frac{4+\pi}{4}$
3. $w_1 = \int_{-1}^{0}(-x)\mathrm{d}x + \int_0^1(-y)\mathrm{d}y = 1/2 - 1/2 = 0$
   $w_2 = \int_{-1}^{0}(2xy + y)\mathrm{d}x\Big|_{y=0} + \int_0^1(x^2 + 1)\mathrm{d}y\Big|_{x=0} = 1$

Notice that $\mathbf{F}_1(\bar{r}) = -\bar{r}$, $\mathbf{F}_1$ is central force, so the work done is path independent (proved in a later section).

Visualized Force Field $\mathbf{F}_1$ (Left) and $\mathbf{F}_2$ (Right)

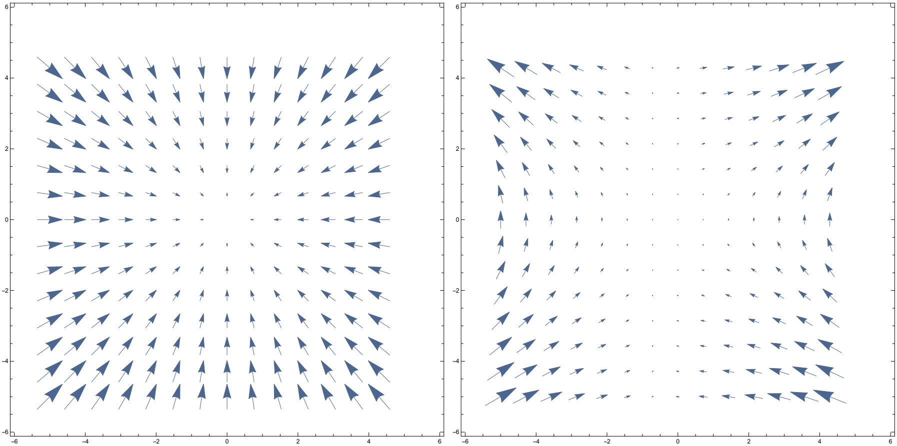


Figure: Force Field $\mathbf{F}_1$ (Left) and $\mathbf{F}_2$ (Right)

$\mathbf{F}_3 = \frac{1}{r^2}\hat{n}_r$, $\mathbf{F}_4 = \sin(r)\hat{n}_r$

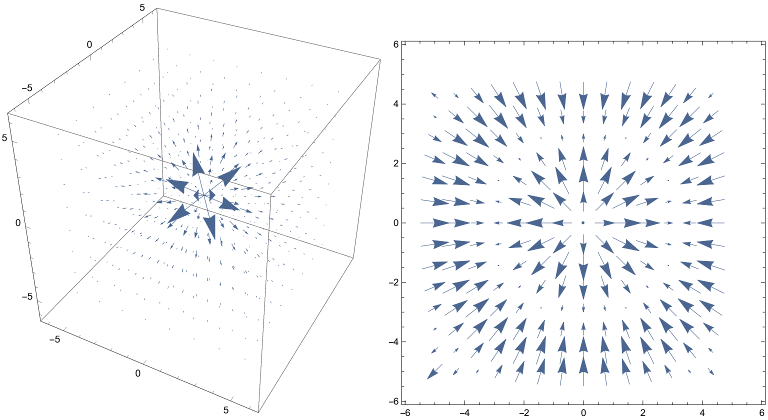


Figure: 3D Vector Plot of $\mathbf{F}_3$ on the left, and 2D Vector Plot of $\mathbf{F}_4$ on the right.

Find Work

Question

Find the work the force $\mathbf{F}(\mathbf{r}) = (x^2 - y, z, 1)$ does on a particle that is being moved from $(0, 0, 0)$ to $(1, 1, 1)$ along

1. straight line connecting these points
2. the curve given in the parametric form: $x(t) = t, y(t) = t^2, z(t) = \frac{1}{2}t(t+1)$, where $0 \le t \le 1$.

Solution

1. A parametrization is given by $\gamma : [0, 1] \to \mathbb{R}^3$, $\gamma(t) = (t, t, t)$, $w = \int_0^1 (t^2 - t, t, 1) \circ (1, 1, 1)\mathrm{d}t = \frac{4}{3}$
2. $w = \int_0^1 (t^2 - t^2, \frac{1}{2}t(t+1), 1) \circ (1, 2t, t + \frac{1}{2})\mathrm{d}t = \frac{1}{4}t^4 + \frac{1}{3}t^3 + \frac{1}{2}t^2 + \frac{1}{2}t\Big|_0^1 = \frac{19}{12}$

Check whether Conservative

Question

Check whether the following force fields are conservative. Find the corresponding potential energy for those that are.

1. $\mathbf{F}(\mathbf{r}) = (-y^2z - 3y, -xz^2 + 4yz - 3x, -2xyz + 2y^2 + 1)$
2. $\mathbf{F}(\mathbf{r}) = (x^2 + y^2, y^2 + z^2, z)$

Solution

1. $\nabla \times \overline{F} = ((-2xz+4y)-(-2xz+4y), (-y^2)-(-2yz), (-z^2-3)-(-2yz-3))$ not conservative.
2. $\nabla \times \overline{F} = ((0) - (2z), (0) - (0), (0) - (2y))$ not conservative.

Central Forces are Conservative

$\mathbf{F}(\mathbf{r}) = f(r)\hat{n}_r$ is an expression given in the spherical coordinate. `http://hyperphysics.phy-astr.gsu.edu/hbase/curl.html`, so

$$\nabla \times \overline{F} = \begin{vmatrix} \frac{\hat{n}_r}{r^2 \sin\theta} & \frac{\hat{n}_\theta}{r \sin\theta} & \frac{\hat{n}_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f(r) & 0 & 0 \end{vmatrix} = 0$$

Otherwise, we need to convert to the Cartesian Coordinates and use chain rule on $f(r)$.

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x f(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}} & \frac{y f(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}} & \frac{z f(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

Central Forces are Conservative (Continued)

$$
\begin{aligned}
\left\langle \nabla \times \overline{F}, \hat{n}_x \right\rangle = & \left[\frac{z f_r(r) \frac{2y}{2\sqrt{x^2+y^2+z^2}} \sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)} - \frac{z f(r) \frac{2y}{2\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)} \right] \\
& - \left[\frac{y f_r(r) \frac{2z}{2\sqrt{x^2+y^2+z^2}} \sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)} - \frac{y f(r) \frac{2z}{2\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)} \right] \\
= & \, 0
\end{aligned}
$$

where $f_r(r) = \left.\frac{\mathrm{d}f(\cdot)}{\mathrm{d}r}\right|_r$, and the other three components can also be shown as 0 in an identical manner.

- Elements of Lagrangian Mechanics
- Momentum
- Center-of-Mass FoR
- Exercises

Generalized Coordinates and Velocities; Degrees of Freedom

Definition

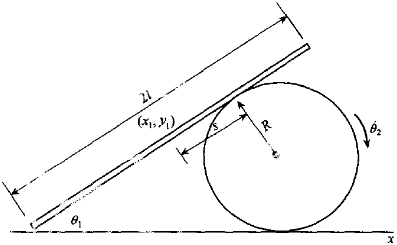
Generalized Coordinates are any coordinates describing position of a particle (or a system of particles). Usually denoted by $q_1, q_2, \ldots$. Then $\dot{q}_i$ denote **generalized velocities**.

Definition

Number of **degrees of freedom** of a particle (or a system of particles): the minimum number of independent generalized coordinates needed to uniquely describe position of a particle (or a system of particles). Usually denoted by f.

Example for Generalized Coordinates and DoF

A uniform disk with radius R is rolling without sliding along the x axis. A uniform thin stick with length $2l$ stays in contact with the disk without sliding. One end of the stick is sliding along the x axis. When the system is in motion, the disk and the stick stay in the same vertical plane. Choose appropriate coordinates, write down the constraint relations, and state the number of degree of freedom of this system.



Use (x_1, y_1) to express the position of the center of mass of the stick, the angle θ_1 the stick forms with the x axis to express the inclination of the stick, x_2 to express the position of the center of mass of the disk,and s to express the distance from the tangential point of the stick and the disk and the center of mass on the stick.

$$y_1 = l \sin\theta_1$$

$$\dot{x}_2 - R\dot{\theta}_2 = 0 \text{ due to pure rolling } \implies x_2 - R\theta_2 = C$$

Since there is no sliding between the stick and the disk,

$$\dot{x}_1 \hat{n}_x + \dot{y}_1 \hat{n}_y + \dot{\theta}_1 \hat{n}_z \times s(\cos\theta_1 \hat{n}_x + \sin\theta_1 \hat{n}_y)$$

$$= \dot{x}_2 \hat{n}_x - \dot{\theta}_2 \hat{n}_z \times R(-\sin\theta_1 \hat{n}_x + \cos\theta_1 \hat{n}_y)$$

so $\dot{x}_1 - s\dot{\theta}_1 \sin\theta_1 = \dot{x}_2 + R\dot{\theta}_2 \cos\theta_1$ and $\dot{y}_1 + s\dot{\theta}_1 \cos\theta_1 = R\dot{\theta}_2 \sin\theta_1$
Geometrically, $x_2 - x_1 + l\cos\theta_1 = l + s$, so there are only three independent generalized coordinates.

Expressing K Using Generalized Coordinates

A particle with mass m is moving on a plane. Use r and $\sin\varphi$ instead of the polar coordinates r and φ to express the kinetic energy of this particle.

$x = r\cos\varphi$, $y = r\sin\varphi$. Use r and $q = \sin\varphi$ as generalized coordinates. $x = r\cos\varphi = r\sqrt{1-q^2}$, $y = r\sin\varphi = rq$, so $\dot{x} = \dot{r}\sqrt{1-q^2} - \frac{rq\dot{q}}{\sqrt{1-q^2}}$, and $\dot{y} = \dot{r}q + r\dot{q}$.

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + \frac{r^2\dot{q}^2}{1-q^2})$$

Lagrangian, Hamilton's Action, Hamilton's Principle

Definition

Lagrangian $L := K - U$

For any trajectory $\overline{q} = \overline{q}(t) = (q_1(t), q_2(t), \dots, q_f(t))$ we can define **Hamilton's Action**

$$S := S[\overline{q}] = \int_{t_A}^{t_B} L(\overline{q}, \dot{\overline{q}}, t)\mathrm{d}t$$

Hamilton's Principle The real trajectory extremizes Hamilton's action. $\delta S = 0$. Similar to chain rule in ordinary differentiation, (Noticing that variation of trajectory is independent of time)

$$\delta \int_{t_A}^{t_B} L(\overline{q}, \dot{\overline{q}}, t)\mathrm{d}t = \int_{t_A}^{t_B} \delta L(\overline{q}, \dot{\overline{q}}, t)\mathrm{d}t = \int_{t_A}^{t_B} \left(\sum_{i=1}^{f} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right) \mathrm{d}t$$

Euler-Lagrange Equations

The f equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

are called the **Euler-Lagrange Equations**

Mass, Rope and Cylinder

A particle with mass m is tied to the edge of a fixed cylinder with radius R via a weightless, non-elastic rope. Initially, the rope is winded on the cylinder tightly where the particle is in contact with the cylinder. Now we give the particle an initial radial velocity v_0, and the particle is constrained on a smooth horizontal surface. Find the relation of length l of the rope that is not winded on the cylinder with time t.
As was promised on Slide 126, a comparison is made in this exercise.

Solution using a Non Inertial FoR

Recall that the acceleration in Cylindrical coordinates is given by

$$\bar{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{n}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{n}_\varphi + \ddot{z}\hat{n}_z$$

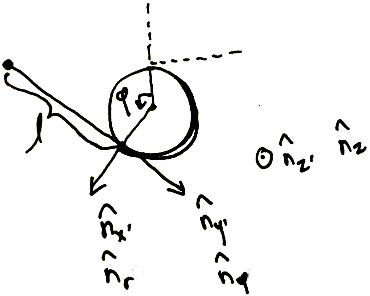
and that the acceleration in Non-inertial FoR is given by

$$m\overline{a'} = \overline{F} - m\bar{a}_{O'} - m\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t} \times \overline{r'} - 2m(\overline{\omega} \times \overline{v'}) - m\overline{\omega} \times (\overline{\omega} \times \overline{r'})$$

Consider the non inertial FoR: origin O' is the intersection of straight rope and winded rope, and $O'Y'$ is along the straight rope. $\hat{n}_{x'} = \hat{n}_r$, $\hat{n}_{y'} = \hat{n}_\varphi$, and $\hat{n}_{z'} = \hat{n}_z$ The position of the particle in this non-inertial FoR is $y' = -l$. Geometrically, $l = R\varphi$, so $\dot{l} = R\dot{\varphi}$, and $\ddot{l} = R\ddot{\varphi}$. Furthermore, $m\overline{a'} = m\ddot{l}(-\hat{n}_{y'})$, $\overline{F} = T\hat{n}_{y'}$,

$$-m\bar{a}_{O'} = -m[(-R\dot{\varphi}^2)\hat{n}_r + (R\ddot{\varphi})\hat{n}_\varphi]$$

$-m\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t}\times\overline{r'} = -m\ddot{\varphi}\hat{n}_z\times(-l\hat{n}_{y'}) = m\ddot{\varphi}l\hat{n}_z\times\hat{n}_{y'} = -m\ddot{\varphi}l\hat{n}_{x'}$



$-2m(\overline{\omega}\times\overline{v'}) = -2m(\dot{\varphi}\hat{n}_z\times(-\dot{l}\hat{n}'_y)) = -2m\dot{\varphi}\dot{l}\hat{n}_{x'}$
$-m\overline{\omega}\times(\overline{\omega}\times\overline{r'}) = -m(\dot{\varphi}\hat{n}_z)\times(\dot{\varphi}\hat{n}_z\times(-l)\hat{n}_{y'}) = -m(\dot{\varphi}\hat{n}_z)\times(\dot{\varphi}l\hat{n}'_x) = -m\dot{\varphi}^2l\hat{n}_{y'}$ Now look at the x' direction ($\hat{n}_r$ and $\hat{n}_{x'}$):

$$mR\dot{\varphi}^2 - m\ddot{\varphi}l - 2m\dot{\varphi}\dot{l} = 0 \implies \dot{l}^2 + l\ddot{l} = 0$$

Using $\ddot{l} = \frac{\mathrm{d}\dot{l}}{\mathrm{d}l}\dot{l}$ (by chain rule), we get $\dot{l} + l\frac{\mathrm{d}\dot{l}}{\mathrm{d}l} = 0$, $\dot{l}\mathrm{d}l + l\mathrm{d}\dot{l} = 0$, so $l\dot{l} = C$.

To find $l\dot{l} = C$ at $t = 0$, we need to use $\dot{l} = R\dot{\varphi}$. $l\dot{l} = lR\dot{\varphi} = Rl\dot{\varphi}$. Now $\overline{v} = \overline{v}_{O'} + \overline{v'} + (\overline{\omega} \times \overline{r'})$. At $t = 0$, $\overline{v} = v_0\hat{n}_r$ is perpendicular to $\hat{n}_{y'}$, and $\overline{\omega} \times \overline{r'} = \dot{\varphi}\hat{n}_z \times y'\hat{n}_{y'} = \dot{\varphi}(-l)(-\hat{n}_{x'})$ is also perpendicular to $\hat{n}_{y'}$. Besides, $\overline{v'}$ is along $\hat{n}'_y$ because our choice of the non-inertial FoR ensures that the particle is always on the $O'Y'$ axis. Furthermore, O' slides on the edge of the cylinder, so $\overline{v}_{O'}$ is also along $\hat{n}_{y'}$, so $\overline{v'} + \overline{v}_{O'} = 0$, and $\overline{v} = \overline{\omega} \times \overline{r'}$. $v_0 = l\dot{\varphi}$. Furthermore, $\overline{v}_{O'} = R\dot{\varphi}\hat{n}_\varphi$ by the velocity in the polar coordinates, so $\overline{v'} = -R\dot{\varphi}\hat{n}_\varphi$. Therefore, $C = l\dot{l}\Big|_{t=0} = Rv_0$. $l\dot{l} = Rv_0$, so $l\mathrm{d}l = Rv_0\mathrm{d}t$, $\frac{1}{2}l^2 = Rv_0t$, $l = \sqrt{2Rv_0t}$.

Solution Using Lagrangian Mechanics

Use the length l of the straight component of the rope as the generalized coordinate. $L = K - U = \frac{1}{2}mv^2$. v consists of two components: v_φ (along the straight rope) and v_r (perpendicular to the rope). $v_\varphi = R\dot{\varphi} - \dot{l} = 0$, and $v_r = l\dot{\varphi} = \frac{l\dot{l}}{R}$. $L = \frac{1}{2}ml^2\dot{l}^2/R^2$.

$$\frac{\partial L}{\partial \dot{l}} = \frac{ml^2\dot{l}}{R^2} \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{l}} = \frac{2ml\dot{l}^2}{R^2} + \frac{ml^2\ddot{l}}{R^2}$$

$$\frac{\partial L}{\partial l} = \frac{m\dot{l}^2 l}{R^2}$$

so by the Euler Lagrange Equations, $\frac{ml\dot{l}^2}{R^2} + \frac{ml^2\ddot{l}}{R^2} = 0$, $\dot{l}^2 + l\ddot{l} = 0$.

- Elements of Lagrangian Mechanics
- **Momentum**
- Center-of-Mass FoR
- Exercises

Momentum

Definition

Momentum $\overline{P} = m\overline{v}$
Newton's second law in terms of linear momentum: $\overline{F} = \frac{\mathrm{d}\overline{P}}{\mathrm{d}t}$

Conservation of Momentum

If the sum of all external forces on the system is equal to zero, then the total momentum of the system is constant.
The total momentum of a system can only be changed by external forces.

Collisions

Two objects interact (directly or non-directly) over a finite time interval.

Elastic

Internal forces involved are potential, hence mechanical energy is conserved. Approach speed is equal to departure speed.

Inelastic

Internal forces are non-conservative, so mechanical energy is not conserved. Departure speed is zero.

In both cases, the total momentum is conserved.

Center of Mass

Discrete distributions of mass $\overline{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \overline{r}_i}{\sum_{i=1}^{N} m_i}$

Continuous distributions of mass

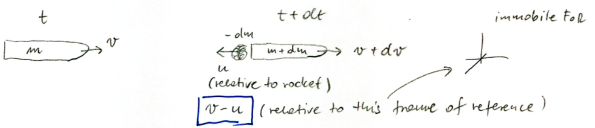
$x_{cm} = \frac{\int_\Omega x \mathrm{d}m}{\int_\Omega \mathrm{d}m}$ $y_{cm} = \frac{\int_\Omega y \mathrm{d}m}{\int_\Omega \mathrm{d}m}$ $z_{cm} = \frac{\int_\Omega z \mathrm{d}m}{\int_\Omega \mathrm{d}m}$

The total momentum of the system is equal to the momentum of a hypothetical particle of mass M moving with velocity $\overline{v}_{cm}$

$$M\overline{v}_{cm} = \sum_{i=1}^{N} \overline{P_i} = \overline{P}$$

This property of the center of mass motivates a new Frame of Reference: the center-of-mass Frame of Reference.

Rocket Propulsion



$(u, \frac{\mathrm{d}m}{\mathrm{d}t}$ const.)

By the conservation of momentum in the immobile frame of reference,

$$mv = (m + \mathrm{d}m)(v + \mathrm{d}v) - \mathrm{d}m(v - u)$$

$$mv = mv + v\mathrm{d}m + m\mathrm{d}v - v\mathrm{d}m + u\mathrm{d}m$$

$$0 = m\mathrm{d}v + u\mathrm{d}m \implies \mathrm{d}v = -\frac{u\mathrm{d}m}{m} \implies v(t) - v(0) = -u\ln\left(\frac{m(t)}{m(0)}\right)$$

- Elements of Lagrangian Mechanics
- Momentum
- Center-of-Mass FoR
- Exercises

Center-of-Mass FoR

It is often convenient to consider impacts in a translational FoR whose origin is attached to the center of mass of the system. The kinetic energy of the system can be decomposed into the translational kinetic energy of the center of mass and the kinetic energy of the mass in the system with respect to the center of mass.

Proof.

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i (\overline{v}_c + \overline{v}_{i,c})^2 =$$
$$\sum_{i=1}^{N} \frac{1}{2} m_i v_c^2 + \sum_{i=1}^{N} \frac{1}{2} m_i v_{i,c}^2 + \sum_{i=1}^{N} m_i \overline{v}_c \circ \overline{v}_{i,c} =$$
$$\underbrace{\frac{1}{2} M v_c^2}_{\text{K CoM}} + \underbrace{\sum_{i=1}^{N} \frac{1}{2} m_i v_{i,c}^2}_{\text{K w.r.t. CoM}} + \underbrace{\overline{v_c} \circ \sum_{i=1}^{N} m_i \overline{v}_{i,c}}_{\text{Zero}}$$

□

Particle down a Wedge

Question

A point particle of mass m moves without friction down a wedge of mass M that is free to slide on a frictionless table. The wedge is inclinded at the angle α to the horizontal. How many degrees of freedom does the particle have here? Identify the generalized coordinates here.

Solution

We need two independent generalized coordinates:

1. Position of the tip of the edge x
2. Height of the particle h

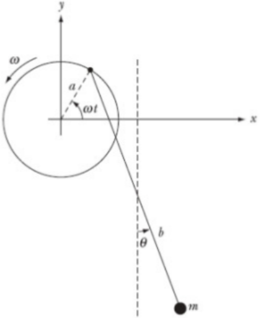
Now let's solve this problem using Lagrangian Mechanics.

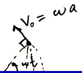
The Power of Lagrangian's (over Newton's) Mechanics

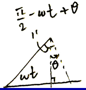
$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \dot{h}/\tan\alpha)^2 + (\dot{h})^2)$
$U = mgh$
$L = K - U = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \dot{h}/\tan\alpha)^2 + \dot{h}^2) - mgh$
$\frac{\partial L}{\partial x} = 0$
$\frac{\partial L}{\partial h} = -mg$
$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + m\dot{h}/\tan\alpha$
$\frac{\partial L}{\partial \dot{h}} = m\dot{h} + m\dot{x}/\tan\alpha + m\dot{h}/\tan^2\alpha$
Hence using the Euler-Lagrangian Equations,
$(M + m)\ddot{x} + \frac{m\ddot{h}}{\tan\alpha} = 0$
$m\ddot{h} + \frac{m\ddot{x}}{\tan\alpha} + m\ddot{h}/\tan^2\alpha + mg = 0$
It is then easy to solve for $\ddot{x}$ and $\ddot{h}$:
$\ddot{h} = \frac{g\tan^2\alpha}{\frac{m}{M+m} - 1 - \tan^2\alpha}$
$\ddot{x} = \frac{mg\cos\alpha\sin\alpha}{M + m\sin^2\alpha}$

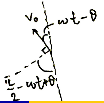
Simple Pendulum on a Rim

A simple pendulum of length b and mass m moves on a massless rim of radius a rotating with constant angular velocity ω. How many degrees of freedom do we have here? Find the Lagrangian.









Simple Pendulum on a Rim

There is only one degree of freedom θ for this particle on the end of the simple pendulum.

$$U = mg(a\sin(\omega t) - b\cos\theta)$$

$$K = \frac{1}{2}m[(\dot{\theta}b)^2 - 2\dot{\theta}b\omega a\sin(\omega t - \theta) + (\omega a)^2]$$

Now $L = K - U$. Here the constraint is more complicated and requires some more sophisticated knowledge to obtain the EoM.

Particle on the Surface of a Sphere

Question

Find the equations of motion of a particle of mass m constrained to move on the surface of a sphere, acted upon a conservative force $\mathbf{F} = F_0 \hat{n}_\theta$, with F_0 a constant.

Solution

On this particular sphere, we are able to define potential for this force $\mathbf{F}$ (similar to the proof of central force). Now in the spherical coordinates, $\nabla U = \frac{\partial U}{\partial r}\hat{n}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{n}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}\hat{n}_\varphi$, so $U = -r\int F_0 \mathrm{d}\theta = -rF_0\theta + C$. Furthermore, $K = \frac{1}{2}m[(r\dot{\theta})^2 + (r\sin\theta\dot{\varphi})^2]$, so the Lagrangian

$$L = K - U = \frac{1}{2}m[(r\dot{\theta})^2 + (r\sin\theta\dot{\varphi})^2] + rF_0\theta + C$$

For the general coordinate φ,

$$\frac{\partial L}{\partial \varphi} = 0 \quad \frac{\partial L}{\partial \dot{\varphi}} = m(r\sin\theta\dot{\varphi})r\sin\theta = mr^2\sin^2\theta\dot{\varphi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = mr^2[2\dot{\varphi}\sin\theta\cos\theta\dot{\theta} + \sin^2\theta\ddot{\varphi})] \overset{!}{=} 0$$

For the general coordinate θ,

$$\frac{\partial L}{\partial \theta} = rF_0 \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = mr^2\ddot{\theta}$$

so

$$mr^2\ddot{\theta} - rF_0 = 0 \quad \ddot{\theta} = \frac{F_0}{mr}$$

The conclusion is that $\varphi = 0$ and θ satisfies $\ddot{\theta} = \frac{F_0}{mr}$

Double Pendulum

The generalized coordinates are θ_1 and θ_2.

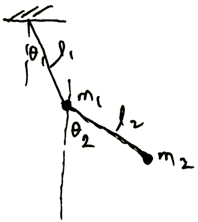
$U = -m_1 g l_1 \cos\theta_1 - m_2 g(l_2 \cos\theta_2 + l_1 \cos\theta_1)$, $K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

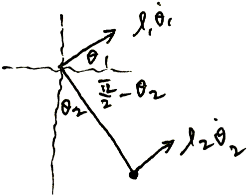
where $v_1 = l_1\dot{\theta}_1$, $v_2^2 = v_{2,\tau}^2 + v_{2,n}^2$

$v_{2,n} = l_1\dot{\theta}_1 \cos(\theta_1 + \frac{\pi}{2} - \theta_2)$, and

$v_{2,\tau} = l_1\dot{\theta}_1 \sin(\theta_1 + \frac{\pi}{2} - \theta_2) + l_2\dot{\theta}_2$.

Hence $L = K - U$, and the calculations can be done.





Block Mass Oscillation After Impact with Suspended Scale

Question

A block with mass m_1 falls down from height h on a horizontal plane with mass m_2 suspended on a spring with spring constant k, and remains on the plane. Find the amplitude of resulting oscillations.

Solution

Upon the non elastic impact, the speed v_0 of the two masses become the speed of their center of mass right before impact. $v_0 = \frac{\sqrt{2gh}m_1}{m_1+m_2}$. Be aware that when the two masses come together, the equilibrium position changes. Initial displacement from equilibrium $x_0 = \frac{m_1 g}{k}$, so the amplitude of resulting oscillation is

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} = \sqrt{\left(\frac{m_1 g}{k}\right)^2 + \left(\frac{\sqrt{2gh}m_1}{m_1+m_2}\sqrt{\frac{m_1+m_2}{k}}\right)^2}$$

Find the Center of Mass

Question

Find the center of mass of a non-uniform cylinder with the z axis as the axis of symmetry and $\rho(\mathbf{r}) = \alpha z^2$

Solution

Due to symmetry, $x_{CoM} = y_{CoM} = 0$. Now $z_{CoM} = \frac{\int_0^H (z)(\alpha z^2)\pi R^2 \mathrm{d}z}{\int_0^H (\alpha z^2)\pi R^2 \mathrm{d}z} = \frac{3}{4}H$

Angular Momentum

For a single particle, the angular momentum is defined as $\overline{L} = \overline{r} \times \overline{P}$. Torque is defined as $\overline{\tau} = \overline{r} \times \overline{F}$. Now $\dot{\overline{r}} = \overline{v}$, $\overline{P} = m\overline{v}$, $\mathrm{d}\overline{r} \times \overline{P} = 0$, so

$$\overline{\tau} = \frac{\mathrm{d}\overline{L}}{\mathrm{d}t} \tag{1}$$

Now consider central force $\overline{F}(\overline{r}) = f(r)\overline{r}$. They are conservative, as is proved on Slide 170. They also produce zero torque, because $\tau = \overline{r} \times \overline{F} = 0$. These two characteristics give rise to the two conservation properties of central force

- Mechanical Energy is preserved
- Angular Momentum is preserved

Aerial Velocity $\overline{\sigma} = \frac{1}{2}(\overline{r} \times \overline{v})$ is equivalent to angular momentum for constant-mass heavenly bodies.

Momentum of Inertia for a Particle About a Point

The angular momentum and angular velocity has the following relation:

$$\overline{L} = I\overline{\omega}$$

Here I, moment of inertia, is a one-by-one tensor quantity (a scalar).
$I = mr^2$. If $I = const$ (particle in circular motion), then $\overline{\tau} = I\overline{\varepsilon}$
For a system of particles, the total angular momentum can only be changed by a non-zero external torque.

Rigid Body

Definition

A body is called rigid if $|\overline{r} - \overline{r'}| = const$ for any two points on the body.

Momentum in Lab FoR

$$\overline{P} = \underbrace{M\overline{v_{O'}}}_{\text{translational motion}} + \underbrace{M\overline{\omega} \times \overline{r_{cm'}}}_{\text{rotational motion}}$$

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$$\overline{P} = \underbrace{M\overline{v_{O'}}}_{\text{translational motion}} + \underbrace{M\overline{\omega} \times \overline{r_{cm'}}}_{\text{rotational motion}}$$

Angular momentum about the origin of Lab FoR $\overline{L} = \sum_{i=1}^{N} m_i \overline{r_i} \times \overline{v_i}$

$$\overline{L} = M\overline{r_{O'}} \times \overline{v_{O'}} + M\overline{r_{O'}} \times (\overline{\omega} \times \overline{r_{cm'}}) + M\overline{r_{cm'}} \times \overline{v_{O'}} + \sum_{i=1}^{N} m_i \overline{r_i'} \times (\overline{\omega} \times \overline{r_i'})$$

where in the FoR associated with the rigid body, $\overline{r_i'}$ is the position vector of point mass $\overline{r_{cm'}}$ is the position vector of the center of mass.

Rigid Body with Pure Rotation

If we choose $\overline{v_{O'}} = 0$, O' at the center of mass of the body, and $O = O'$, then using the back-cab identity of vectors (i.e., $\overline{a} \times (\overline{b} \times \overline{c}) = \overline{b}(\overline{a} \circ \overline{c}) - \overline{c}(\overline{a} \circ \overline{b})$), The angular momentum with pure rotation $\overline{L} = \sum_{i=1}^{N} m_i \overline{r_i'} \times (\overline{\omega} \times \overline{r_i'})$ in the CoM FoR is rewritten as

$$\overline{L} = \sum_{i=1}^{N} m_i [\overline{\omega} r_i'^2 - \overline{r_i'}(\overline{\omega} \circ \overline{r_i'})]$$

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$$\overline{L} = \sum_{i=1}^{N} m_i [\overline{\omega} r_i'^2 - \overline{r_i'}(\overline{\omega} \circ \overline{r_i'})]$$

Decomposing the linear terms in the CoM FoR of the rigid body (i.e, $\overline{\omega} = \overline{\omega_{x'}} + \overline{\omega_{y'}} + \overline{\omega_{z'}}$, $\overline{r_i'} = \overline{r_{ix'}} + \overline{r_{iy'}} + \overline{r_{iz'}}$), for $\alpha' = x', y', z'$

$$L_{\alpha'} = \left\langle \overline{L}, \hat{n}_{\alpha'} \right\rangle = \sum_{i=1}^{N} m_i \left(\omega_{\alpha'} r_i'^2 - r_{i\alpha'} \left(\sum_{\beta'} \omega_{\beta'} r_{i\beta'} \right) \right)$$

Next: Try to find I that $\overline{L} = I\overline{\omega}$, where I is a tensor quantity.

$L_{\alpha'} = \sum_{i=1}^{N} m_i \left(\omega_{\alpha'} r_i'^2 - r_{i\alpha'} \left(\sum_{\beta'} \omega_{\beta'} r_{i\beta'}\right)\right)$ To sum over β', rewrite

$$\omega_{\alpha'} r_i'^2 = \sum_{\beta'} \omega_{\beta'} r_i'^2 \delta_{\alpha'\beta'} (\delta_{\alpha'\beta'} = \begin{cases} 1 & \alpha' = \beta' \\ 0 & \alpha' \neq \beta' \end{cases})$$

so that $L_{\alpha'} = \sum_{i=1}^{N} m_i \left(\sum_{\beta'} \omega_{\beta'} r_i'^2 \delta_{\alpha'\beta'} - r_{i\alpha'} \left(\sum_{\beta'} \omega_{\beta'} r_{i\beta'}\right)\right)$

$L_{\alpha'} = \sum_{i=1}^{N} m_i \left(\omega_{\alpha'} r_i'^2 - r_{i\alpha'} \left(\sum_{\beta'} \omega_{\beta'} r_{i\beta'}\right)\right)$ To sum over β', rewrite

$$\omega_{\alpha'} r_i'^2 = \sum_{\beta'} \omega_{\beta'} r_i'^2 \delta_{\alpha'\beta'} \left(\delta_{\alpha'\beta'} = \begin{cases} 1 & \alpha' = \beta' \\ 0 & \alpha' \neq \beta' \end{cases}\right)$$

so that $L_{\alpha'} = \sum_{i=1}^{N} m_i \left(\sum_{\beta'} \omega_{\beta'} r_i'^2 \delta_{\alpha'\beta'} - r_{i\alpha'} \left(\sum_{\beta'} \omega_{\beta'} r_{i\beta'}\right)\right)$ Taking out the sum iterator β' (both $\sum$ and $\omega_{\beta'}$),

$$L_{\alpha'} = \sum_{\beta'} \left[\sum_{i=1}^{N} m_i (r_i'^2 \delta_{\alpha'\beta'} - r_{i\alpha'} r_{i\beta'})\right] \omega_{\beta'}$$

$$L_{\alpha'} = \sum_{\beta'=x',y',z'} I_{\alpha'\beta'} \omega_{\beta'}$$

The 3×3 matrix $I_{\alpha'\beta'}$ is called the tensor of the moment of inertia

$$I_{\alpha'\beta'} = \sum_{i=1}^{N} m_i (\underbrace{r_i'^2 \delta_{\alpha'\beta'}}_{\text{Diagonal Terms}} - \underbrace{r_{i\alpha'} r_{i\beta'}}_{\text{Off-Diagonal Terms}})$$

Tensor of Inertia $[I_{\alpha'\beta'}]_{\alpha',\beta'=x',y',z'}$

Note that $I_{\alpha'\beta'} = I_{\beta'\alpha'}$, so this tensor quantity is symmetric. In the **Center-of-Mass** Frame of Reference,

$$\begin{bmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{bmatrix} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{bmatrix} \begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix}$$

where $[I_{\alpha'\beta'}]_{\alpha',\beta'=x',y',z'}$ is explicitly given as

$$\begin{bmatrix} \sum_{i=1}^N m_i(y_i'^2 + z_i'^2) & -\sum_{i=1}^N m_i x'y' & -\sum_{i=1}^N m_i x'z' \\ -\sum_{i=1}^N m_i y'x' & \sum_{i=1}^N m_i(x_i'^2 + z_i'^2) & -\sum_{i=1}^N m_i y'z' \\ -\sum_{i=1}^N m_i z'x' & -\sum_{i=1}^N m_i z'y' & \sum_{i=1}^N m_i(x_i'^2 + y_i'^2) \end{bmatrix}$$

In case of a continuous mass distribution, the summations are replaced by integrations.

Physical Significance of Diagonal Terms and Off Diagonal Terms

It is instructive to assume you have an axis along $O'X'$ so that the rigid body is rotating along it at $\overline{\omega} = \begin{pmatrix} \omega_{X'} \\ 0 \\ 0 \end{pmatrix}$.

Physical Significance of Diagonal Terms and Off Diagonal Terms

It is instructive to assume you have an axis along $O'X'$ so that the rigid body is rotating along it at $\overline{\omega} = \begin{pmatrix} \omega_{x'} \\ 0 \\ 0 \end{pmatrix}$.The angular momentum is

$$\overline{L} = \begin{pmatrix} I_{x'x'}\omega_{x'} \\ I_{y'x'}\omega_{x'} \\ I_{z'x'}\omega_{x'} \end{pmatrix}$$

Notice that the y' component and the z' component are rotating with the rigid body, whereas x' is in a fixed direction. The axis is providing torque to change the direction of the angular momentum, causing the axis to wear out.

The Spectral Theorem

Reference: Page 222 Vv286 FA2015. Eigenvalue λ and eigenvector u satisfies: $Au = \lambda u$.

Spectral Theorem

Let $A = A^* \in \mathrm{Mat}(n \times n; \mathbb{R})$ be a self-adjoint matrix. Then there exists an orthonormal basis of $\mathbb{R}^n$ consisting of *eigenvectors* of A.

Corollary

Every self-adjoint matrix A is diagonalizable. Furthermore, if $(v_1, \ldots, v_n)$ is an orthonormal basis of eigenvectors and $U = (v_1, \ldots, v_n)$, then $U^{-1} = U^*$. Hence, if A is self-adjoint, there exists an orthogonal matrix U such that $D = U^*AU$ is the diagonalization of A.

Notice that our tensor of inertia I is real and symmetric, so it is self-adjoint. We can *always* diagonalize it.

Principal Axes

Definition

For any tensor of inertia we can find three axes $\tilde{x}'$, $\tilde{y}'$, and $\tilde{z}'$ such that $[I_{\tilde{\alpha}'\tilde{\beta}'}]$ only has diagonal terms. Then we have $L_{\tilde{\alpha}'} = I_{\tilde{\alpha}'\tilde{\alpha}'}\omega_{\tilde{\alpha}'}$, where $\overline{L} \parallel \overline{\omega}$. Such axes are called **principal axes** of the tensor of inertia. The corresponding values of $I_{\tilde{\alpha}'\tilde{\alpha}'}$ are called *principal moments of inertia*.

General Steps

1. Find the Center of Mass of the rigid body
2. Set up a Cartesian Coordinate whose origin is at the CoM
3. Find the tensor of inertia
4. Diagonalize the tensor of inertia (find the eigenvalues and eigenvectors)

Eigenvalues and Eigenvectors

Eigenvalues λ_i and eigenvalues u_i for matrix I come in pairs: $\mathrm{I}u_i = \lambda_i u_i$.

Theorem

Eigenvectors u_i define directions of principal axes, and in the new coordinate system of principal axes (unit vectors are $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$), tensor of inertia is diagonal, and the eigenvalues line up on the main diagonal (i.e., $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$).

To find these eigenvalues, we need to solve

$$(\mathrm{I} - \lambda \mathbb{1})u_i = 0 \tag{2}$$

i.e., $u_i \in \ker(\mathrm{I} - \lambda \mathbb{1})$

Finding Eigenvalues

By *Fredhom Alternative* 1.7.21 on Slide 233 of Vv 285 SU 2016, for our matrix $A = \mathrm{I} - \lambda\mathbb{1}$, either

- $\det A = 0$, in which case $Ax = 0$ has a non-zero solution $x \in \ker A$, or
- $\det A \neq 0$, then $Ax = b$ has a unique solution $x = A^{-1}b$ for any $b \in \mathbb{R}^n$.

Since we need to find eigenvalues, we need the first case, i.e., we need to find such λ that $\det(\mathrm{I} - \lambda\mathbb{1}) = 0$
Then we plug back each λ_i into Eqn. 2 to find its corresponding eigenvector.

Finding Eigenvalues

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Since we need to find eigenvalues, we need the first case, i.e., we need to find such λ that $\det(\mathrm{I} - \lambda\mathbb{1}) = 0$
Then we plug back each λ_i into Eqn. 2 to find its corresponding eigenvector.If at least two principal moments are equal, the rigid body is called a symmetrical top; If all three principal moments are equal, it is called a spherical top.

Theorem

Kinetic Energy of a Rigid Body is given by
$K = \frac{1}{2}\sum_{\alpha',\beta'} I_{\alpha',\beta'}\omega_{\alpha'}\omega_{\beta'} = \frac{1}{2}\langle\overline{\omega}, I\overline{\omega}\rangle$

Moment of Inertia and Angular Momentum

After choosing the principal axes x, y, z, we omit the $'$.

$$I_{xx} = \sum_{i=1}^{N} m_i(y_i^2 + z_i^2), I_{yy} = \sum_{i=1}^{N} m_i(x_i^2 + z_i^2), I_{zz} = \sum_{i=1}^{N} m_i(x_i^2 + y_i^2)$$

Given $\omega = (0, 0, \omega_z)$ (no translational motion),

$$\overline{L} = I_{zz}\overline{\omega}, \text{ and } K = \frac{1}{2} I_{zz}\omega_z^2$$

Easier Configuration: Fixed Axis

For rotation of the rigid body around a fixed axis, we are only interested in the torque and angular momentum along the axis. The moment of inertia is a scalar defined by $I = \int_\Omega r^2 \mathrm{d}m$ because now the angular momentum has a fixed direction, all elementary mass are in planar motion, the speed given by $\omega r_\perp$, and angular momentum $L = \int_\Omega \omega r_\perp^2 \mathrm{d}m = \omega \int_\Omega r_\perp^2 \mathrm{d}m$, where $r_\perp$ is the distance from the elementary mass to the axis.

Steiner's Theorem (Parallel Axis Theorem)

Suppose A is an axis through the center of mass, and A' is an axis parallel to A and b from A.

$$I_{A'} = I_A + mb^2$$

Useful because we can traverse the rigid body more easily in a symmetric coordinate system (e.g., a torus).

2nd Law of Dynamics, Kinetic Energy

For rotation $\overline{\omega} = (0, 0, \omega)$, $\overline{L} = I_{zz}\overline{\omega}$. But $\frac{\mathrm{d}\overline{L}}{\mathrm{d}t} = \overline{\tau}^{ext}$, so

$$I_{zz}\frac{\mathrm{d}\omega}{\mathrm{d}t} = \tau^{ext}$$

CAUTION: $\frac{\mathrm{d}\overline{L}}{\mathrm{d}t} = \overline{\tau}^{ext}$ is generally valid, but $I_{zz}\frac{\mathrm{d}\omega}{\mathrm{d}t} = \tau^{ext}$ is valid only when the rigid body is given a fixed axis z, so that $\overline{\omega}$ does not change its orientation.

Work and Power in Rotational Motion (Fixed Axis)

In a rotational motion, $\overline{F}_{tan} \parallel \mathrm{d}\overline{r}$, so

$$\delta w = \tau_z \mathrm{d}\theta \quad w = \int_{\theta_1}^{\theta_2} \tau_z \mathrm{d}\theta$$

Note: Axis and radial components do no work. Nor do they contribute to torque.

Rotational Analogue of work-kinetic energy theorem

$$\delta w = \mathrm{d}\left(\frac{1}{2} I \omega_z^2\right) = \mathrm{d}K_{rot} \quad w = K_2 - K_1$$

Power

$$P = \tau_z \omega_z$$

Combined Translational and Rotational Motion

Kinetic Energy

For a rigid body in combined translational and rotational motion at angular velocity ω whose center of mass is in a translational motion $\overline{v}_{cm}$

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

Compare with the kinetic energy in Center-of-Mass FoR given on Slide 191.

Angular Momentum Theorem

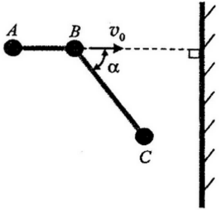
$$\tau_z = I\varepsilon_z$$

still holds true if axis

1. passes through center of mass
2. axis does not change orientation

Rigid Body Hitting a Wall, Inducing a Rotation

Two light rigid rods *AB* and *BC* are glued together at *B*. $\overline{AB}$ and $\overline{BC}$ form angle $\alpha \in (0, \pi/2)$, $|\overline{BC}| = l$, and $|\overline{AB}| = l\cos\alpha$. One small ball with mass *m* is fixed at each of *A*, *B*, and *C*. The balls and the rods form a rigid body. The entire system is placed on a smooth horizontal desk, and there is a fixed smooth vertical wall on the desk. Initially, *AB* is perpendicular to the wall, and the system is in a translational motion at v_0 along $\overline{AB}$ toward the wall. At one instant, ball *C* hit the wall, and right after impact, ball *C* has a zero velocity component perpendicular to the wall. Ball *C* does not stick to the wall. If after ball *C* hitting the wall, ball *B* hits the wall before ball *A* does, what condition does α satisfy?



Rigid Body Hitting a Wall, Inducing a Rotation (Sol.)

Suppose upon impact, the wall provides impulse J to the system at C. The effect of this impulse is to reduce the velocity of the Center of Mass of the system and to provide an angular momentum around the center of mass.

$$3mv_0 - J = 3mv_c \quad J \cdot (\frac{2}{3} l \sin\alpha) = \mathbf{I}\omega \tag{3}$$

In order that B hits the wall before A does, consider the situation where they hit the wall at the same time, i.e., the system has rotated $\pi/2$, and the center of mass has traveled $l\cos\alpha - \frac{1}{3} l \sin\alpha$. B hitting earlier means the time it would take the system to rotate $\pi/2$ is longer than the time it would take the center of mass to travel $l\cos\alpha - \frac{1}{3} l\sin\alpha$, should there be no secondary impact (which is possible if J is large).

$$\frac{l\cos\alpha - \frac{1}{3} l \sin\alpha}{v_c} < \frac{\frac{\pi}{2}}{\omega} \tag{4}$$

Rigid Body Hitting a Wall, Inducing a Rotation (Sol.)

Now we do not know J, but there is a constraint on it: the velocity of C after impact, which is the sum of the velocity of the center of mass and the velocity of C in the center of mass FoR.

$$v_C - \omega(\frac{2}{3}l\sin\alpha) = 0$$

The moment of inertia is contributed by the three balls. Ball A contributes $m\left[\left(\frac{1}{3}l\sin\alpha\right)^2 + (l\cos\alpha)^2\right]$, Ball C contributes $m\left[\left(\frac{2}{3}l\sin\alpha\right)^2 + (l\cos\alpha)^2\right]$, and Ball B contributes $m\left(\frac{1}{3}l\sin\alpha\right)^2$

$$\mathbf{I} = ml^2(\frac{2}{3} + \frac{4}{3}\cos^2\alpha)$$

Rigid Body Hitting a Wall, Inducing a Rotation (Sol.)

it then follows that (plugging **I** into Equation 3)

$$3\sin\alpha(v_0 - v_c) = \omega l(1 + 3\cos^2\alpha)$$

so $v_c = \frac{2v_0\sin^2\alpha}{4-\sin^2\alpha}$, and $\omega = \frac{3v_0\sin\alpha}{(4-\sin^2\alpha)l}$. Plugging these into Equation 4,

$$(\pi + 1)\sin\alpha > 3\cos\alpha$$

$$\tan\alpha > \frac{3}{\pi + 1}$$

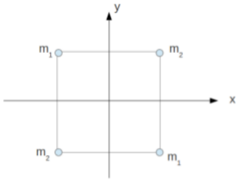
$$\alpha > 36^\circ$$

Principal Axes Transformation

Question

A square with side length a lies in plane $z = 0$ and has masses m_1 and m_2 in its vertices.

- Find the components of the tensor of inertia with respect to axes x, y, z.
- Diagonalize this tensor, giving directions of the principal axes.



Tensor of Inertia

$$I = \begin{bmatrix} 2(m_2+m_1)(\frac{a}{2})^2 & 2(m_1-m_2)(\frac{a}{2})^2 & 0 \\ 2(m_1-m_2)(\frac{a}{2})^2 & 2(m_2+m_1)(\frac{a}{2})^2 & 0 \\ 0 & 0 & 2(m_1+m_2)\frac{a^2}{2} \end{bmatrix}$$

The characteristic equation is

$$\det\begin{bmatrix} 2(m_2+m_1)(\frac{a}{2})^2-\lambda & 2(m_1-m_2)(\frac{a}{2})^2 & 0 \\ 2(m_1-m_2)(\frac{a}{2})^2 & 2(m_2+m_1)(\frac{a}{2})^2-\lambda & 0 \\ 0 & 0 & 2(m_1+m_2)\frac{a^2}{2}-\lambda \end{bmatrix}=0$$

The eigenvalues are

$$\lambda_1 = 2(m_1+m_2)\frac{a^2}{2} \quad \lambda_2 = m_2 a^2 \quad \lambda_3 = m_1 a^2$$

and their corresponding unit eigenvectors are

$$u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

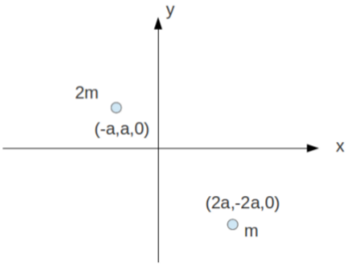
The tensor of inertia in the principal axes FoR is given by the eigenvalues on the diagonal:

$$D = diag(\lambda_1, \lambda_2, \lambda_3)$$

Degenerate Eigenvalues

Albegraic Multiplicity

Then the multiplicity of the zero in $p(\lambda) = 0$ is called the algebraic multiplicity of λ.



Using symmetry, the three unit eigenvectors are

$$\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}, \text{ and } \begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}$$

The tensor of inertia is

$$I = \begin{bmatrix}6ma^2 & 6ma^2 & 0\\6ma^2 & 6ma^2 & 0\\0 & 0 & 12ma^2\end{bmatrix}$$

Characteristic Equation

$$p(\lambda) = (6ma^2 - \lambda)^2(12ma^2 - \lambda) - (12ma^2 - \lambda)(6ma^2)^2 = 0$$

$$\lambda_1 = 12ma^2, \lambda_2 = 12ma^2, \lambda_3 = 0$$

Eigenspace and Geometric Multiplicity

Geometric Multiplicity

The subspace $V_\lambda = \{x \in V : Ax = \lambda x\}$ is called the eigenspace for eigenvalue λ. The dimension $\dim V_\lambda$ is called the geometric multiplicity of λ.

Notice that with $\lambda = 12ma^2$ we get $u_x - u_y = 0$ and no control over u_z.

Remarks

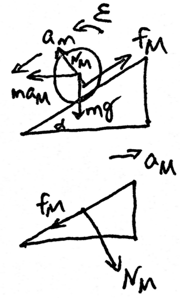
Since we can always diagonalize the tensor of inertia, we anticipate the algebraic multiplicity of each eigenvalue to be equal to its geometric multiplicity, in which case we choose orthonormal vectors that span the eigenspace as the direction of our principal axes.

With $\lambda = 12ma^2$ you can get two eigenvectors: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

Cylinder down a Movable Wedge

Question

A wedge with mass *M* and angle α rests on a frictionless horizontal surface. A cylinder with mass *m* rolls down the wedge without slipping. Find the acceleration of the wedge.



Cylinder down a Movable Wedge (sol.)

Solution

The cylinder: No slipping constraint: $\varepsilon = \frac{a_m}{R}$
Rotation around the center of mass: $f_M R = \varepsilon(\frac{1}{2}mR^2)$
Translational force along the surface: $ma_M \cos\alpha + mg\sin\alpha - f_M = ma_m$
Translational force perpendicular to the surface:
$N_M + ma_M \sin\alpha = mg\cos\alpha$
Notice we don't have $f_M = N_M$ in these rolling without slipping problems. Instead, we use the no slipping constraint.

Cylinder down a Movable Wedge (sol. contd.)

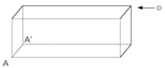
Then we analyze the wedge in the FoR attached to the ground.
Horizontal forces: $Ma_M = N_M \sin\alpha - f_M \cos\alpha$
We get $f_M = \frac{1}{2} ma_m$ from the first two equations,
$N_M = mg\cos\alpha - ma_M \sin\alpha$ from the fourth equation, and
$ma_M \cos\alpha + mg\sin\alpha = \frac{3}{2} ma_m$ from the third equation. Finally, plugging in everything into the last equation, we get

$$Ma_M = (mg\cos\alpha - ma_M\sin\alpha)\sin\alpha - \frac{1}{2}m(\frac{2}{3}(a_M\cos\alpha + g\sin\alpha))\cos\alpha$$

$(M + m\sin^2\alpha + \frac{1}{3}m\cos^2\alpha)a_M = \frac{2}{3}mg\sin\alpha\cos\alpha$,
$a_M = \frac{mg\sin 2\alpha}{3(M+m\sin^2\alpha+\frac{1}{3}m\cos^2\alpha)}$

Ball hitting a Fixed-Axis Box

Question



A ball with mass m, moving within the horizontal direction with speed v, hits the upper edge of a rectangular box with dimensions $l \times l \times 2l$. Assuming that the box can rotate about a fixed axis containing the edge AA', and the collision of the ball with the box is elastic (and the ball moves back in the horizontal direction), find

1. angular velocity of the box starts moving at the moment of collision
2. equation of motion of the box after the collision
3. the minimum speed of the ball needed to put the box in the upright position

The angular momentum of the box around axis AA' is $I_{AA'}$, and the mass of the box is M (uniform distribution).

Conservation of angular momentum around AA'

$$I_{AA'}\omega - mv_1 l = mv_0 l$$

Conservation of mechanical energy

$$\frac{1}{2}I_{AA'}\omega^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_0^2$$

Get a quadratic equation about ω:

$$(I_{AA'} + \frac{I_{AA'}^2}{ml^2})\omega^2 - \frac{2I_{AA'}mv_0 l}{ml^2}\omega + C = 0$$

Mathematically, sum of the two roots of ω for $a\omega^2 + b\omega + c = 0$ is equal to $-\frac{b}{a}$. Since the two solutions of ω corresponds to the angular velocity of the box before and after the collision, and we already know that before the collision, $\omega = 0$, it follows that after the collisiion,

$$\omega = \frac{2v_0}{l + \frac{I_{AA'}}{ml}}$$

After the collision, the box is under the torque of gravity. Torque changes the angular momentum following Eqn. 1, so

$$I_{AA'}\ddot{\alpha} + Mgl\frac{\sqrt{5}}{2}\cos\alpha = 0$$

After the collision, the mechanical energy of the box is conserved.

Initial: $K_1 = \frac{1}{2}I_{AA'}\left(\frac{2v_0}{l+\frac{I_{AA'}}{ml}}\right)^2$, Maximum height: $K_2 = 0$ (when the center of mass is above AA'). Increased potential energy: $\Delta U = -Mg\frac{l}{2} + Mg\frac{\sqrt{5}}{2}l$. Therefore, using $\Delta K + \Delta U = 0$,

$$-\frac{1}{2}I_{AA'}\left(\frac{2v_0}{l+\frac{I_{AA'}}{ml}}\right)^2 + Mg\frac{\sqrt{5}-1}{2}l = 0$$

The minimal required speed $v_0 = \frac{l+\frac{I_{AA'}}{ml}}{2}\sqrt{\frac{(\sqrt{5}-1)Mgl}{I_{AA'}}}$

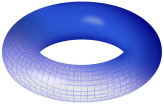
Simple (maybe not) Calculations

Problem

Using symmetry, find the principal axes and corresponding principal moments of inertia for:

1. thin disk
2. thin-walled hollow sphere
3. torus with mean radius R and the radius of cross-section r

assuming total mass is m and is distributed uniformly across the body.



Thin Disk, Axes in the Disk

Thin disk has two axes contained in the disk through the center and a perpendicular axis through the center.
Aerial mass density $\sigma = \frac{m}{\pi R^2}$. For the two axes contained in the disk,

$$I = 2\int_0^R 2\sqrt{R^2 - x^2}\sigma x^2 \,\mathrm{d}x$$
$$= 4\sigma \int_0^R \sqrt{R^2 - x^2} x^2 \,\mathrm{d}x$$
$$= 4\sigma R \int_0^R \sqrt{1 - \frac{x^2}{R^2}} x^2 \,\mathrm{d}x$$
$$= 4\sigma R \int_0^{\frac{1}{2}\pi} \cos\theta R^2 \sin^2\theta R \cos\theta \,\mathrm{d}\theta$$
$$= \sigma R^4 \int_0^{\frac{1}{2}\pi} \sin^2(2\theta) \,\mathrm{d}\theta$$
$$= \sigma R^4 \int_0^{\frac{1}{2}\pi} \frac{1}{2} \,\mathrm{d}\theta - \sigma R^4 \int_0^{\frac{1}{2}\pi} \frac{1}{2}\cos(4\theta) \,\mathrm{d}\theta$$
$$= \sigma R^4 (\tfrac{1}{4}\pi) - 0$$
$$= \frac{1}{4} m R^2$$

Thin Disk, Perpendicular Axis

For the perpendicular axis,

$$\begin{aligned} I &= \sigma \int_0^R 2\pi\rho \cdot \rho^2 \,\mathrm{d}\rho \\ &= \sigma \int_0^R 2\pi\rho^3 \,\mathrm{d}\rho \\ &= \sigma \, \frac{1}{4}(2\pi)\rho^4 \Big|_0^R \\ &= \sigma(\frac{1}{2}\pi)R^4 \\ &= \frac{1}{2}mR^2 \end{aligned}$$

Thin-walled hollow sphere has three mutually perpendicular axes through the center. Aerial mass density $\sigma = \frac{m}{4\pi R^2}$.

$$
\begin{aligned}
I &= 2\int_0^{\frac{1}{2}\pi} \sigma 2\pi (R\sin\theta)^3 R\,\mathrm{d}\theta \\
&= 4\pi\sigma R^4 \int_0^{\frac{1}{2}\pi} \sin^2\theta \sin\theta\,\mathrm{d}\theta \\
&= 4\pi\sigma R^4 \int_0^{\frac{1}{2}\pi} (1-\cos^2\theta)(-\mathrm{d}\cos\theta) \\
&= 4\pi\sigma R^4 \left[\int_0^{\frac{1}{2}\pi} -\mathrm{d}\cos\theta + \int_0^{\frac{1}{2}\pi} \cos^2\theta\,\mathrm{d}\cos\theta\right] \\
&= 4\pi\sigma R^4 \left[-(0-1) + \frac{1}{3}(0-1)\right] \\
&= 4\pi\sigma R^4 (\frac{2}{3}) = \frac{2}{3} m R^2
\end{aligned}
$$

Torus has two axes crossing the torus and the center and one perpendicular axis through the center. We need to calculate its volume first. The coordinate system is shown in Figure 10.

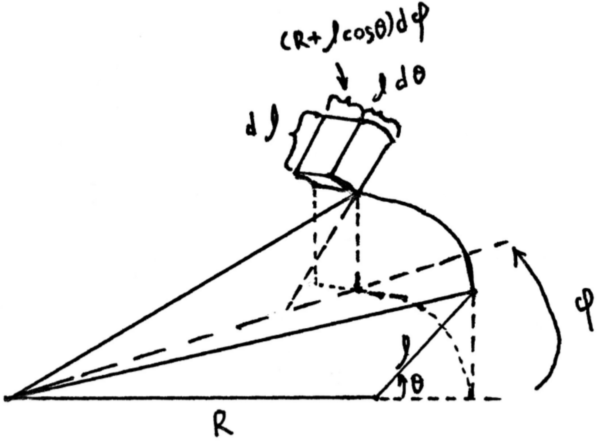


Figure: Coordinates for torus on Slide 241.

Torus Geometry

$$\begin{aligned} V &= \left[\int_0^{2\pi} \mathrm{d}\varphi\right] \int_0^r \left[\int_0^{2\pi} Rl\,\mathrm{d}\theta + \int_0^{2\pi} l^2 \cos\theta\,\mathrm{d}\theta\right] \mathrm{d}l \\ &= (2\pi) \int_0^r [Rl(2\pi)]\,\mathrm{d}l \\ &= (4\pi^2)R\,\frac{1}{2}l^2\Big|_0^r \\ &= 4\pi^2 R(\frac{1}{2})r^2 \\ &= (2\pi R)(\pi r^2) \end{aligned}$$

$$\rho = \frac{m}{V} = \frac{m}{(2\pi R)(\pi r^2)}$$

Torus, Perpendicular Axis

For the perpendicular axis,

$$\mathrm{d}I = \rho(R + l\cos\theta)^3 l\,\mathrm{d}l\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

$$\begin{aligned}
I &= \rho\left[\int_0^{2\pi}\mathrm{d}\varphi\right]\int_0^r[\int_0^{2\pi}R^3 l\,\mathrm{d}\theta + \dots \\
&+ \int_0^{2\pi}3R^2l^2\cos\theta\,\mathrm{d}\theta + \int_0^{2\pi}3Rl^3\cos^2\theta\,\mathrm{d}\theta + \int_0^{2\pi}l^4\cos^3\theta\,\mathrm{d}\theta]\,\mathrm{d}l \\
&= \rho(2\pi)\int_0^r[2\pi R^3 l + 0 + 3Rl^3\pi + 0]\,\mathrm{d}l \\
&= \rho(2\pi)[\pi R^3 r^2 + \frac{3}{4}R\pi r^4] \\
&= m\left[R^2 + \frac{3}{4}r^2\right]
\end{aligned}$$

For the axis through the torus,

$$d^2 = (l\sin\theta)^2 + [(R + l\cos\theta)\sin\varphi]^2$$

$$\mathrm{d}I = \rho(R + l\cos\theta)l[(l\sin\theta)^2 + [(R + l\cos\theta)\sin\varphi]^2]\,\mathrm{d}l\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

$$\begin{aligned}
I &= \int_0^r \int_0^{2\pi} \int_0^{2\pi} \rho(R + l\cos\theta)l[(l\sin\theta)^2 + [(R + l\cos\theta)\sin\varphi]^2]\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}l \\
&= \int_0^r \int_0^{2\pi} \rho(R + l\cos\theta)l[l^2\sin^2\theta(2\pi) + (R + l\cos\theta)^2\pi]\,\mathrm{d}\theta\,\mathrm{d}l \\
&= \int_0^r (\rho R)[l^3(2\pi)(\pi) + \pi R^2 l(2\pi) + 2\pi R l^2(0) + \pi l^3(\pi)] + \ldots \\
&\quad + (\rho l)[l^3(2\pi)(0) + \pi R^2 l\cos\theta(0) + 2\pi R l^2(\pi) + \pi l^3(0)]\,\mathrm{d}l \\
&= (\rho R)[(2\pi^2)(\frac{1}{4}r^4) + 2\pi^2 R^2(\frac{1}{2}r^2) + \pi^2(\frac{1}{4}r^4)] + \rho[2\pi^2 R\frac{1}{4}r^4] \\
&= m\left[\frac{1}{2}R^2 + \frac{5}{8}r^2\right]
\end{aligned}$$

Conditions for Equilibrium

The two conditions required for the rigid body to be in equilibrium:

1. Net external force is equal to zero (translational motion of the center of mass):

$$\mathbf{F}^{ext} = 0$$

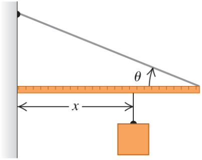
2. Net external torque is equal to zero (rotational motion around the center of mass):

$$\tau^{ext} = 0$$

$\overline{R} = \overline{N} + \overline{f}$

11.70 ••• One end of a uniform meter stick is placed against a vertical wall (Fig. P11.70). The other end is held by a lightweight cord that makes an angle θ with the stick. The coefficient of static friction between the end of the meter stick and the wall is 0.40. (a) What is the maximum value the angle θ can have if the stick is to remain in equilibrium? (b) Let the angle θ be 15°. A block of the same weight as the meter stick is suspended from the stick, as shown, at a distance x from the wall. What is the minimum value of x for which the stick will remain in equilibrium? (c) When $\theta = 15°$, how large must the coefficient of static friction be so that the block can be attached 10 cm from the left end of the stick without causing it to slip?

Figure **P11.70**



When $f = \mu N$, the direction of the total reactive force $\overline{R}$ is governed by the coefficient of friction μ. The balance of gravity, tension, and reactive force requires torque $\tau = 0$ about any point, so the lines of the three forces have to intersect at the same point.

Pull Wheel upstairs

11.76 •• You are trying to raise a bicycle wheel of mass m and radius R up over a curb of height h. To do this, you apply a horizontal force $\vec{F}$ (Fig. P11.76). What is the smallest magnitude of the force $\vec{F}$ that will succeed in raising the wheel onto the curb when the force is applied (a) at the center of the wheel and (b) at the top of the wheel? (c) In which case is less force required?

Figure **P11.76**

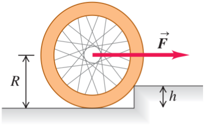


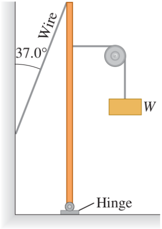
Figure **P11.77**

Be aware that as the wheel creeps up the stair, the moment arm of gravity is reducing, and the moment arm of F is increasing. Therefore, the minimal constant force of is given by a balance of torque initially with respect to the contact point on the stair.

Torque Balance and Force Balance

11.82 • A weight W is supported by attaching it to a vertical uniform metal pole by a thin cord passing over a pulley having negligible mass and friction. The cord is attached to the pole 40.0 cm below the top and pulls horizontally on it (Fig. P11.82). The pole is pivoted about a hinge at its base, is 1.75 m tall, and weighs 55.0 N. A thin wire connects the top of the pole to a vertical wall. The nail that holds this wire to the wall will pull out if an *outward* force greater than 22.0 N acts on it. (a) What is the greatest weight W that can be supported this way without pulling out the nail? (b) What is the *magnitude* of the force that the hinge exerts on the pole?

Figure **P11.82**



(a) Torque balance with respect to the hinge. (b) Force balance of the pole.

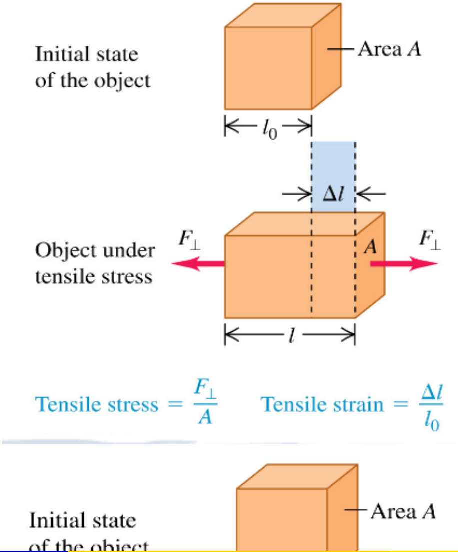
- Conditions for Equilibrium
- Elasticity
- Fluid Statics
- Fluid in Motion
- Gravitation
- Additional Exercises

Strain, Stress, and Elastic Modulus

Stress is the force per unit area.
Strain is the fractional deformation due to the stress.
Elastic modulus is stress divided by strain.
Hooke's Law: Stress and strain are proportional (small deformation).

$$\frac{\text{stress}}{\text{strain}} = \text{elastic modulus}$$

Tensile and Compressive Stress and Strain

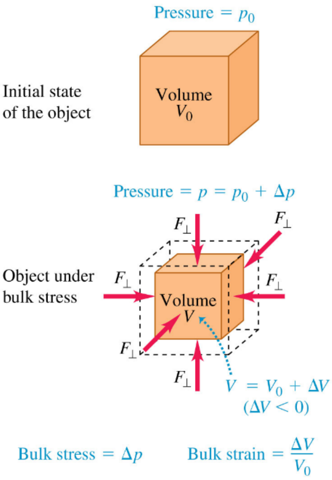


Young's Modulus

Young's modulus Y is tensile stress divided by tensile strain:

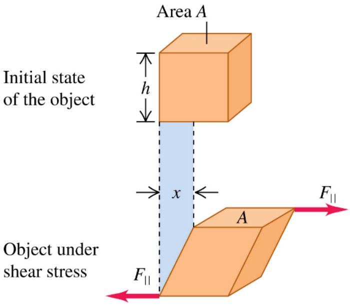
$$Y = \frac{\frac{F_\perp}{A}}{\frac{\Delta l}{l_0}}$$

Bulk Stress and Strain



Pressure in a fluid is force per unit area $p = \frac{F_\perp}{A}$.
Bulk stress is pressure change Δp upon volume change from V_0 to $V = V_0 + \Delta V$
Bulk strain is fractional volume change $\frac{\Delta V}{V_0}$
Bulk modulus is bulk stress divided by bulk strain:
$B = -\frac{\Delta p}{\Delta V / V_0}$

Shear stress and strain



Shear stress $= \frac{F_{||}}{A}$ Shear strain $= \frac{x}{h}$

Shear stress is $\frac{F_{||}}{A}$
Shear strain is $\frac{x}{h}$
Shear modulus is shear stress divided by shear strain: $S = \frac{\frac{F_{||}}{A}}{\frac{x}{h}}$

Elasticity and Plasticity

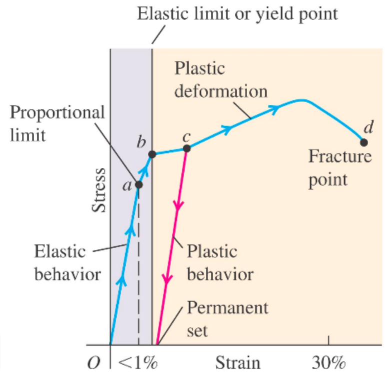


Table 11.3 Approximate Breaking Stresses

Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.2×10^8
Brass	4.7×10^8
Glass	10×10^8
Iron	3.0×10^8
Phosphor bronze	5.6×10^8
Steel	$5 - 20 \times 10^8$

Hooke's law applies to point *a*. Beyond elastic limit, the material demonstrates plastic behavior. You may try this with the spring in your used pens.

Pressure in a Fluid

For a fluid at rest,

$$p = \frac{\Delta F_\perp}{\Delta A}$$

Pressure at depth h:

$$p = p_0 + \rho g h$$

Pascal's law

Pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the liquid and the walls of the container.
Cause: work done on the fluid is zero.

Absolute pressure: total pressure $p = p_{atm} + p_{gauge}$. (e.g., gauge pressure at depth $p_{gauge} = p - p_0 = \rho g h$)

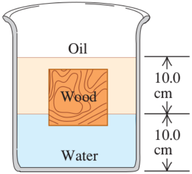
Buoyancy and Archimedes's law

When a body is immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body. Justification: the liquid was originally there in static, so the buayancy force has to balance the weight of that portion of liquid (replaced by the body).

Block in Fluids

12.31 •• A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (Fig. E12.31). The density of the oil is 790 kg/m^3. (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure at the lower face of the block? (c) What are the mass and density of the block?

Figure **E12.31**



(a) $p_{gauge,upper} = \rho_{oil} g h_{upper}$

(b) $p_{gauge,lower} = \rho_{oil} g h_{oil} + \rho_{water} g h_{lower}$

(c) $mg = (p_{gauge,lower} - p_{gauge,upper})S$; $m = \rho_{wood} V_{block}$

Ideal Fluid, Flow lines, Stream lines

Ideal Fluid

Fluid density does not change, experiences no internal friction (incompressible and no viscosity).

Flow Lines

Trajectories of individual particles in a fluid.

Stream Lines

Family of curves that are instantaneously tangential to the velocity vector field.

Steady Flow

The Flow lines coincide the stream lines.

Continuity Equation

Flow Tube

A tube formed by flow lines passing through the edge of an imaginary element of area. In steady flow

1. No fluid can cross the side walls of a flow tube
2. fluids in different flow tubes cannot mix

Continuity Equation

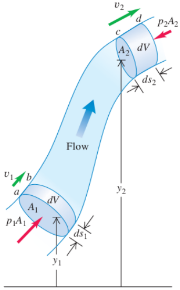
For homogeneous incompressible fluid:

$$A_1 v_1 = A_2 v_2$$

Bernoulli's Equation

$$p + \frac{1}{2}\rho v^2 + \rho g y = const$$

Bernoulli's Equation



Work done by pressure difference: $(p_1 - p_2)\mathrm{d}V$
Work done by gravity: $\rho \mathrm{d}Vg(y_1 - y_2)$
Change in Kinetic energy: $\frac{1}{2}\rho \mathrm{d}V(v_2^2 - v_1^2)$

Work-Kinetic energy theorem:

$$\frac{1}{2}\rho \mathrm{d}V(v_2^2 - v_1^2) = (p_1 - p_2)\mathrm{d}V + \rho \mathrm{d}Vg(y_1 - y_2)$$

Bernoulli's Equation:

$$\frac{1}{2}\rho v^2 + p + \rho gy = const$$

Continuity Equation and Bernoulli's Equation

Question

At one point in a pipeline the water's speed is 3.00 m/s and the gauge pressure is 5.00×10^4 Pa. Find the gauge pressure at a second point in the line, 11.0 m lower than the first, if the pipe diameter at the second point is twice that at the first.

Solution

$v_1 A_1 = v_2 A_2$ due to the continuity equation. The speed $v_1 = 3.00 \mathrm{~m/s}$, and down there, speed is $v_2 = 0.75 \mathrm{~m/s}$.

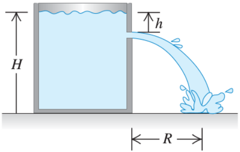
$$\frac{1}{2}\rho v_1^2 + p_1 + \rho g h_1 = \frac{1}{2}\rho v_2^2 + p_2 + \rho g h_2$$

$h_1 - h_2 = 11 \mathrm{~m}$, $p_1 = 5.00 \times 10^4 \mathrm{~Pa}$

Water out of an Open Tank

12.89 • **CP** Water stands at a depth H in a large, open tank whose side walls are vertical (Fig. P12.89). A hole is made in one of the walls at a depth h below the water surface. (a) At what distance R from the foot of the wall does the emerging stream strike the floor? (b) How far above the bottom of the tank could a second hole be cut so that the stream emerging from it could have the same range as for the first hole?

Figure **P12.89**



(a) $v = \sqrt{2gh}$

$R = \sqrt{2gh}\sqrt{2(H-h)/g}$

(b) $h^* = H - h$ will give the same range.

Bucket with Hole

Question

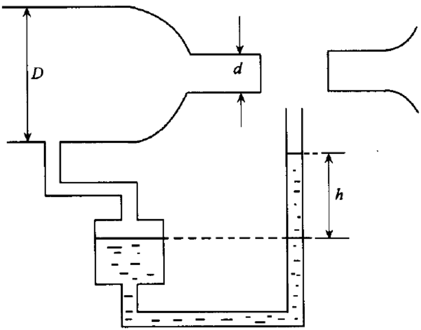
A cylindrical bucket, open at the top, is 25.0 cm high and 10.0 cm in diameter. A circular hole with a cross-sectional area 1.50 cm^2 is cut in the center of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of $2.40 \times 10^{-4}\ \mathrm{m}^3/\mathrm{s}$. How high will the water in the bucket rise?

Solution

At stabilized height, flow out rate is $2.40 \times 10^{-4}\ \mathrm{m}^3/\mathrm{s}$, and flow speed at the top is equal to zero. Hence $h = \frac{v^2}{2g}$, with $v = \frac{2.40\times10^{-4}}{1.50\times10^{-4}}\ \mathrm{m}/\mathrm{s}$.

Tube with Open Experimental Segment

Question



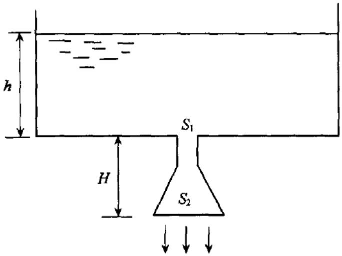
The open segment has cross-sectional diameter d, and the thick segment (cross-sectional diameter D) is connected to an alcohol (density ρ') pressure meter. When ideal incompressible fluid (density ρ) flows through, the pressure meter has a reading of height h. The atmospheric pressure is p_0. Find the speed of the liquid in the open segment.

Solution

$p_2 = 0$, $p_1 = \rho' gh$, $v_1 D^2 = v_2 d^2$, $\frac{1}{2}\rho v_1^2 + p_1 = \frac{1}{2}\rho v_2^2 + p_2$.

Water from Container to Conduit

Question



Water (ρ) flows from a large container to a trumpet-shaped conduit. The cross-sectional area of entrance and exit are S_1 and S_2, and the conduit has a length of H. The atmospheric pressure is p_0, and the flow is steady. For what length of H will the pressure of the liquid at the entrance of the conduit be zero?

Solution

By equation of continuity, $S_1 v_1 = S_2 v_2$; by Bernoulli's equation,

$$p_0 + \rho g(h + H) = p_0 + \frac{1}{2}\rho v_2^2 \quad p_2 + \frac{1}{2}\rho v_2^2 = \frac{1}{2}\rho v_1^2 + \rho g H$$

Newton's Law of Gravitation

The particle m_1 at $\bar{r}_1$ exerts gravitation force $\bar{F}_{12}$ on particle m_2 at $\bar{r}_2$ is

$$\bar{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\frac{\bar{r}_{12}}{|\bar{r}_{12}|}$$

where $\bar{r}_{12} = \bar{r}_1 - \bar{r}_2$. Gravitation force is a central force, so it is conservative and conserves angular momentum. Conservation of the angular momentum means planar motion (e.g. planets). Define gravitation interaction due to M on unit mass as a vector field in space:

$$\bar{E}_G = -G\frac{M}{r^2}\frac{\bar{r}}{|\bar{r}|}$$

$\nabla \circ \overline{E}_G$ Due to Point Mass at the Origin

For a point mass at the origin, the divergence of $\overline{E}_G$ everywhere else is zero: $\nabla \circ \overline{E}_G = -GM\nabla \circ \frac{\overline{r}}{r^3} = -GM\sum_{\alpha=x,y,z}\left(\frac{\partial}{\partial\alpha}\frac{\overline{r}\circ\hat{n}_\alpha}{r^3}\right)$ Now $\overline{r}\circ\hat{n}_\alpha = \alpha$, so $\frac{\partial}{\partial\alpha}\frac{\alpha}{r^3} = \frac{1}{r^3} + \alpha\left(-\frac{3}{r^4}\right)\frac{2\alpha}{2\sqrt{\sum_{\beta=x,y,z}\beta^2}} = \frac{1}{r^3} - \frac{3\alpha^2}{r^5}$, it follows that $\sum_{\alpha=x,y,z}\frac{\partial}{\partial\alpha}\frac{\alpha}{r^3} = \frac{3}{r^3} - \frac{3\sum_{\alpha=x,y,z}\alpha^2}{r^5} = 0$
Now choose a sphere Σ_1, radius R, centered at the origin, so $\int_{\Sigma_1}\overline{E}_G\circ\mathrm{d}\overline{S} = -\frac{GM}{R^2}(4\pi R^2) = -4\pi GM$, and by the theorem of Gauss that $\int_{\Sigma_1}\overline{E}_G\circ\mathrm{d}\overline{S} = \int_{\Omega_1}(\nabla\circ\overline{E}_G)\mathrm{d}^3r$, ($\Omega_1$ is the region enclosed by surface Σ_1), so the divergence of $\overline{E}_G$ at the origin satisfies

$$\int_{\Omega_1}(\nabla\circ\overline{E}_G)\Big|_{r=0}\delta^3(0)\mathrm{d}^3r = -4\pi GM$$

Now rewrite $M = \int_{\Omega_1}\rho(0)\delta^3(0)\mathrm{d}^3r$ (point mass at the origin), we get $\left(\nabla\circ\overline{E}_G\right) = -4\pi G\rho(0)$.

Gauss' Law for Gravitational Field

$\left(\nabla \circ \overline{E}_G\right) = -4\pi\rho(0)$ generalizes to a mass distribution $\rho(\overline{r})$ as $\nabla \circ \overline{E}_G(\overline{r}) = -4\pi G\rho(\overline{r})$ which is the differential form of Gauss' Law for Gravitational Field. Back into the integral form,

$$\int_{\Sigma_2} \overline{E}_G \circ \mathrm{d}\overline{S} = \int_{\Omega_2} (\nabla \circ \overline{E}_G)\mathrm{d}^3 r = \int_{\Omega_2} (-4\pi G\rho(\overline{r}))\mathrm{d}^3 r = -4\pi G M_{\Sigma_2}$$

where M_{Σ_2} is the mass enclosed by surface Σ_2.

Potential Energy and Potential

Potential Energy $U(r) = -G\frac{Mm}{r} + C$ where C depends on the choice of zero potential. Gravitational potential (potential energy of unit mass) with $U(\infty) = 0$:

$$V(r) = -G\frac{M}{r}$$

Note: there is a useful fact about the gradient of $\frac{1}{r}$:

$$\nabla\frac{1}{r} = \sum_{\alpha=x,y,z} -\frac{1}{r^2}\frac{\partial r}{\partial\alpha}\hat{n}_\alpha = \sum_{\alpha=x,y,z} -\frac{1}{r^2}\frac{2\alpha}{2\sqrt{\sum_{\beta=x,y,z}\beta^2}}\hat{n}_\alpha$$

but $\sum_{\beta=x,y,z}\beta^2 = r^2$ and $\sum_{\alpha=x,y,z}\alpha\hat{n}_\alpha = \bar{r}$, so

$$\nabla\frac{1}{r} = -\frac{1}{r^2}\frac{\bar{r}}{r}$$

which conforms to $\overline{F} = -\nabla U$ and $\overline{E}_G = -\nabla V$

Satellites on Circular Orbits

Gravitation force provides centripetal force:

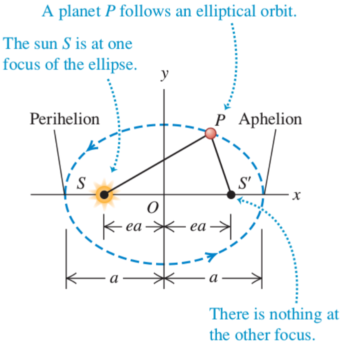
$$-\frac{GMm}{r^2}\frac{\bar{r}}{r} = -m\frac{v^2}{r}\hat{n}_r$$

$$v = \sqrt{\frac{GM}{r}}$$

Period on a circular orbit:

$$T = \frac{2\pi r}{v} = 2\pi\frac{r^{3/2}}{\sqrt{GM}}$$

Kepler's Laws



1. Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.
2. A line from the sun to a given planet sweeps out equal areas in equal times (constant aerial velocity $\overline{\sigma} = \frac{1}{2}(\overline{r} \times \overline{v})$).
3. $\frac{T^2}{a^3} = \frac{4\pi^2}{Gm_s}$

Ellipse's a, b, c versus planet's E and L

Given the mechanical energy $E < 0$ of the planet and the angular momentum L of the planet, we need to find the parameters semi-major axis length a, semi-minor axis length b, and semi-focal length c of the ellipse ($e = \frac{c}{a}$ is the eccentricity).

When the planet is on one end of the minor axis, $v = \sqrt{\frac{2}{m}\left[E + \frac{GMm}{a}\right]}$.

The angular momentum is $L = mvb$, so

$$\frac{L}{mb} = \sqrt{\frac{2}{m}\left[E + \frac{GMm}{a}\right]}$$

Then when the planet is at its perihelion or at its aphelion,

$$\frac{1}{2}mv_p^2 = E + \frac{GMm}{a-c} \quad \frac{1}{2}mv_a^2 = E + \frac{GMm}{a+c} \implies$$

$\frac{1}{2}mv_p^2(a-c)^2 = E(a-c)^2 + GMm(a-c)$
$\frac{1}{2}mv_a^2(a+c)^2 = E(a+c)^2 + GMm(a+c)$

$\frac{1}{2}mv_p^2(a-c)^2 = E(a-c)^2 + GMm(a-c)$
$\frac{1}{2}mv_a^2(a+c)^2 = E(a+c)^2 + GMm(a+c)$
Using $(a+c)v_a = (a-c)v_p$ by constant aerial velocity, we subtract one equation from the other and get

$$E(-4ac) + GMm(-2c) = 0 \implies E = -\frac{GMm}{2a} \implies a = -\frac{GMm}{2E}$$

Plugging this back to $\frac{L}{mb} = \sqrt{\frac{2}{m}\left[E + \frac{GMm}{a}\right]}$, we get $b = \sqrt{\frac{L^2}{-2mE}}$. It then follows that $c = \sqrt{a^2 - b^2} = \sqrt{\left(\frac{GMm}{2E}\right)^2 + \frac{L^2}{2mE}}$

Tunnel through the Earth

Question

A shaft is drilled from the surface through a straight tunnel *d* from the center of the earth. Assume the mass distribution of the earth is uniform, find the time it takes an object that is released from one end of the tunnel to travel to the other end (frictionless).

Solution

Suppose the object is x from equilibrium. The net force on the object has a magnitude of $\frac{M\frac{4}{3}\pi(d^2+x^2)^{3/2}}{\frac{4}{3}\pi R^3}\frac{Gmx}{(d^2+x^2)^{3/2}} = \frac{GMmx}{R^3}$, so the motion is simple harmonic.

A Little Line Integral

Question

A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod.

Figure **E13.32**

Calculate the gravitational potential energy of the rod-sphere system. Find the force exerted on the sphere by the rod.

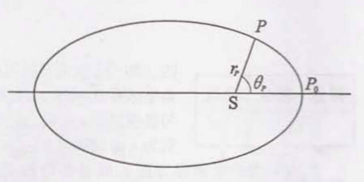
Solution

$U = \int_{x+L}^{x} -\frac{G\lambda m}{r}(-\mathrm{d}r) = G\lambda m \ln\left(\frac{x}{x+L}\right),$

$\overline{F} = -\nabla U = -G\lambda m \left(\frac{1}{x} - \frac{1}{x+L}\right)\hat{n}_x$

Halley's Comet

Halley's Comet is on an ellipse trajectory around the sun in a counter clockwise motion, whose period is 76.1 years. In 1986, when it was at its perihelion P_0, it was $r_0 = 0.590 \text{ AU}$ from the sun S. Some years later, the comet reached point P on the orbit, and the angle it has traversed is $\theta_p = 72.0°$. The following quantities are known:
$1 \text{ AU} = 1.50 \times 10^{11} \text{ m}$, gravitational constant
$G = 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$, the mass of the sun
$m_s = 1.99 \times 10^{30} \text{ kg}$. Find the distance r_p of P from S and the velocity of the comet at P.



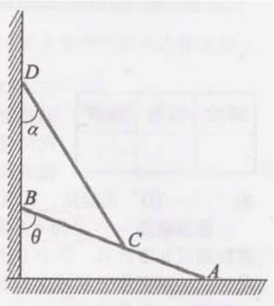
Kepler's third law: $a = \sqrt[3]{\frac{GT^2 m_s}{4\pi^2}}$ Mechanical energy $E = \frac{1}{2}mv_0^2 - \frac{Gm_s m}{r_0}$

Then using $x = c + r_p \cos\theta_p$ and $y_p = r_p \sin\theta_p$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$(a^2 \sin^2\theta_p + b^2\cos^2\theta_p) r_p^2 + 2b^2 c r_p \cos\theta_p - b^4 = 0$$

$r_p = \frac{-b^2 c\cos\theta_p + b^2 a}{a^2\sin^2\theta_p + b^2\cos^2\theta_p}$ Plugging in data, $a = 2.685\times 10^{12}\ \mathrm{m}$,
$b = \sqrt{a^2 - (a - r_0)^2} = 6.837\times 10^{11}\ \mathrm{m}$, $c = 2.597\times 10^{12}\ \mathrm{m}$, so
$r_p = 1.340\times 10^{11}\ \mathrm{m}$
Aerial velocity $\sigma = \frac{1}{2} r_p v_{p,transversal} = \frac{\pi ab}{T}$, so
$v_{p,transversal} = \frac{2\pi ab}{r_p T} = 3.587\times 10^4\ \mathrm{m/s}$
$v_p = \sqrt{-\frac{Gm_s}{a} + \frac{2Gm_s}{r_p}} = 4.395\times 10^4\ \mathrm{m/s}$ Hence
$v_{p,radial} = \sqrt{v_p^2 - v_{p,transversal}^2} = 2.540\times 10^4\ \mathrm{m/s}$
$\arctan(v_{p,radial}/v_{p,transversal}) = 35.3^\circ$, so the velocity has a direction that forms 126.7° from $\hat{n}_x$

Two Rods Static Balance



Two uniform rods AB and CD are placed as are shown in the figure. The vertical wall which B and D are in contact with are smooth, and the horizontal ground which A is in contact with has static coefficient of friction μ_A. The point where AB and CD are in contact has static coefficient of friction μ_C. Both rods have mass m and length l. Suppose AB forms θ with the vertical wall, find the constraint α that CD forms with the wall so that the system is in static balance.