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# Goodness-of-fit test for specification of semiparametric copula dependence models

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ABSTRACT

This paper concerns goodness-of-fit tests for semiparametric copula models. Our contribution is two-fold: we first propose a new test constructed via the comparison between "in-sample" and "out-of-sample" pseudo-likelihoods. Under the null hypothesis that the copula model is correctly specified, we show that the proposed test statistic converges in probability to a constant equal to the dimension of the parameter space. We establish the asymptotic normality and investigate the local power of the test. We also extend the proposed test to the specification test of a class of multivariate time series models, and propose a new bootstrap procedure to establish the finite-sample null distribution, which is shown to have better control of type I error than the commonly used bootstrap. Secondly, we introduce a Bonferroni-based hybrid mechanism to combine several test statistics, which yields a useful test. This hybrid method is particularly appealing when there exists no single dominant optimal test. We conduct comprehensive simulation experiments to compare the proposed new test and hybrid approach with two of the best "blanket" tests in the literature. For illustration, we apply the proposed tests to analyze two real datasets.


## 1. Introduction

Assessing dependency among multiple variables is a primary task in business economics or financial applications. Copula is becoming increasingly popular in such fields due to its flexibility in seamlessly integrating sophisticated dependence structures and varying marginal distributions of multivariate random variables. For example, in finance, copulas are widely applied to study dependency in asset pricing, asset allocation and risk management; see Klugman and Parsa (1999) and Cherubini et al. (2004, 2011), among others. More examples in other fields can be found in Frees and Valdez (1998), Wang and Wells (2000), Song (2007) and Danaher and Smith (2011), just to name a few.

Essentially, a parametric copula is a cumulative distribution function (CDF) specified by a certain known functional form up to some unknown dependence parameters. When a parametric copula is used in applications, misspecification on any of its parametric structure may cause erroneous statistical estimation and inference. To check for the adequacy of a copula model, specification tests have been extensively investigated in the literature. Wang and Wells (2000) proposed a rank based test for bivariate copulas. Malevergne and Sornette (2003) developed a test for the specification of Gaussian copulas. Fermanian (2005) and Scaillet (2007) established goodness-of-fit tests through kernel techniques. Other types of specification tests include Panchenko's (2005) $V$-statistic type test, Prokhorov and Schmidt's (2009) conditional moment based test, Mesfioui et al.'s (2009) Spearman dependence based test, and Genest et al.'s (2011) Pickands dependence based test. Very recently, Huang and Prokhorov (2014) adopted White's test based on the information matrix test (White, 1982) to derive a test for copula model specification. With the utility of either Kendall's or Rosenblat's probability integral transformations, several other versions of specification tests have been proposed in the literature, including those proposed by Breymann et al. (2003), Dobrić and Schmid (2007) and Genest and Favre (2007), among others.

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In a recent paper, Genest et al. (2009) made a thorough comparison for most of the existing "blanket tests". A blanket test refers to a test whose implementation does not require either an arbitrary categorization of data or any strategic choice of smoothing parameter, weight function, kernel or bandwidth. It is demonstrated by Genest et al. (2009) that none of these blanket tests perform uniformly the best. It is interesting to note that almost all of them had illustrated nearly no power in differentiating Gaussian copulas from Student's $t$ copulas, both of which are very important symmetric copulas with different tail dependence properties. Another challenge in the use of the blanket tests considered in Genest et al. (2009) is that they rely on certain probability integral transformations, which may be difficult to derive analytically in many popular copula dependence models, e.g. Student's $t$ copulas and vine copulas (e.g. Kurowicka and Joe, 2011).

To overcome the difficulties above, we propose an alternative specification test for semiparametric copulas in this paper. The proposed test statistic takes a form of ratio constructed via two types of pseudo-likelihoods: one is "in-sample" pseudo-likelihood and the other is "out-of-sample" pseudo-likelihood. The idea behind the construction of the new test is rooted in the fact that, heuristically, a goodness-of-fit test is to examine how a model fits the data. Thus, we vary data by the means of jackknife and quantify how sensitive the pseudo likelihood is to the varying data. Naturally, a comparison of pseudo likelihoods over different datasets are utilized to characterize how well the model fits the data. Inspired by Presnell and Boos's (2004) likelihood based in-and-out-of-sample test, we term our proposed test as the pseudo in-and-out-of-sample (PIOS) test. In comparison to the blanket tests in Genest et al. (2009), which are all indeed rank-based tests, our PIOS test is a pseudo likelihood based test, which does not require any probability integral transformation. Thus, as demonstrated later in the paper, the PIOS test is computationally simple and numerically stable.

Under the null hypothesis of the assumed copula model being correctly specified, we show that under some mild regularity conditions, the PIOS test statistic converges in probability to a constant equal to the dimension of the parameter space of the null copula model. Also, we establish both consistency and asymptotic normality for the PIOS test statistic. Compared to the fully parametric in-and-out-of-sample test proposed by Presnell and Boos (2004), our work makes the following new contributions. First, the PIOS test is applicable to a semiparametric copula model in which the marginal CDFs may be fully unspecified. Secondly, Presnell and Boos's (2004) test is based on a single point data in-and-out-of-sample procedure. As a useful extension, the PIOS test is based on a data block in-sample and out-of-sample procedure, where the size of block is allowed to increase with the sample size. Such flexibility is useful to extend the original method to serially dependent time series data. Thirdly, the development of asymptotic properties of the PIOS test involves the use of the theory of empirical processes with varying block size, and therefore such theoretical work is new and fundamentally different from that established in Presnell and Boos (2004). Fourthly, we develop the asymptotic local power theory in the Pitman sense. Finally, the PIOS test is extended to the case of semi-parametric copula based multivariate dynamic (SCOMDY) model. However, the commonly used bootstrap procedure (Chen and Fan, 2006), based on resampling from estimated innovation processes, may fail to attain the nominal test sizes. We propose a new bootstrap procedure, which involves resampling from the time series data and re-estimating the dynamic parameters of the SCOMDY model in each bootstrap path. The simulation studies have shown that our proposed bootstrap would better control type I error due to accounting for uncertainty in estimating the dynamic parameters.

Another primary focus of the paper is the adoption of Bonferroni correction in combining several test statistics and the resulting test is termed as the hybrid test in this paper. As demonstrated in Genest et al. (2009), there exists no single dominant asymptotically optimal test against general alternatives; see also Freedman (2009). The hybrid test offers a compromise of several different tests, which is particularly appealing when there is no *a priori* knowledge about the top performer in the hypothesis test. We show that the hybrid test can control type I error, as long as each of them does, and that it will be a consistent test as long as there exists one consistent test among the involved tests, regardless of the performance of the remaining tests. The basic setup for the hybrid test is different from that for multiple testing. The difference between these two settings is rooted in the number of null hypotheses involved in the analysis. In our case of hybrid test, there is only one null hypothesis versus one alternative hypothesis, to which several different test statistics (e.g. $S_n, J_n, R_n, T_n$ defined in the following sections) are applied on the same data, so that the test statistics are intrinsically correlated and thus Bonferroni procedure is deemed to control the size of hybrid test. On contrary, in the case of multiple testing, many different null hypotheses are considered and tested simultaneously for whether or not all these null hypotheses hold together, in which only one test statistic is used repeatedly in each hypothesis; see an example of goodness-of-fit test proposed by Hofert and Mächler (2013). Although our setting appears to be different from the multiple testing, the method of Bonferroni procedure is applicable to the hybrid test for the type I error control. Our simulation studies clearly illustrate that, in general, the proposed hybrid test enjoys the desirable finite sample performance.

This paper is organized as follows. Section 2 is devoted to the details for the construction of the PIOS test. Section 3 discusses the hybrid test. Section 4 presents the large sample properties of the proposed PIOS test statistic. Section 5 presents an extension of the PIOS test to multivariate time series data. Section 6 concerns Monte Carlo simulation studies to evaluate finite sample performances of the proposed PIOS test and hybrid test. In Section 7, the proposed tests are applied to two real datasets. The final section provides some concluding remarks. All technical details are included in the appendices.

## 2. Pseudo in-and-out-of-sample (PIOS) test

Suppose that $X_1 = (X_{11}, \ldots, X_{1d})^T, \ldots, X_n = (X_{n1}, \ldots, X_{nd})^T$ is a random sample of size $n$ drawn from a multivariate distribution $H(x) = H(x_1, x_2, \ldots, x_d)$ with continuous marginal CDF $F(x) \triangleq \{F_1(x_1), \ldots, F_d(x_d)\}$. According to Sklar's theorem (Sklar, 1959), we suppose that the joint distribution $H(\cdot)$ can be expressed by the following representation:

$$H(x_1, x_2, \ldots, x_d) \triangleq C_0\{F(x)\} = C_0\{F_1(x_1), \ldots, F_d(x_d)\},$$

where $C_0(\cdot)$ is the true copula function. The corresponding joint density function of $H(\cdot)$, denoted by $h(\cdot)$, takes the form of

$$h(x_1, x_2, \ldots, x_d) = c_0\{F_1(x_1), \ldots, F_d(x_d)\} \prod_{k=1}^{d} f_k(x_k),$$

where, $c_0(u)$, $u = (u_1, \ldots, u_d) \in (0, 1)^d$ is the resulting copula density function of copula $C_0(\cdot)$ and $f_k(\cdot)$ are the corresponding marginal density functions of $F_k(\cdot)$, $k = 1, \ldots, d$. Throughout this paper, the marginal CDF $F(\cdot)$ is not specified by any parametric forms.

In practice, we often assume that the underlying true copula $C_0$ belongs to a parametric class, *say*,

$$\mathcal{C} \triangleq \{C(\cdot; \theta), \theta \in \Theta\},$$

where $\Theta \subset \mathcal{R}^p$ is a $p$-dimensional parameter space. It is well known that misspecification on any of its parametric structure of $C(\cdot;\theta)$ may ruin likelihood based statistical estimation and inference. Hence, checking the model specification is an important task in model diagnosis. In the following, we are interested in the development of a goodness-of-fit test on the hypotheses

$$\mathcal{H}_0 : C_0 \in \mathfrak{C} = \{C(\cdot;\theta) : \theta \in \Theta\} \quad vs.$$
$$\mathcal{H}_1 : C_0 \notin \mathfrak{C} = \{C(\cdot;\theta) : \theta \in \Theta\}.$$

To begin, we first apply the so-called two-step pseudo maximum likelihood estimation (PMLE) method (e.g. Oakes (1994), Genest et al. (1995), Shih and Louis (1995) and Chen and Fan (2005)) to estimate the dependence parameter $\theta$. In order to avoid the estimated copula function from blowing up at the boundary of 0 or 1, let $\tilde{F}(x) = \{\tilde{F}_1(x_1), \ldots, \tilde{F}_d(x_d)\}$ be the set of rescaled empirical marginal distributions, where the $k$th component is given by

$$\tilde{F}_k(x_k) = \frac{1}{n+1}\sum_{t=1}^{n} I(X_{tk} \le x_k), \quad k = 1, \ldots, d, \tag{1}$$

where $I(\cdot)$ is the indicator function. Let $l\{\tilde{F}(X_t);\theta\} = \log c\{\tilde{F}_1(X_{t1}), \ldots, \tilde{F}_d(X_{td});\theta\}$, and let $\hat{\theta}$ be the two-step PMLE of $\theta$ given by

$$\hat{\theta} = \arg\max_{\theta\in\Theta} \sum_{t=1}^{n} l\left\{\tilde{F}(X_t);\theta\right\}. \tag{2}$$

Genest et al. (1995) investigated large sample properties of the above PMLE (2) under the assumption of the copula function being correctly specified. Chen and Fan (2005) and Chen and Fan (2006) have investigated the asymptotic properties of this estimator $\hat{\theta}$ under a misspecified model and proposed a model selection approach based on a pseudo likelihood ratio test for possibly misspecified copulas for i.i.d. data and time series data, respectively. In this paper, we begin with our development first for the i.i.d. data and later extend it to time series data.

To present our new test, let us randomly divide the original i.i.d. data, $\{X_1, \ldots, X_n\}$, into $B$ blocks and denote the $b$th block with block size $n_b$ as $X^b = (X^b_1, \ldots, X^b_{n_b})$, $b = 1, \ldots, B$. Without loss of generality, suppose $X^b_i = X_{n_1+\cdots+n_{b-1}+i}$ and the $k$th element of $X^b_i$ is denoted by $X^b_{ik}$, $k = 1, \ldots, d$, $i = 1, \ldots, n_b$, and $n_1 + \cdots + n_B = n$. For the simplicity of exposition, we assume that all the blocks have an equal size, *say*, $n_b \equiv m$, and hence $mB = n$. With little technical effort, all arguments presented in the rest of this paper can be easily extended to the case of unequal block sizes. In a similar spirit to the "jackknife" resampling method (e.g. Efron, 1982), we can yield a set of delete-one-block PLMEs $\hat{\theta}_{-b}$, $1 \le b \le B$, according to the following procedure:

$$\hat{\theta}_{-b} = \arg\max_{\theta\in\Theta} \sum_{b'\ne b}^{B}\sum_{i=1}^{m} l\{\tilde{F}(X^{b'}_i);\theta\}, \quad b = 1, \ldots, B. \tag{3}$$

Note that the delete-one-block pseudo likelihood $\prod_{i=1}^m l\{\tilde{F}(X^b_i);\hat{\theta}_{-b}\}$ measures how well the hypothesized model predicts the $b$th block of observations, $X^b = (X^b_1, \ldots, X^b_m)$. If the full pseudo likelihood (a.k.a. the in-sample likelihood), $\prod_{i=1}^m l\{\tilde{F}(X^b_i);\hat{\theta}\}$, appears to be much larger than the out-of-sample counterpart, $\prod_{i=1}^m l\{\tilde{F}(X^b_i);\hat{\theta}_{-b}\}$, then the fitted model is very sensitive to the deleted $b$th block of observations, implying that the hypothesized model may be inadequate to fit the data. Thus, we can establish a global measure for goodness-of-fit using a comparison between the "in-sample" pseudo-likelihood and the "out-of-sample" pseudo-likelihood. Precisely, we propose a test statistic of the following form:

$$T_n(m) \triangleq \sum_{b=1}^{B}\sum_{i=1}^{m}\left[l\{\tilde{F}(X^b_i);\hat{\theta}\} - l\{\tilde{F}(X^b_i);\hat{\theta}_{-b}\}\right]. \tag{4}$$

The resulting test is termed as the pseudo in-and-out-of-sample (PIOS) test. It is worth pointing out that, when the margins are known and the block size is fixed at $m \equiv 1$, $T_n(m)$ in (4) reduces to the IOS test statistic proposed by Presnell and Boos (2004).

Under the null hypothesis of correct model specification, the statistic $T_n(m)$ in (4) is shown to converge in probability to $p$, the dimension of the parameter vector $\theta$. Here, we present in a heuristic argument as to why its limiting value is $p$. First, we define two types of Fisher information matrices (Song, 2007, Chapter 3), negative sensitivity matrix and variability matrix as follows:

$$S(\theta) \triangleq -\mathbb{E}_0\left[l_{\theta\theta}\{F(X_1);\theta\}\right],$$
$$V(\theta) \triangleq \mathbb{E}_0\left[l_\theta\{F(X_1);\theta\}l^T_\theta\{F(X_1);\theta\}\right],$$

where $l_\theta(u;\theta) = \frac{\partial}{\partial\theta}\log c(u;\theta)$, $l_{\theta\theta}(u;\theta) = \frac{\partial^2}{\partial\theta\partial\theta^T}\log c(u;\theta)$, and $\mathbb{E}_0(\cdot)$ represents the expectation under the true copula $C_0$. Throughout this paper, we assume we work on hypothetical models and, there exists a parameter value $\theta^* \in \Theta$ such that $\hat{\theta} \to \theta^*$ in probability and satisfies the central limit theorem under some regularity conditions. Refer to Chen and Fan (2005) for the regularity conditions required to establish such large sample properties under misspecified models. The point of interest is that, under suitable regularity conditions given in Theorem 2 in Section 4.2, we can show that

$$T_n(m) \xrightarrow{pr} \mathbb{E}_0\left[l^T_\theta\{F(X_1);\theta^*\}S(\theta^*)^{-1}l_\theta\{F(X_1);\theta^*\}\right]$$
$$= \operatorname{tr}\left\{S(\theta^*)^{-1}V(\theta^*)\right\}, \quad \text{as } n \to \infty,$$

where $\operatorname{tr}(A)$ denotes the trace of a matrix $A$. As a result of the Bartlett's identity (White, 1982), a correct model specification implies $V(\theta^*) = S(\theta^*)$, so $\operatorname{tr}\left\{S(\theta^*)^{-1}V(\theta^*)\right\} = p$, the trace of the $p$-dimensional identity matrix. Given some further conditions, $T_n(m) - p$ is shown to be asymptotically normally distributed, which is the theoretical basis to define our rejection rule for the hypothesis test.

To implement the proposed test statistic $T_n(m)$ in practice we need to estimate parameter $\theta$ in $[n/m]$ (the largest integer less than $n/m$) times, which may be computationally demanding. Indeed, we can approximate $T_n(m)$ by the following test statistic $R_n$, which is shown to be asymptotically equivalent to $T_n(m)$ in Theorem 2(ii):

$$R_n \triangleq \frac{1}{n}\sum_{t=1}^{n} l^T_\theta\{\tilde{F}(X_t);\hat{\theta}\}\hat{S}^{-1}(\hat{\theta})l_\theta\{\tilde{F}(X_t);\hat{\theta}\}$$
$$= \operatorname{tr}\left\{\hat{S}^{-1}(\hat{\theta})\hat{V}(\hat{\theta})\right\}, \tag{5}$$

where $\hat{S}(\hat{\theta})$ and $\hat{V}(\hat{\theta})$ are the sample counterparts of the negative sensitivity matrix and variability matrix, respectively, defined by

$$\hat{S}(\hat{\theta}) = -\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\{\tilde{F}(X_t);\hat{\theta}\},$$

$$\hat{V}(\hat{\theta}) = \frac{1}{n}\sum_{t=1}^{n} l_\theta\{\tilde{F}(X_t);\hat{\theta}\}l^T_\theta\{\tilde{F}(X_t);\hat{\theta}\}.$$

The statistic $R_n$ given in (5) is similar to the information ratio (IR) test statistic proposed by Zhou et al. (2012) for cross-sectional and longitudinal data in the framework of estimating equations, which was later extended to time series data using martingale estimating equations in Zhang et al. (2012).

## 3. Hybrid test

In most scenarios of goodness-of-fit tests, including those developed for copula models (e.g. Genest et al., 2009), there exists no single dominate optimal test. It is often the case that at one occasion, one test is more powerful, but at other occasions, other tests are more powerful. See also Freedman (2009). The same phenomenon also occurs in our simulation studies. At some occasions, PIOS outperforms others, but at other settings, other tests (such as the one proposed by Genest et al., 2009) perform better. Following the method of Bonferroni correction for multiple testing, here we propose the following hybrid test that enables us to combine several different tests to achieve certain compromise in the test power. This approach has been studied in the linear model by Zhou et al. (2015). This strategy is particularly appealing when there is no *a priori* knowledge regarding the top performer at a given occasion.

Consider $q$ test statistics, denoted by $T_n^{(1)}, T_n^{(2)}, \dots, T_n^{(q)}$, where the subscript $n$ is the sample size. Suppose that all of them have type I error controlled at a given significance level $\alpha$ under a common null hypothesis involving the same parameter. In general these test statistics may follow different types of distributions (asymptotical), so it is imperative to convert them into comparable quantities under a common probability measure. The $p$-value is the choice of method used in this paper. Similar to the Bonferroni procedure, a hybrid test is constructed as follows: Let $p_n^{(i)}$ denote the corresponding $p$-value obtained from the test statistic $T_n^{(i)}$, $i = 1, \dots, q$. A hybrid test, denoted by $T_n^{hybrid}$, will make decision according to a $p$-value, defined as

$$p_n^{hybrid} = \min\{q \times \min(p_n^{(1)}, \dots, p_n^{(q)}), 1\}.$$

Consequently, the rejection rule of the hybrid test is that if $p_n^{hybrid} \le \alpha$, the null hypothesis is rejected. This is equivalent to the situation where there is at least one test but not necessary all tests rejecting the null at the level of $\alpha/q$.

Under the null hypothesis $H_0$ and a significance level $\alpha$, we have the type I error for the hybrid test:

$$\begin{aligned} p_r(p_n^{hybrid} \le \alpha | H_0) &= p_r(p_n^{(1)} \le \alpha/q \text{ or } \cdots \text{ or } p_n^{(q)} \le \alpha/q | H_0) \\ &\le \sum_{i=1}^{q} p_r(p_n^{(i)} \le \alpha/q | H_0) \\ &\le \alpha. \end{aligned}$$

The above inequality shows that, provided that all of the test $T_n^{(i)}, i = 1, \dots, q$, have controlled type I errors, the hybrid test $T_n^{hybrid}$ has its type I error controlled at $\alpha$.

Let $\beta_n^{(i)}(\alpha)$ be the power function of test $T_n^{(i)}$ at a given significance level $\alpha$ and sample size $n$, $i = 1, \dots, q$. That is, under the alternative hypothesis $H_A$, $\beta_n^{(i)}(\alpha) = p_r(p_n^{(i)} \le \alpha | H_A)$. The power function of the hybrid test $T_n^{hybrid}$ has the following lower bound:

$$\begin{aligned} \beta_n^{hybrid}(\alpha) &= p_r\left(p_n^{hybrid} \le \alpha | H_A\right) \\ &= p_r\left(p_n^{(1)} \le \frac{\alpha}{q} \text{ or } \cdots \text{ or } p_n^{(q)} \le \frac{\alpha}{q} | H_A\right) \\ &\ge \max\left\{\beta_n^{(1)}\left(\frac{\alpha}{q}\right), \dots, \beta_n^{(q)}\left(\frac{\alpha}{q}\right)\right\}. \end{aligned}$$

The above inequality implies that if there is at least one test that is consistent (namely, the power tends to 1 as the sample size increases to $\infty$), then the hybrid test is consistent. Our simulation studies also show that the hybrid test behave more desirably than any of the individual tests.

## 4. Asymptotic properties of PIOS test

In this section, we establish several asymptotic properties of the proposed PIOS test as well as the relationship between $T_n(m)$ in (4) and $R_n$ in (5). Throughout this paper, we denote $\|x\|$ as the usual Euclidean metric of any vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, namely, $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$ and for any $d \times d$ matrix $A$, $\|A\| = \sqrt{\sum_{i,j=1}^d A_{ij}^2}$, where $A_{ij}$ is the $(i,j)$th element of $A$. Let $\mathcal{N}(\theta^*)$ denote an open neighborhood of $\theta^*$. For simplicity of notations, we denote $l_{\theta,j}(u_1, \dots, u_d; \theta) = \frac{\partial l_\theta(u_1,\dots,u_d;\theta)}{\partial u_j}$ and $l_{\theta\theta,j}(u_1, \dots, u_d; \theta) = \frac{\partial l_{\theta\theta}(u_1,\dots,u_d;\theta)}{\partial u_j}$, $j = 1, \dots, d$.

### 4.1. Law of large numbers theorem

Firstly, we establish the law of large numbers theorem for the test statistics $R_n$. To proceed, we need the following regularity conditions.

(A1) The first-order and second-order derivatives, $l_\theta(u;\theta)$ and $l_{\theta\theta}(u;\theta)$, are continuous with respective to $\theta$ for any $u \in (0,1)^d$; and there exist integrable functions $G_1(u)$ and $G_2(u)$ such that $\|l_\theta(u;\theta) l_\theta^T(u;\theta)\| \le G_1(u)$ and $\|l_{\theta\theta}(u;\theta)\| \le G_2(u)$ for all $\theta \in \mathcal{N}(\theta^*)$.

(A2) Matrix $S(\theta^*) = -\mathbb{E}_0\left[l_{\theta\theta}\{F(X_1;\theta^*)\}\right]$ is finite and nonsingular.

Assumption (A1) is the so-called dominating condition, which is commonly imposed in order to establish the uniform law of large numbers theorem (e.g. Wooldridge (1994)). Assumption (A2) requires the sensitivity matrix $S(\theta^*)$ to be invertible, so that the test statistic $R_n$ in (5) will be well-defined.

**Theorem 1.** *Under conditions* (A1)–(A2), *we have*

$$R_n \xrightarrow{pr} tr\left\{S(\theta^*)^{-1} V(\theta^*)\right\}, \quad \text{as } n \to \infty,$$

where $\theta^*$ is the limiting value of PMLE $\hat\theta$ given in (2).

### 4.2. Central limit theorem

The following regularity conditions are used to establish the central limit theorem for both $R_n$ and $T_n(m)$.

(B1) Denote $J_i(u) = const \times \prod_{k=1}^d \{u_k(1-u_k)\}^{-\xi_{ik}}$, where $\xi_{ik} \ge 0$, $i = 1, 2$, $\xi_{ik}$ are some constants. Suppose that for all $\theta \in \mathcal{N}(\theta^*)$, $\|l_\theta(u;\theta) l_\theta^T(u;\theta)\| \le J_1(u)$, $\|l_{\theta\theta}(u;\theta)\| \le J_2(u)$, and $\mathbb{E}_0\left[J_i^2\{F(X_1)\}\right] < \infty$.

(B2) Suppose that both $l_{\theta,k}(u;\theta)$ and $l_{\theta\theta,k}(u;\theta)$, $k = 1, 2, \dots, d$ exist and are continuous. Denote $\tilde J_i^k(u) = const \times \{u_k(1-u_k)\}^{-\tilde\xi_{ik}} \prod_{j=1, j\ne k}^d \{u_j(1-u_j)\}^{-\xi_{ij}}$, where $\tilde\xi_{ij} > \xi_{ij}$ are some constants, such that for all $\theta \in \mathcal{N}(\theta^*)$, $\|l_{\theta,k}(u;\theta)\| \le \tilde J_1^k(u)$ and $\|l_{\theta\theta,k}(u;\theta)\| \le \tilde J_2^k(u)$, and furthermore, $\mathbb{E}_0\left[\tilde J_i\{F(X_1)\}\right] < \infty$, $i = 1, 2$ and $k = 1, 2, \dots, d$.

(B3) Suppose $\frac{\partial l_{\theta\theta}(u;\theta)}{\partial \theta_k}$, $k = 1, 2, \dots, p$ exist and are continuous with $\theta \in \mathcal{N}(\theta^*)$, and there exists an integrable function $G_3(u)$ such that $\|\frac{\partial l_{\theta\theta}(u;\theta)}{\partial \theta_k}\| \le G_3(u)$ for all $\theta \in \mathcal{N}(\theta^*)$, $k = 1, \dots, d$.

Assumptions (B1) and (B2) are similar to the conditions in Lemma 2 of Chen and Fan (2005). Obviously, assumption (B1) implies assumption (A1). Assumption (B3) is commonly required in the literature to establish the uniform law of large numbers theorem.

(C1) The block size $m$ is of order $o(n^a)$ with $0 \le a \le \frac{1}{4}$.

Assumption (C1) is needed to bound the difference between $R_n$ and $T_n(m)$, so that these two statistics have the same limiting distribution. Under the above regularity conditions, we have the following results.

**Theorem 2.** (i) *Under the null hypothesis, if* (A2) *and* (B1)–(B3) *hold, then we have*

$$\sqrt{n}\,\{R_n - p\} \xrightarrow{d} N(0, \sigma_R^2), \quad \text{as } n \to \infty,$$

*where $\sigma_R^2$ is the asymptotic variance given by Eq.* (16) *in the Appendix A, which can be consistently estimated by Eq.* (17) *in the Appendix A.*

(ii) *Under assumptions* (A2), (B1)–B(3) *and* (C1), *we have*

$$R_n - T_n(m) = o_p(n^{-1/2}).$$

**Remark 1.** One issue in the use of the above PIOS test is how to select block-size $m$ to achieve better performance. Our Monte Carlo simulations show that the choice of $m$ depends on the underlying data generating process, and in most cases the PIOS test with blocksize $m = 1$ behaves satisfactorily for independent cross-sectional data.

### *4.3. Local power of evaluation*

To establish a theory of consistency for the proposed PIOS test, we investigate the asymptotic power of the test statistic $R_n$ against a local alternative in the Pitman sense (Nikitin and Nikitin, 1995), which takes the following mixture form, for a constant $\delta > 0$,

$$\begin{aligned} H_{1,n} : P_n^{C_1,\delta}(x) &= \left(1 - \frac{\delta}{\sqrt{n}}\right) C_0\{F(x); \theta_0\} + \frac{\delta}{\sqrt{n}} C_1\{F(x)\} \\ &= C_0\{F(x); \theta_0\} + \frac{\delta}{\sqrt{n}} \left[C_1\{F(x)\} - C_0\{F(x); \theta_0\}\right], \end{aligned} \tag{6}$$

where both $C_0(\cdot;\cdot)$ and $C_1(\cdot)$ are copulas. To ensure that the mixture in (6) is a copula for $0 < \delta \le n^{1/2}$ and the departure from the null $C_0(F(x); \theta_0)$ increases as $\delta$ increases, we consider the concordance ordering given in Berg and Quessy (2009), namely $C_1\{F(x)\} \ge C_0\{F(x); \theta_0\}$ for all $x \in \mathcal{R}^d$; this suggests that the dependence obtained from $C_1$ is weakly stronger than that obtained from $C_0$ (Nelsen, 2006, page 39).

A size $\alpha$ test is defined as to reject $H_0$ when $R_n$ is larger than a threshold value $\tau_\alpha$ that satisfies $\lim_{n\to\infty} Pr(R_n \ge \tau_\alpha | H_0) \le \alpha$. For such a test, its asymptotic local power function (ALPF) is given by van der Vaart (2000)

$$\Pi(\alpha; C_1, \delta) = \lim_{n\to\infty} Pr\left(R_n \ge \tau_\alpha | H_{1,n}\right).$$

In order to ensure that the probability law induced by mixture $P_n^{C_1,\delta}$ under $H_{1,n}$ is contiguous with respect to that induced by $P_0 \equiv C(\cdot; \theta_0)$ under $H_0$, in the sense that if for any sequence of measurable sets $A_n$, $P_0(A_n) \to 0$ implies $P_n^{C_1,\delta}(A_n) \to 0$ as $n \to \infty$, following van der Vaart and Wellner (1996), we assume the following condition:

(D1) Both the copulas $C_0(\cdot; \theta_0)$ and $C_1(\cdot)$ in (6) are absolutely continuous with respect to square integrable densities $c_0(\cdot; \theta_0)$ and $c_1(\cdot)$. Moreover

$$\int_{u\in[0,1]^d} \left[\sqrt{n}\left\{\sqrt{p_n^{c_1,\delta}(u)} - \sqrt{p_0(u)}\right\} - \frac{1}{2}\delta g(u)\sqrt{p_0(u)}\right]^2 du \to 0, \quad \text{as } n \to \infty, \tag{7}$$

where $p_n^{c_1,\delta}(u) = (1 - \frac{\delta}{\sqrt{n}}) c_0(u; \theta_0) + \frac{\delta}{\sqrt{n}} c_1(u)$, $p_0(u) = c_0(u; \theta_0)$ and $g(u) = \frac{c_1(u) - c_0(u;\theta_0)}{c_0(u;\theta_0)}$.

**Theorem 3.** *Suppose* (D1) *holds in addition to the assumptions* (A2) *and* (B1)–(B3). *Then under $H_{1,n}$, the test statistic $\sqrt{n}\,(R_n - p)$ converges to a normal distribution with mean $\delta m(c_0, c_1)$ and variance $\bar{\sigma}_R^2$ defined by* (22) *in the Appendix A, where*

$$m(c_0, c_1) = \mathbb{E}_{c_0}\left\{W(X_t) g\left(F(X_t); \theta_0\right)\right\},$$

*and $\mathbb{E}_{c_0}(\cdot)$ denotes the expectation under the null distribution $c_0$ or $P_0$, and $W(\cdot)$ is defined in Eq.* (21) *in the Appendix A. That is, $m(c_0, c_1)$ is a weighted expectation of $g\left(F(X_t); \theta_0\right)$ under $P_0$.*

This theorem implies that as long as $m(c_0, c_1) \ne 0$, the proposed test statistic $R_n$ will yield power locally and the asymptotic local power increases to 1 as $\delta$ increases to infinity. This means that $R_n$ is a consistent test under the alternative hypothesis specified by Berg and Quessy's concordance ordering. Using Theorem 2(ii), we can show that $T_n$ has the same asymptotic local power function as $R_n$ and hence $T_n$ is also a consistent test in the Pitman sense under the concordance ordering assumption.

## 5. Extension of PIOS test

In this section, we extend the PIOS test to cases of time series data. Following Chen and Fan (2006), we consider a class of multivariate time series models constructed by a semi-parametric copula of the following form,

$$Y_t = \mu_t(\eta_1^0) + \Sigma_t^{1/2}(\eta^0)\epsilon_t, \tag{8}$$

where $Y_t = (Y_{t1}, \ldots, Y_{td})^T$ is a $d$-dimensional vector, $\mu_t(\eta_1^0) = \left\{\mu_{t1}(\eta_1^0), \ldots, \mu_{td}(\eta_1^0)\right\}^T = \mathbb{E}(Y_t|\mathcal{F}_{t-1})$ is the true conditional mean of $Y_t$ given the filtration $\mathcal{F}_{t-1}$ up to time $t-1$, where $\mathcal{F}_t$ is the sigma-field generated by $\{Y_{t-1}, Y_{t-2}, \ldots; Z_t, Z_{t-1}, \ldots\}$, and $Z_t$ is a vector of predetermined or exogenous variables. Assume $\mu_t(\eta_1^0)$ is parametrized by a finite-dimensional unknown parameter $\eta_1^0 \in \Psi_1 \subset \mathcal{R}^{p_1}$, and $\Sigma_t(\eta^0) = diag\left\{\Sigma_{t1}(\eta^0), \ldots, \Sigma_{td}(\eta^0)\right\}$, where $\Sigma_{tj}(\eta^0) = \mathbb{E}\left[\left\{Y_{tj} - \mu_{tj}(\eta_1^0)\right\}^2 |\mathcal{F}_{t-1}\right]$, $j = 1, \ldots, d$, are the conditional variances of $Y_{tj}$ given $\mathcal{F}_{t-1}$, and parametrized by a finite-dimensional unknown parameter $\eta^0 = \left\{(\eta_1^0)^T, (\eta_2^0)^T\right\}^T$, with $\eta_1^0$ and $\eta_2^0$ being exclusive, $\eta_2^0 \in \Psi_2 \subset \mathcal{R}^{p_2}$. The innovation process $\epsilon_t = (\epsilon_{t1}, \ldots, \epsilon_{td})^T$, $t = 1, \ldots, n$ are i.i.d. $d$-dimensional vectors with zero mean and unity variance according to a joint CDF $H(\epsilon) = C_0\{F(\epsilon)\} = C_0\{F_1(\epsilon_1), \ldots, F_d(\epsilon_d)\}$, where $F_j(\cdot)$ is the true but unknown continuous marginal CDF of $\epsilon_j$, $j = 1, \ldots, d$, and $C_0$ is the true but unknown copula function. Here we assume $C_0$ belongs to a certain parametric class of copulas, say, $\mathcal{C} \triangleq \{C(\cdot; \theta), \theta \in \Theta\}$, where $\Theta \subset \mathcal{R}^p$ is a compact $p$-dimensional parameter space. Being the same in the previous sections, $f_j(\cdot)$ and $c(\cdot; \theta)$ denote the density function of $F_j(\cdot)$ and $C(\cdot; \theta)$, respectively, $j = 1, \ldots, d$.

Model (8) is called by Chen and Fan (2006) as the semi-parametric copula based multivariate dynamic (SCOMDY) model, which is flexible to capture a wide range of serial and contemporaneous dependence patterns as well as marginal behaviors of a multivariate time series. For example, vector autoregression models (VAR), multivariate ARMA models and multivariate GARCH models are special cases of SCOMDY models.

We adopt a three-stage procedure proposed by Chen and Fan (2006) to estimate the SCOMDY model parameters, and then we use residuals to construct the PIOS test statistics to test the specification of a parametric copula. The estimation steps are given as follows. First, we use the univariate quasi-maximum likelihood to estimate the dynamic parameters $\eta = \left(\eta_1^T, \eta_2^T\right)^T$ under the assumption of normality of the standardized innovations $\epsilon_{tj}, j = 1, \ldots, d$ and $t = 1, \ldots, n$. That is,

$$\hat{\eta}_1 = \arg\min_{\eta_1 \in \Psi_1} \left[ \frac{1}{n} \sum_{t=1}^{n} \{Y_t - \mu_t(\eta_1)\}^T \{Y_t - \mu_t(\eta_1)\} \right], \tag{9}$$

and

$$\hat{\eta}_2 = \arg\min_{\eta_2 \in \Psi_2} \left[ \frac{1}{n} \sum_{t=1}^{n} \sum_{j=1}^{d} \left\{ \Sigma_{tj}^{-1}(\hat{\eta}_1, \eta_2) \left(Y_t - \mu_t(\hat{\eta}_1)\right)^2 + \log \Sigma_{tj}(\hat{\eta}_1, \eta_2) \right\} \right]. \tag{10}$$

Secondly, given the consistent estimator $\hat{\eta} = (\hat{\eta}_1^T, \hat{\eta}_2^T)^T$, we obtain standardized residuals $\tilde{\epsilon}_{tj} = \Sigma_{tj}^{-1/2}(\hat{\eta}) \left\{y_{tj} - \mu_{tj}(\hat{\eta}_1)\right\}, j = 1, \ldots, d$ and $t = 1, \ldots, n$. We estimate the marginal distribution $F_j(\cdot)$ using the rescaled empirical distribution of the standardized residuals $\tilde{\epsilon}_{tj}$, by

$$\check{F}_j(x) = \frac{1}{n+1} \sum_{t=1}^{n} I\left\{\tilde{\epsilon}_{tj} \le x\right\}, \quad x \in \mathcal{R}, j = 1, \ldots, d. \tag{11}$$

Finally, the dependent parameter $\theta$ of a given copula is estimated by maximizing the corresponding pseudo log-likelihood function, i.e., $\hat{\theta} = \arg\max_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} l\{\check{F}(\tilde{\epsilon}_t); \theta\}$. The PIOS test statistics $T_n$ and $R_n$ are obtained by replacing $\tilde{F}(X_i^b)$ in (4) and (5) for i.i.d data with $\check{F}(\tilde{\epsilon}_t)$. For convenience, the resulting statistics are denoted by $\tilde{T}_n(m)$ and $\tilde{R}_n$ at the presentation of their large-sample theory in this section and the presentation of their proofs in the Appendix.

To establish the large-sample properties of $\tilde{T}_n(m)$ and $\tilde{R}_n$ for time series, we need the following conditions:

(E1) $\left\{\left(Y_t^T, Z_t^T\right), t = 1, \ldots, n\right\}$ is stationary $\beta$-mixing with a serial decay rate of order $O(t^{-\xi/(\xi-1)})$ for some $\xi > 1$;

(E2) $\hat{\eta}$ is a root-$n$ consistent estimator of $\eta_0$;

(E3) for all $t \ge 1$ and $j = 1, \ldots, d$, $\epsilon_{tj} = \Sigma_{tj}^{-1/2}(\eta^0) \left\{Y_{tj} - \mu_{tj}(\eta_1^0)\right\}$ is continuously differentiable in the neighborhood of $\eta^0$, and $\omega_1 = \mathbb{E}_0 \left\{\Sigma_{tj}^{-1/2}(\eta^0) \dot{\mu}_{tj}(\eta_1^0)\right\} < \infty$ and $\omega_2 = \mathbb{E}_0 \left\{\Sigma_{tj}^{-1}(\eta^0) \dot{\Sigma}_{tj}(\eta^0)\right\} < \infty$, where $\dot{\mu}_{tj}(\eta_1^0) = \frac{\partial \mu_{tj}(\eta_1^0)}{\partial \eta_1}$ and $\dot{\Sigma}_{tj}(\eta^0) = \frac{\partial \Sigma_{tj}(\eta^0)}{\partial \eta}$.

The conditions (E1)–(E3) are similar to the conditions (D1)–(D3) in Chen and Fan (2006). Following Chen and Fan (2006), we have $\tilde{F}_j(x) = \frac{1}{n+1} \sum_{t=1}^{n} I\left\{\epsilon_{tj} \le x\right\}, j = 1, \ldots, d$, and

$$\check{F}_j(x) - \tilde{F}_j(x) = f_j(x) \omega(x)^T \left(\hat{\eta} - \eta_0\right) + o_p(n^{-1/2}), \tag{12}$$

where $\omega(x) = \omega_1 + \frac{x}{2}\omega_2$, and Eq. (12) holds uniformly over $x \in \mathcal{R}^d$.

In addition, we assume, that $\hat{\theta}$ satisfies the following condition (Chen and Fan, 2006):

(E4) The PMLE $\hat{\theta}$ has the following asymptotic expansion

$$\hat{\theta} - \theta^* = \frac{1}{n} \sum_{t=1}^{n} \phi_\theta \left(U_t; \theta^*\right) + o_p(n^{-1/2}), \tag{13}$$

where $U_t = (U_{t1}, \ldots, U_{td})^T, U_{tj} = F_j(\epsilon_{tj}), j = 1, \ldots, d, t = 1, \ldots, n$, and

$$\phi_\theta \left\{U_t; \theta^*\right\} = S(\theta^*)^{-1} \left( l_\theta \left\{U_t; \theta^*\right\} + \sum_{j=1}^{d} \mathbb{E}_0 \left[ l_{\theta,j}\left(U_s; \theta^*\right) \times \left\{I(U_{tj} \le U_{sj}) - U_{sj}\right\} |U_{tj} \right] \right).$$

**Theorem 4.** (i) *Under conditions* (A1)–(A2) *and* (E1)–(E4), *we have*

$$\tilde{R}_n \xrightarrow{pr} tr\left\{S(\theta^*)^{-1} V(\theta^*)\right\}, \quad \text{as } n \to \infty.$$

(ii) *Under the null hypothesis, if* (A2), (B1) − (B3) *and conditions* (E1)–(E4) *hold, we have*

$$\sqrt{n}\left(\tilde{R}_n - p\right) \xrightarrow{d} N(0, \tilde{\sigma}_R^2), \quad \text{as } n \to \infty,$$

*where $\tilde{\sigma}_R^2$ is the asymptotic variance defined in Eq.* (25) *in the* Appendix A.

(iii) *Under assumptions* (A2), (B1)–(B3), (C1) *and* (E1)–(E4), *we have*

$$\tilde{R}_n - \tilde{T}_n(m) = o_p(n^{-1/2}).$$

The proof of Theorem 4 is given in the Appendix A.

## 6. Simulation study

This section presents several simulation studies to demonstrate the finite-sample behaviors of the proposed tests $T_n(m)$ in (4) and $R_n$ in (5) in terms of type I error control, global power under fixed alternatives and local power under drift alternatives.

We consider both cross-sectional data and multivariate time series data. To implement our tests, the asymptotic variance $\sigma_R^2$ needs to be estimated, which is very challenging analytically. For cross-sectional data, semi-parametric bootstrap procedure was used to numerically establish the null distribution of $R_n$. Similar bootstrap approaches were considered in Genest et al. (2009) and Scaillet (2007). For multivariate time series data, the bootstrap procedures considered in the literature such as Chen and Fan (2006) is to resample the estimated innovations $\tilde{\epsilon}_t$ (see Section 5). The justification of this procedure is rooted in the fact that the estimation error of $\hat{\eta}$ has ignorable impact asymptotically on the estimation of the dependence parameter of copula, and hence $\tilde{\epsilon}_t$ would be regarded as if observed. We call it a residual-based bootstrap procedure in this paper. Even though ignoring the estimation error of the dynamic parameters $\eta$ is asymptotically valid according to Theorem 4(ii) (see also Chen and Fan (2006)), in the finite-sample situation, the bootstrap method subject to the estimation errors could lead to poor approximations to the true null distribution of the test statistic, and therefore results in inadequate control of type I error. This issue is shown in Table 4 in Section 6.3. To address this challenge, we proposed an alternative bootstrap procedure, termed as the observation-based bootstrap for convenience, where bootstrap resampling of time series data is undertaken directly from the assumed SCOMDY time series model (8).

### 6.1. Two types of bootstrap procedures

We now describe the two types of bootstrap procedures, observation-based bootstrap and residual-based bootstrap. For the ease of exposition, in this section, we use a generic notation $T_n(m)$ and $R_n$ to represent the PIOS test statistics in either the case of i.i.d. data or in the case of time-series data, whichever case is applicable. The residual-based bootstrap proceeds as follows:

Step 1 Generate a bootstrap sample $\left\{\epsilon_t^{(k)}, t = 1, \ldots, n\right\}$ from the copula $C(u; \hat{\theta})$ under the null hypothesis with the PMLE $\hat{\theta}$ and the estimated marginal distribution $\check{F}$ in (11) obtained from the original data;

Step 2 Based on $\left\{\epsilon_t^{(k)}, t = 1, \ldots, n\right\}$ from Step 1, estimate the dependence parameter $\theta$ of the copula under the null hypothesis by the two-step PMLE method, and compute the test statistic $R_n$, denoted by $R_n^k$;

Step 3 Repeat Steps 1–2 $N$ times and obtain $N$ statistics $R_n^{(k)}, k = 1, \ldots, N$;

Step 4 Compute empirical $p$-value as $p_e = \frac{1}{N}\sum_{k=1}^{N} I\left(|R_n^{(k)}| \geq |R_n|\right)$.

The observation-based bootstrap procedure draws time series data from the assumed SCOMDY model (8) and the parameter $\eta$ is estimated with each bootstrap path of the process. Specifically,

Step 1 Generate a time series $\{Y_t^{(k)}, t = 1, \ldots, n\}$ from the SCOMDY model (8) with the parameter values $\hat{\eta}_1$ and $\hat{\eta}_2$ estimated from the original data, and with the innovation process generated from the assumed copula under the null hypothesis with the parameter value $\hat{\theta}$ and marginal distribution $\check{F}$.

Step 2 Based on $\{Y_t^{(k)}, t = 1, \ldots, n\}$, estimate $\hat{\eta}_1^{(k)}$ and $\hat{\eta}_2^{(k)}$ by (9) and (10). Estimate the residuals $\tilde{\varepsilon}_{tj}^{(k)} = \{y_{tj}^{(k)} - \mu_{tj}(\hat{\eta}_1^{(k)})\}/\Sigma_{tj}^{1/2}(\hat{\eta}_2^{(k)}), t = 1, \ldots, n, j = 1, \ldots, d$.

Step 3 Based on the $\{\tilde{\varepsilon}_t^{(k)}, t = 1, \ldots, n\}$, estimate the parameter $\theta$ of the copula under the null hypothesis by the two-step PMLE method and compute the test statistic $R_n^{(k)}$.

Step 4 Repeat Steps 1–3 $N$ times and obtain $N$ statistics $R_n^{(k)}, k = 1, \ldots, N$.

Step 5 Compute empirical $p$-value as $p_e = \frac{1}{N}\sum_{k=1}^{N} I\left(|R_n^{(k)}| \geq |R_n|\right)$.

Note that observation-based bootstrap without Steps 1 and 2 reduces to the residual-based bootstrap. Both of these two types of bootstrap procedures can be also applied for the test $T_n(m)$.

### 6.2. Simulation setup

For the first setting of cross-sectional data, we investigate the performance of $R_n$ and $T_n(m)$ with two block sizes $m = 1, 3$. For the purpose of comparison, we include two tests proposed respectively by Genest et al. (2009) and Scaillet (2007), denoted in short by $S_n$ and $J_n$ tests. The $S_n$ test has been shown to be one of the best performers on average among all the existing "blanket tests". Being another top competitor, the Scaillet (2007)'s test $J_n$ is a kernel-based goodness-of-fit test with fixed smooth parameter. The detailed descriptions of these two tests are provided in the Appendix B. To make a fair comparison we use the same number of bootstrap samples across different tests. All tests have been implemented based on the residual-based bootstrap according to Genest et al. (2009). In addition, whenever it is needed to estimate parameters in a parametric copula, we use the PLME instead of an inversion of the Kendall's $\tau$ proposed by Genest et al. (2009).

We consider several versions of hybrid tests by combining two or three single tests in the comparison. For hybrids of pairs, we include (i) a hybrid of $S_n$ and $R_n$, denoted as $SR_n$; (ii) a hybrid of $S_n$ and $T_n(m)$, denoted as $ST_n(m)$; (iii) a hybrid of $J_n$ and $R_n$, denoted as $JR_n$; (iv) a hybrid of $J_n$ and $T_n(m)$, denoted as $JT_n(m)$. In addition, we consider two hybrids of triplets: (i) a hybrid of $S_n$, $J_n$ and $R_n$, denoted as $SJR_n$; and (ii) a hybrid of $S_n$, $J_n$ and $T_n(m)$, denoted as $SJT_n(m)$. In these hybrid tests $ST_n(m)$, $JT_n(m)$ and $SJT_n(m)$, for the sake of brevity, we fix $m = 1$.

In the second setting of multivariate time series, we also run simulation experiments to evaluate how the proposed PIOS tests $R_n$ and $T_n(1)$ behave in the models discussed in Section 5. Because both $S_n$ and $J_n$ are not established in the literature for the time series data, we did not examine hybrid tests in this setting. In particular, we hope to demonstrate and compare the performances of the two types of bootstrap methods to establish the null distribution. We generate bivariate time series data from the following GARCH(1,1) process of the form:

$$x_{it} = \sigma_{it}\varepsilon_{it}$$
$$\sigma_{it}^2 = \omega + \alpha x_{i,t-1}^2 + \beta\sigma_{i,t-1}^2, \quad i = 1, 2$$

where bivariate innovation vector $(\varepsilon_{1t}, \varepsilon_{2t}) \overset{i.i.d}{\sim} C\{\Phi(\cdot), \Phi(\cdot); \theta\}$ with $\Phi(\cdot)$ being the CDF of the standard normal distribution $N(0, 1)$. We set $\omega = 10^{-6}$, $\alpha = 0.1$ and $\beta = 0.8$ in the simulation experiments.

For both scenarios of cross-sectional data and time series data, we consider four most popular bivariate copula families, namely Gaussian, Student's $t$, Clayton and Gumbel. All of them have been investigated extensively in a vast literature; see for example, Song (2000), Chen and Fan (2005), Cossin and Schellhorn (2007), Song et al. (2009) and Genest et al. (2009), just to name a few. The former two copulas are prominent examples of the elliptical families and the latter two are important Archimedean copulas. In the whole simulation study all these copula models have just one dependence parameter $\theta$ to estimate. For the $t$ copula we fix the number of degrees of freedom at $\nu = 4$, following Genest et al. (2009). Thus, Gaussian and $t$ copulas are not nested in our settings.

To investigate the impact of dependence strength on the finite performance of the tests, we set three values of dependence parameters in terms of Kendall's tau, $\tau = 0.25, 0.50$ and $0.75$, for both scenarios of cross-sectional data and multivariate time series. We set two sample sizes $n = 100$ and $300$ for the case of cross-sectional data, and one sample size $n = 300$ for the case of multivariate time series data. In each experiment, we conduct $M = 1000$ rounds of simulations, in which, $N = 1000$ bootstrap sample paths are generated for each simulation replication to yield the null distribution. All numerical calculations in the designed simulation study have been undertaken simultaneously on the high-performance servers of the Humboldt University in Berlin and the computing clusters of Simon Fraser University in Vancouver over a period of three months.

### 6.3. Simulation results

#### 6.3.1. Type I error control

Tables 1–3 report the empirical type I errors in the scenarios of cross-sectional data at nominal level 5% for all four copulas being true under $H_0$ hypothesis. The three tables are evidential that the proposed tests $R_n$, $T_n(m)$ and all the hybrid tests $SR_n$, $ST_n(m)$, $JR_n$, $JT_n(m)$, $SJR_n$, $SJT_n(m)$ perform well on type I error control. The empirical type I error rates are marked with bold font for all cases and are located on the top of each panel of the tables. Regardless of the choice of the sample size, the choice of the dependence strength or the choice of the copula family, the type I error is satisfactorily controlled at the level close to the nominal level. In this aspect, our new tests $R_n$ and $T_n(m)$ are clearly comparable to the existing $S_n$ and $J_n$ tests.

In the scenarios of multivariate time series data, we implement both residual-based bootstrap and observation-based bootstrap procedures. Table 4 reports the empirical type I errors obtained by these two types of bootstrap methods respectively. It is clear that the type I error is not well controlled by the residual-based bootstrap procedure, especially in the setting with strong dependency, $\tau = 0.75$. For example, under the null hypothesis of the Clayton copula with $\tau = 0.75$, the empirical type I errors are 0.07 for both $R_n$ and $T_n(1)$, which is much higher than the nominal level 0.05. It is worth noting that increasing sample size or the number of bootstrap samples does not alleviate the problem of inflated type I error based on some additional simulation experiments that are not shown due to the space limitations. The results clearly demonstrate that for cross-sectional multivariate data, the residual-based bootstrap procedure works adequately, whereas for multivariate time series data, the residual-based bootstrap could perform poorly if the uncertainly of estimation for the model parameters is not accounted for in the resampling scheme. In contrast, when such uncertainty is incorporated in the bootstrap procedure, the nominal type I error is satisfactorily attained in all the settings.

**Table 1**
Percentage of rejection of $H_0$ by various tests with cross-sectional data of sizes $n = 100$ and $n = 300$ from different copula models with $\tau = 0.25$. The number of bootstrap samples $M = 1000$ and the number of replicates $N = 1000$.

| | True | $H_0$ | $S_n$ | $J_n$ | $R_n$ | $T_n(1)$ | $T_n(3)$ | $SR_n$ | $ST_n(1)$ | $JR_n$ | $JT_n(1)$ | $SJR_n$ | $SJT_n(1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Ga. | Ga. | **4.5** | **4.2** | **6.1** | **6.0** | **5.1** | **5.6** | **5.9** | **4.2** | **4.6** | **5.0** | **4.8** |
| | $t$ | $t$ | **5.2** | **4.0** | **4.1** | **5.2** | **3.7** | **3.9** | **4.5** | **4.2** | **3.4** | **3.0** | **3.0** |
| | Cl. | Cl. | **4.9** | **4.2** | **4.5** | **4.0** | **4.7** | **4.1** | **3.8** | **3.8** | **4.0** | **5.2** | **5.6** |
| | Gu. | Gu. | **4.7** | **5.0** | **5.8** | **5.4** | **4.6** | **4.5** | **4.7** | **4.0** | **3.8** | **3.8** | **3.8** |
| | Ga. | $t$ | 2.9 | 9.0 | 28.9 | 5.3 | 1.5 | 18.1 | 2.5 | 21.2 | 7.2 | 16.8 | 5.4 |
| | Ga. | Cl. | 14.7 | 11.6 | 5.4 | 4.3 | 4.8 | 10.3 | 10.0 | 10.0 | 9.4 | 12.8 | 12.2 |
| | Ga. | Gu. | 4.3 | 8.0 | 4.4 | 2.2 | 3.3 | 3.7 | 2.6 | 6.4 | 4.8 | 5.0 | 4.4 |
| $n = 100$ | $t$ | Ga. | 11.1 | 9.4 | 55.4 | 54.9 | 37.8 | 49.8 | 48.9 | 49.2 | 50.0 | 44.6 | 46.6 |
| | $t$ | Cl. | 22.2 | 16.4 | 30.4 | 30.4 | 20.5 | 29.4 | 27.3 | 30.8 | 29.8 | 31.4 | 26.8 |
| | $t$ | Gu. | 7.8 | 7.4 | 29.9 | 31.7 | 18.2 | 24.6 | 24.2 | 25.0 | 23.8 | 22.4 | 21.2 |
| | Cl. | Ga. | 22.9 | 11.6 | 10.5 | 9.3 | 9.1 | 18.3 | 18.1 | 14.6 | 13.4 | 18.8 | 18.4 |
| | Cl. | $t$ | 7.4 | 14.0 | 12.6 | 1.5 | 1.2 | 10.8 | 3.8 | 12.8 | 6.8 | 10.0 | 6.2 |
| | Cl. | Gu. | 34.1 | 25.8 | 16.9 | 15.1 | 10.7 | 31.7 | 29.7 | 25.8 | 23 | 35.6 | 34.4 |
| | Gu. | Ga. | 6.4 | 7.0 | 15.0 | 14.2 | 12.3 | 12.9 | 13.0 | 14.6 | 15.6 | 12.6 | 13.4 |
| | Gu. | $t$ | 8.6 | 9.4 | 10.8 | 0.9 | 1.5 | 8.6 | 4.1 | 9.0 | 5.2 | 9.2 | 7.2 |
| | Gu. | Cl. | 36.8 | 29.6 | 11 | 9.0 | 7.6 | 30.0 | 29.1 | 25.8 | 24.8 | 32.2 | 32.2 |
| | Ga. | Ga. | **5.1** | **6.8** | **5.1** | **5.4** | **5.0** | **4.7** | **5.1** | **7.3** | **7.5** | **7.0** | **6.9** |
| | $t$ | $t$ | **4.5** | **5.7** | **5.0** | **6.0** | **5.1** | **5.5** | **5.1** | **4.9** | **5.7** | **5.1** | **4.5** |
| | Cl. | Cl. | **4.5** | **4.6** | **5.8** | **6.2** | **4.2** | **4.9** | **5.3** | **4.3** | **4.7** | **4.6** | **4.4** |
| | Gu. | Gu. | **6.7** | **4.0** | **5.3** | **5.5** | **4.1** | **5.3** | **5.5** | **3.3** | **3.2** | **3.7** | **3.7** |
| | Ga. | $t$ | 9.7 | 16.9 | 80.8 | 69.8 | 21.2 | 70.3 | 51.6 | 71.6 | 53.6 | 64.3 | 55.3 |
| | Ga. | Cl. | 41.7 | 31.1 | 5.8 | 5.6 | 5.9 | 30.9 | 30.8 | 24.5 | 24.4 | 40.0 | 40.1 |
| | Ga. | Gu. | 19.8 | 15.3 | 6.0 | 4.7 | 3.8 | 15.7 | 14.9 | 12.5 | 11.3 | 17.6 | 17.1 |
| $n = 300$ | $t$ | Ga. | 21.5 | 21.1 | 90.9 | 92.9 | 77.3 | 87.8 | 89.9 | 87.7 | 89.5 | 86.0 | 87.5 |
| | $t$ | Cl. | 63.6 | 44.4 | 61.9 | 64.9 | 44.2 | 71.8 | 72.9 | 63.7 | 64.8 | 72.9 | 73.2 |
| | $t$ | Gu. | 22.2 | 17.6 | 64.5 | 67.0 | 40.9 | 61.3 | 63.4 | 59.1 | 60.1 | 57.9 | 59.1 |
| | Cl. | Ga. | 61.5 | 26.5 | 24.7 | 25.7 | 16.1 | 54.6 | 55.1 | 30.2 | 30.6 | 55.4 | 56.1 |
| | Cl. | $t$ | 35.9 | 35.3 | 40.9 | 22.8 | 9.0 | 46.0 | 32.2 | 45.8 | 34.1 | 49.2 | 38.3 |
| | Cl. | Gu. | 96.3 | 70.2 | 50.3 | 53.2 | 29.0 | 94.6 | 94.6 | 72.4 | 73.4 | 94.9 | 94.9 |
| | Gu. | Ga. | 10.3 | 13.7 | 32.5 | 33.7 | 20.2 | 30.5 | 30.2 | 27.5 | 27.6 | 27.1 | 28.0 |
| | Gu. | $t$ | 22.6 | 14.0 | 31.3 | 16.4 | 5.3 | 30.3 | 18.7 | 28.1 | 14.7 | 30.9 | 22.1 |
| | Gu. | Cl. | 92.5 | 69.6 | 18.8 | 20.8 | 13.6 | 87.0 | 87.0 | 62.7 | 62.8 | 91.5 | 91.5 |

**Table 2**
Percentage of rejection of $H_0$ by various tests with cross-sectional data of sizes $n = 100$ and $n = 300$ from different copula models with $\tau = 0.50$. The number of bootstrap samples $M = 1000$ and the number of replicates $N = 1000$.

| | True | $H_0$ | $S_n$ | $J_n$ | $R_n$ | $T_n(1)$ | $T_n(3)$ | $SR_n$ | $ST_n(1)$ | $JR_n$ | $JT_n(1)$ | $SJR_n$ | $SJT_n(1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Ga. | Ga. | **5.7** | **5.6** | **5.8** | **5.1** | **5.1** | **5.7** | **6.2** | **5.8** | **5.8** | **4.6** | **4.4** |
| | $t$ | $t$ | **4.7** | **4.2** | **4.6** | **5.1** | **3.9** | **3.5** | **4.1** | **3.8** | **4.4** | **5.8** | **6.0** |
| | Cl. | Cl. | **4.2** | **4.4** | **6.2** | **5.8** | **4.6** | **5.7** | **5.7** | **5.4** | **5.4** | **5.4** | **5.6** |
| | Gu. | Gu. | **4.3** | **4.6** | **5.3** | **4.7** | **4.8** | **3.2** | **3.4** | **5.0** | **5.2** | **4.8** | **4.6** |
| | Ga. | $t$ | 2.9 | 9.8 | 26.3 | 10.8 | 2.4 | 16.3 | 3.8 | 19.2 | 7.8 | 16.2 | 8.6 |
| | Ga. | Cl. | 27.3 | 50.2 | 13.5 | 15.9 | 10.6 | 21.9 | 22.4 | 42.2 | 42.2 | 42.0 | 42.0 |
| | Ga. | Gu. | 10.0 | 13.0 | 5.0 | 3.1 | 2.0 | 6.6 | 6.5 | 10.0 | 9.4 | 10.4 | 10.2 |
| $n = 100$ | $t$ | Ga. | 20.0 | 9.6 | 62.7 | 64.3 | 47.5 | 55.0 | 55.8 | 48.8 | 49.4 | 45.0 | 46.6 |
| | $t$ | Cl. | 38.1 | 56.8 | 44.8 | 46.4 | 33.9 | 45.7 | 46.2 | 61.4 | 61.8 | 59.8 | 59.8 |
| | $t$ | Gu. | 18.8 | 9.8 | 30.8 | 32.1 | 20.7 | 29.3 | 30.0 | 28.2 | 29.0 | 25.0 | 25.6 |
| | Cl. | Ga. | 79.0 | 42.8 | 36.5 | 32.0 | 23.7 | 73.7 | 73.4 | 46.8 | 44.6 | 76.4 | 75.4 |
| | Cl. | $t$ | 38.0 | 40.6 | 4.5 | 4.3 | 3.5 | 25.0 | 24.2 | 30.4 | 29.8 | 39.6 | 38.2 |
| | Cl. | Gu. | 92.5 | 85.2 | 44.8 | 48.0 | 27.4 | 90.6 | 90.6 | 85.0 | 85.4 | 97.2 | 97.2 |
| | Gu. | Ga. | 7.6 | 12.4 | 27.8 | 27.5 | 23.4 | 22.5 | 22.1 | 23.6 | 23.2 | 21.4 | 21.8 |
| | Gu. | $t$ | 10.7 | 10.8 | 7.1 | 3.0 | 2.3 | 7.8 | 5.6 | 10.2 | 7.6 | 9.2 | 6.4 |
| | Gu. | Cl. | 68.1 | 85.2 | 30.5 | 35.3 | 27.2 | 62.5 | 62.8 | 82.8 | 83.0 | 88.0 | 88.0 |
| | Ga. | Ga. | **4.3** | **4.3** | **5.2** | **5.3** | **5.3** | **4.9** | **4.8** | **3.9** | **3.5** | **4.5** | **3.9** |
| | $t$ | $t$ | **4.8** | **5.2** | **4.4** | **5.8** | **2.8** | **4.9** | **5.8** | **5.2** | **4.9** | **5.3** | **7.3** |
| | Cl. | Cl. | **5.1** | **4.7** | **4.1** | **4.4** | **3.7** | **3.8** | **3.9** | **4.7** | **4.5** | **3.6** | **3.9** |
| | Gu. | Gu. | **3.6** | **4.7** | **6.3** | **5.7** | **6.1** | **4.6** | **4.7** | **5.9** | **6.3** | **5.3** | **5.3** |
| | Ga. | $t$ | 6.7 | 17.1 | 75.0 | 73.9 | 27.1 | 64.9 | 58.8 | 68.7 | 63.7 | 60.8 | 48.9 |
| | Ga. | Cl. | 88.3 | 91.7 | 31.8 | 33.9 | 21.2 | 81.3 | 81.4 | 88.9 | 89.3 | 95.2 | 95.3 |
| | Ga. | Gu. | 42.4 | 33.6 | 8.3 | 6.1 | 4.8 | 33.8 | 32.7 | 25.1 | 24.5 | 42.3 | 41.7 |
| $n = 300$ | $t$ | Ga. | 45.8 | 22.3 | 93.9 | 94.5 | 84.7 | 91.2 | 92.6 | 92.5 | 94.5 | 90.9 | 92.7 |
| | $t$ | Cl. | 90.4 | 94.8 | 81.1 | 82.4 | 66.7 | 91.8 | 91.8 | 97.3 | 97.3 | 98.7 | 98.8 |
| | $t$ | Gu. | 57.9 | 29.7 | 61.8 | 63.6 | 37.9 | 69.0 | 70.1 | 59.3 | 61.2 | 63.3 | 66.0 |
| | Cl. | Ga. | 99.9 | 92.7 | 78.8 | 72.4 | 55.4 | 99.9 | 99.9 | 95.3 | 95.2 | 100.0 | 100.0 |
| | Cl. | $t$ | 96.0 | 89.9 | 4.6 | 4.1 | 3.3 | 93.2 | 92.5 | 84.1 | 84.0 | 98.4 | 98.4 |
| | Cl. | Gu. | 100.0 | 100.0 | 94.9 | 95.7 | 68.6 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| | Gu. | Ga. | 16.1 | 30.7 | 63.8 | 63.0 | 45.2 | 57.8 | 57.6 | 60.5 | 61.7 | 55.9 | 55.6 |
| | Gu. | $t$ | 36.8 | 25.2 | 15.5 | 5.5 | 3.4 | 29.7 | 26.0 | 21.5 | 18.4 | 36.8 | 35.2 |
| | Gu. | Cl. | 100.0 | 100.0 | 79.2 | 81.5 | 60.5 | 99.9 | 99.9 | 99.9 | 99.9 | 100.0 | 100.0 |

**Table 3**
Percentage of rejection of $H_0$ by various tests with cross-sectional data of sizes $n = 100$ and $n = 300$ from different copula models with $\tau = 0.75$. The number of bootstrap samples $M = 1000$ and the number of replicates $N = 1000$.

| | True | $H_0$ | $S_n$ | $J_n$ | $R_n$ | $T_n(1)$ | $T_n(3)$ | $SR_n$ | $ST_n(1)$ | $JR_n$ | $JT_n(1)$ | $SJR_n$ | $SJT_n(1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n = 100$ | Ga. | Ga. | **4.7** | **6.2** | **4.1** | **4.0** | **4.3** | **4.4** | **4.5** | **5.0** | **4.8** | **4.8** | **4.8** |
| | $t$ | $t$ | **6.3** | **6.0** | **4.3** | **3.8** | **5.5** | **5.6** | **5.4** | **6.2** | **6.2** | **7.4** | **7.0** |
| | Cl. | Cl. | **5.3** | **6.0** | **6.3** | **6.0** | **4.3** | **5.5** | **5.2** | **6.0** | **6.0** | **4.2** | **4.0** |
| | Gu. | Gu. | **5.9** | **5.8** | **5.3** | **5.5** | **4.9** | **5.5** | **5.4** | **5.6** | **5.4** | **6.8** | **6.8** |
| | Ga. | $t$ | 2.1 | 9.4 | 21.5 | 9.1 | 2.3 | 13.7 | 3.8 | 15.2 | 6.4 | 11.4 | 4.8 |
| | Ga. | Cl. | 35.9 | 87.0 | 36.4 | 36.7 | 26.6 | 39.3 | 39.5 | 83.4 | 83.4 | 80.4 | 80.4 |
| | Ga. | Gu. | 18.1 | 15.2 | 5.1 | 4.5 | 5.2 | 12.8 | 13.0 | 12.6 | 12.6 | 18.8 | 18.8 |
| | $t$ | Ga. | 25.4 | 9.0 | 59.2 | 59.9 | 48.0 | 51.7 | 52.5 | 50.2 | 53.0 | 49.4 | 51.0 |
| | $t$ | Cl. | 42.9 | 82.8 | 63.8 | 63.9 | 47.4 | 57.2 | 57.1 | 84.4 | 84.6 | 84.4 | 84.4 |
| | $t$ | Gu. | 36.9 | 12.2 | 32.2 | 32.5 | 24.0 | 38.5 | 38.9 | 32.0 | 32.0 | 38.4 | 38.0 |
| | Cl. | Ga. | 99.9 | 75.4 | 84.9 | 72.5 | 55.9 | 99.2 | 99.2 | 87.6 | 84.6 | 99.2 | 99.2 |
| | Cl. | $t$ | 87.9 | 56.2 | 19.1 | 58.2 | 29.7 | 80.4 | 81.9 | 52.8 | 66.6 | 83.0 | 84.2 |
| | Cl. | Gu. | 100.0 | 98.4 | 97.2 | 97.7 | 81.2 | 100.0 | 100.0 | 99.4 | 99.4 | 100.0 | 100.0 |
| | Gu. | Ga. | 10.1 | 11.6 | 45.4 | 42.4 | 31.7 | 35.7 | 35.0 | 35.2 | 35.6 | 31.4 | 27.6 |
| | Gu. | $t$ | 8.9 | 12.2 | 4.7 | 4.4 | 5.0 | 6.5 | 6.8 | 9.6 | 8.2 | 8.8 | 8.8 |
| | Gu. | Cl. | 82.6 | 99.8 | 80.9 | 81.6 | 61.4 | 86.5 | 86.6 | 99.4 | 99.4 | 99.8 | 99.8 |
| $n = 300$ | Ga. | Ga. | **5.5** | **4.3** | **4.4** | **4.5** | **4.2** | **4.7** | **4.2** | **4.3** | **4.1** | **5.6** | **5.7** |
| | $t$ | $t$ | **4.3** | **5.1** | **5.5** | **4.6** | **6.2** | **5.6** | **4.5** | **4.7** | **5.1** | **5.1** | **4.7** |
| | Cl. | Cl. | **5.0** | **5.9** | **6.6** | **6.5** | **5.0** | **5.5** | **5.5** | **3.5** | **3.5** | **3.2** | **3.2** |
| | Gu. | Gu. | **4.5** | **3.3** | **5.2** | **5.2** | **5.2** | **4.4** | **4.3** | **4.5** | **4.3** | **5.1** | **5.1** |
| | Ga. | $t$ | 5.1 | 12.4 | 66.0 | 61.7 | 22.4 | 55.3 | 46.4 | 58.3 | 50.3 | 51.2 | 42.9 |
| | Ga. | Cl. | 99.1 | 100.0 | 77.7 | 78.8 | 62.5 | 98.3 | 98.3 | 100.0 | 100.0 | 100.0 | 100.0 |
| | Ga. | Gu. | 60.2 | 36.3 | 7.3 | 6.9 | 6.3 | 49.5 | 49.1 | 26.8 | 26.9 | 57.9 | 57.9 |
| | $t$ | Ga. | 65.7 | 12.3 | 95.6 | 96.3 | 88.1 | 92.9 | 93.7 | 93.2 | 94.0 | 91.9 | 92.5 |
| | $t$ | Cl. | 98.3 | 100.0 | 98.0 | 98.0 | 86.5 | 99.6 | 99.6 | 100.0 | 100.0 | 100.0 | 100.0 |
| | $t$ | Gu. | 88.3 | 24.7 | 71.4 | 72.6 | 52.7 | 88.3 | 88.3 | 67.9 | 68.1 | 83.1 | 83.1 |
| | Cl. | Ga. | 100.0 | 100.0 | 100 | 99.8 | 97.2 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| | Cl. | $t$ | 100.0 | 98.5 | 36.6 | 97.7 | 75.9 | 100.0 | 100.0 | 97.9 | 99.6 | 100.0 | 100.0 |
| | Cl. | Gu. | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |
| | Gu. | Ga. | 26.1 | 30.9 | 87.8 | 84.1 | 69.4 | 83.1 | 80.0 | 82.8 | 82.1 | 79.7 | 78.4 |
| | Gu. | $t$ | 47.0 | 25.6 | 5.5 | 4.3 | 5.9 | 32.2 | 31.8 | 19.6 | 19.5 | 30.4 | 29.2 |
| | Gu. | Cl. | 100.0 | 100.0 | 100.0 | 100.0 | 97.5 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |

**Table 4**
Percentages of rejection of $H_0$ by various tests with time series data for different copula models. Sample size $n = 300$, the number of bootstrap samples $M = 1000$, and the number of replicates $N = 1000$ from marginal GARCH(1,1) dependent data. The type I errors (upper panel), were obtained using both the residual-based (in bold) and observation-based bootstrap procedures. Test power (lower panel) was obtained using the observation-based bootstrap alone.

| True | $H_0$ | $\tau = 0.25$ | | $\tau = 0.5$ | | $\tau = 0.75$ | |
|---|---|---|---|---|---|---|---|
| | | $R_n$ | $T_n(1)$ | $R_n$ | $T_n(1)$ | $R_n$ | $T_n(1)$ |
| Ga | Ga | **3.4** | **3.8** | **5.2** | **5.3** | **5.4** | **4.4** |
| | | 5.4 | 5.5 | 3.9 | 3.3 | 3.7 | 4.7 |
| Cl | Cl | **5.2** | **5.1** | **5.3** | **5.5** | **7.0** | **7.0** |
| | | 5.6 | 5.5 | 5.3 | 5.3 | 5.4 | 5.4 |
| $t$ | $t$ | **5.7** | **6.8** | **5.5** | **3.9** | **5.4** | **5.1** |
| | | 4.7 | 3.9 | 4.1 | 4.1 | 5.4 | 4.9 |
| Gu | Gu | **5.2** | **5.1** | **5.7** | **5.6** | **4.5** | **4.3** |
| | | 5.4 | 5.1 | 4.8 | 4.8 | 5.1 | 4.8 |
| Ga | Cl | 7.1 | 6.5 | 30.2 | 32.4 | 66.7 | 66.6 |
| Ga | $t$ | 80.3 | 69.6 | 72.2 | 71.7 | 68.9 | 65.5 |
| Ga | Gu | 7.9 | 5.6 | 6.8 | 4.6 | 7.3 | 7.3 |
| Cl | Ga | 24.4 | 24.3 | 81.6 | 74.6 | 99.9 | 99.4 |
| Cl | $t$ | 42.2 | 21.8 | 3.6 | 3.4 | 29.7 | 95.6 |
| Cl | Gu | 48.9 | 52.2 | 95.7 | 96.5 | 100.0 | 100.0 |
| $t$ | Ga | 93.5 | 94.3 | 94.9 | 96.2 | 94.5 | 95.5 |
| $t$ | Cl | 63.3 | 64.8 | 80.9 | 82.7 | 97.1 | 97.2 |
| $t$ | Gu | 65.6 | 68.3 | 62.9 | 64.1 | 67.8 | 68.0 |
| Gu | Ga | 30.1 | 31.4 | 62.6 | 61.5 | 83.2 | 80.9 |
| Gu | Cl | 18.3 | 20.2 | 79.3 | 81.0 | 99.9 | 99.9 |
| Gu | $t$ | 29.8 | 14.8 | 13.5 | 6.2 | 7.0 | 6.7 |

### 6.3.2. Global power

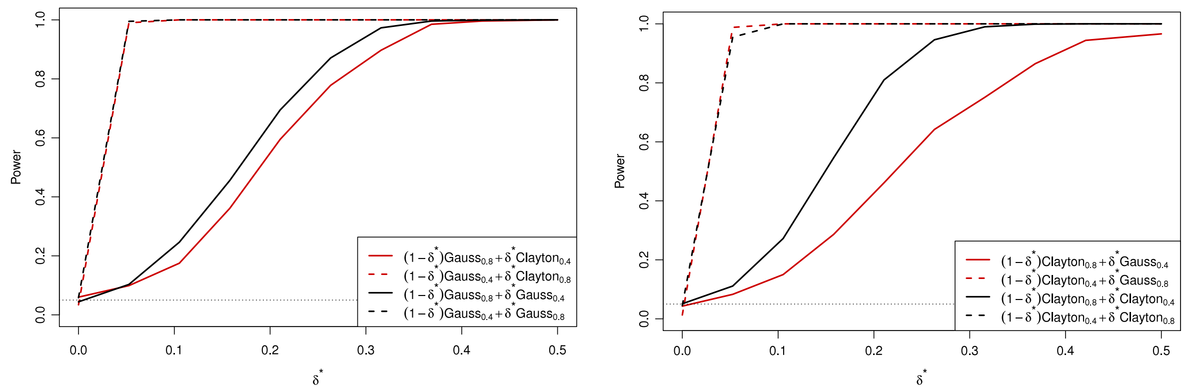
We now evaluate the global power of the proposed tests under a fixed true model. The empirical test power is reported in Tables 1–3. We may draw the following conclusions for the scenario of cross-sectional data:

(1) In general, $T_n(m)$ and $R_n$ exhibit similar global power in most of the settings except a few ones, including the setting where Student's $t$ is under the null hypothesis. When we increase the sample sizes to $n = 1000$, the discrepancies between these two tests disappear (the results are not shown due to the space limitations), which confirms the theoretical results (i.e., Theorem 2(ii)) of asymptotic equivalence between $T_n(m)$ and $R_n$. In addition, there are only marginal differences among the two-element hybrid tests $ST_n(m)$ and $SR_n$, $JT_n(m)$ and $JR_n$, $SJT_n(m)$ and $SJR_n$ regardless of the choice of the dependence strength, the choice of the sample size or the choice of the copula family.

(2) The $T_n(1)$ test has overall better or equal performance to the $T_n(3)$ test, because in the case of cross-sectional data using block size of $m = 3$ may shrink the effective sample size. Thus in the later discussion on the comparison with the other methods we only focus on the test $T_n(1)$. This numerical evidence also serves as the basis for our use of the test $T_n(1)$ in two empirical studies in Section 7.

(3) As demonstrated clearly, the two-element hybrid tests, such as $ST_n(1)$, $JT_n(1)$, $SR_n$ and $JR_n$ show clearly on average the advantage on global power. Moreover, the triplet hybrid tests $SJR_n$ and $SJT_n(1)$ perform on average superior to the two-element hybrid tests. The hybrid tests demonstrate superior performances in all the settings, regardless of the choice of the copula family, or the choice of the dependency strength, and hence they are recommended to be applied in practice.

(4) The performances of the proposed $R_n$, $T_n(1)$, $SR_n$, $ST_n(1)$, $JR_n$, $JT_n(1)$, $SJR_n$, $SJT_n(1)$ tests as well as the existing $S_n$ and $J_n$ tests are relied on the strength of dependence. When $\tau = 0.25$ and sample size $n = 100$, with no surprise, all these tests have almost no power. Up to our knowledge, there exists no single test that has a desirable performance in such a setting




(a) Gauss under $H_0$. (b) Clayton under $H_0$.

**Fig. 1.** Local power curves for the $R_n$ test with (a) Gaussian copula and (b) Clayton copula being under $H_0$ and four different cases of the true mixtures of copulas, where $\delta^* = \delta/\sqrt{n}$.

of low correlation. Similar results are reported by Genest et al. (2009). Challenges arise in the separation of the copulas with $\tau = 0.25$ because with a Kendall's tau close to 0, the simulated data are drawn from a copula that resembles the independent copula. In this case, it becomes hard to differentiate one copula from others. Perhaps in this case making a choice of the copula functional form does not really matter. It is interesting to note that even in this situation of weak dependence, in contrary to $S_n$ and $J_n$ tests, all our proposed tests have demonstrated to have relatively high power of rejecting Gaussian copula against the Student's $t$ copula. When Kendall's tau is not too small ($\tau = 0.5$ or $\tau = 0.75$), and the sample size is large enough ($n = 300$), all the proposed tests exhibit satisfactory global power.

(5) It is interesting to observe that all the proposed tests are significantly superior to $S_n$ and $J_n$ tests to differentiate between Student's $t$ copula and Gaussian copula. These two blanket tests $S_n$ and $J_n$ have similar behaviors in this scenario. When the sample size increases to $n = 300$, the tests $R_n$ and $T_n(1)$ as well as the hybrid tests almost reach 100% power as opposed to the $S_n$ test having power lower than 70% and $J_n$ lower than 25%.

(6) It is worth pointing out that in some scenarios, both $R_n$ and $T_n(1)$ tests perform poorly and are inferior to the $S_n$ test and the hybrid tests involving $S_n$, such as when Gaussian is true and Gumbel is under $H_0$, or when Gumbel is true and $t$ is under $H_0$, or when Gaussian is true and Clayton under $H_0$ with $\tau = 0.5$, or when Clayton is true and Student's $t$ copula under $H_0$ with $\tau = 0.5$. However, in all the remaining cases, the proposed $R_n$ and $T_n(1)$ tests together with the hybrid ones $SR_n$ and $ST_n(1)$ perform comparably or better than $S_n$ test. On the other side, the $J_n$ is superior to $S_n, R_n, T_n(m)$ in most of the cases when Clayton copula is under $H_0$. These scenarios are worth for further investigation.

(7) For the scenario of multivariate time series data, Table 4 reports the empirical global power of the proposed tests $R_n$ and $T_n(1)$ using the observation-based bootstrap procedure. The two tests $R_n$ and $T_n(1)$ show similar performances. For most of the settings, as the strength of the dependence increases, the test power increases.

#### 6.3.3. Local power

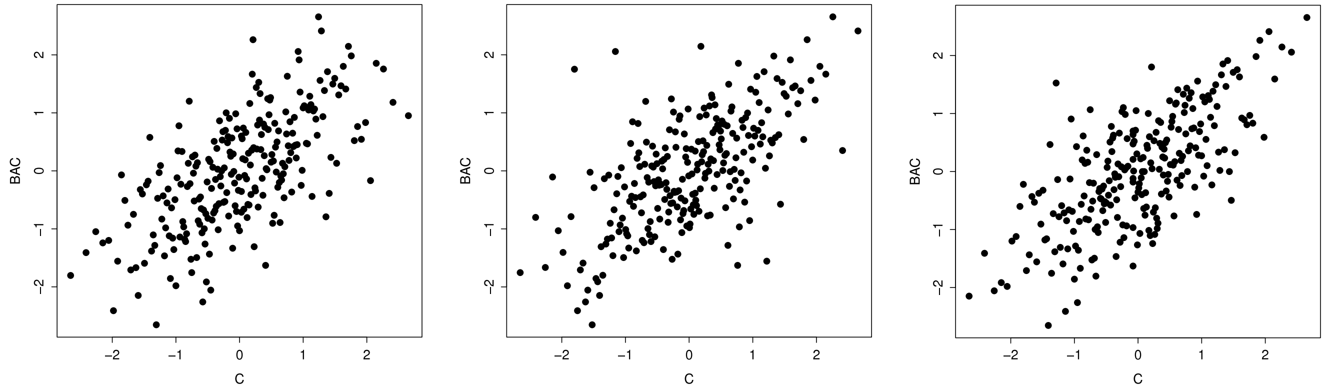
We also run a simulation experiment to illustrate the theory of local power discussed in Section 4.3. The simulation study concerns two settings of mixtures. On setting involves mixtures of two copulas from the same family, either Gaussian or Clayton, with different values of dependence parameter $\theta$, and the other setting involves mixtures of two copulas belonging to two different families, one from Gaussian and the other from Clayton. The sample size is set at $n = 500$, and we perform $M = 1000$ replications to obtain the empirical rejection rate. The margins $F(\cdot)$ are set to be uniform on $(0, 1)$. The Kendall's $\tau$ in $C_0(\cdot; \theta_0)$ and $C_1(\cdot; \theta_1)$ are specified as $(\tau_1, \tau_2) = (0.4, 0.8)$. Fig. 1 displays the local power curves of the $R_n$ statistic with the values of $\delta^* := \delta/\sqrt{n}$ in (6) varying from 0.0 to 0.5. All the curves are approximately equal to 5% for $\delta^* = 0$. This corresponds to the type I error which is well controlled. Also see Section 6.3.2. As $\delta^*$ increases to 0.5 the local power curve rises to 1.0 in all the curves. Dashed lines in both panels of Fig. 1 indicate a fast power growth of test $R_n$ when the copula with higher dependency becomes more dominant, while the solid lines show the power rises relatively slowly when the copula with higher dependency dims off.

## 7. Applications

In this section we present two empirical examples to illustrate the usefulness of the proposed tests in practice. The first example focuses on changes of the dependence structure over time between stock returns, and the second example concerns the dependence structure of insurance data on losses and expenses.

### 7.1. Detecting structural changes in the dependency

We are interested in daily stock returns of Citigroup (C) and Bank of America (BAC) over years 2004, 2006 and 2009. It is known from some recent studies (e.g. Hafner and Manner, 2012, Patton, 2012, Härdle et al., 2013) that during the global financial crisis over years 2008–2009 the dependency between various financial instruments appeared different, which may be used as a benchmark to gauge the performance of the proposed tests. First, similar to the procedures used in the simulation study, we remove temporal dependencies using the GARCH(1,1) model for each year separately, and the residuals are then used to estimate the dependency between the two bank stocks. To remove the marginal distribution of the residuals, we use the empirical cumulative distributions. To visualize potential structure changes in dependency, Fig. 2 displays the scatterplots of the residuals transformed by the standard normal distribution for the three years, 2004 (top left), 2006 (top right) and 2009 (bottom central). Table 5 presents $p$-values of the tests computed via the observation-based bootstrap and values of the maximized log-likelihood, for bivariate copula model specification at these years, when one of four chosen candidate copulas is set for $H_0$. Having different goodness-of-fit tests, the best suited copula is selected via




(a) Year 2004. (b) Year 2006. (c) Year 2009.

**Fig. 2.** Scatterplots of the residuals transformed to the standard normal for Citigroup (C) / Bank of America (BAC) for years 2004 (top left), 2006 (top right) and 2009 (bottom).

**Table 5**
*p*-values of the tests for bivariate copula model specification for the dependency between residuals of Citigroup and Bank of America for years 2004, 2006 and 2009. Selected copulas for each year and test are marked with bold. The values of maximized log-likelihood (loglik) are also provided.

| | | $T_n(1)$ | $R_n$ | $S_n$ | $J_n$ | $ST_n(1)$ | $SR_n$ | $JT_n(1)$ | $JR_n$ | $SJT_n(1)$ | $SJR_n$ | loglik |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2004 | Cl. | 0.450 | 0.574 | 0.001 | 0.002 | 0.002 | 0.002 | 0.004 | 0.004 | 0.003 | 0.003 | 52.467 |
| | Gu. | **0.834** | **0.920** | 0.158 | 0.841 | 0.316 | 0.316 | **1.000** | **1.000** | 0.474 | 0.474 | 64.999 |
| | Ga. | 0.333 | 0.544 | **0.455** | **0.887** | **0.666** | **0.910** | 0.666 | **1.000** | **0.999** | **1.000** | **72.234** |
| | *t* | 0.033 | 0.012 | 0.092 | 0.777 | 0.066 | 0.024 | 0.066 | 0.024 | 0.099 | 0.036 | 65.041 |
| 2006 | Cl. | 0.015 | 0.015 | 0.002 | 0.000 | 0.004 | 0.004 | 0.000 | 0.000 | 0.000 | 0.000 | 55.644 |
| | Gu. | 0.024 | 0.024 | 0.166 | 0.054 | 0.048 | 0.048 | 0.048 | 0.048 | 0.072 | 0.072 | 73.337 |
| | Ga. | 0.000 | 0.000 | **0.446** | 0.089 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 68.705 |
| | *t* | **0.697** | **0.328** | 0.397 | **0.165** | **0.794** | **0.656** | **0.330** | **0.330** | **0.495** | **0.495** | **78.396** |
| 2009 | Cl. | 0.265 | 0.328 | 0.002 | 0.000 | 0.004 | 0.004 | 0.000 | 0.000 | 0.000 | 0.000 | 60.479 |
| | Gu. | **0.478** | **0.409** | **0.686** | 0.514 | **0.956** | **0.818** | **0.956** | **0.818** | **1.000** | **1.000** | **87.449** |
| | Ga. | 0.145 | 0.301 | 0.599 | **0.759** | 0.290 | 0.602 | 0.290 | 0.602 | 0.435 | 0.903 | 85.814 |
| | *t* | 0.127 | 0.360 | 0.572 | 0.608 | 0.254 | 0.720 | 0.254 | 0.720 | 0.381 | **1.000** | 85.604 |

the largest *p*-value, following the suggestions by Murtaugh (2014), de Valpine (2014), and Gneiting and Raftery (2007), among others. Based on these residuals, Fig. 2(a) of the residuals for year 2004 shows a usual Gaussian elliptical shape, which is supported by the tests $S_n$ and $J_n$, with the largest *p*-value and the largest value of the log-likelihood, but not chosen by the $T_n(1)$ and $R_n$ (see Table 5). Fig. 2(b) for year 2006 shows a shape of *t*-copula with Gaussian margins, highlighted by a few outliers lying far from the diagonal on the two wings close to $(2, -2)$ and $(-2, 2)$. This is confirmed by the $T_n(1)$, $R_n$ and $J_n$ tests and by the largest log-likelihood, but not by the $S_n$. Fig. 2(c) for year 2009 shows an asymmetric shape like a "water-drop", which signifies the Gumbel copula. This observation is verified by the tests $T_n(1)$, $R_n$ and $S_n$ and by the log-likelihood based criteria but not by the $J_n$ test. These discrepant results given by the individual single tests are harmonized by the triplet-hybrid tests $SJT_n(1)$ and $SJR_n$; both consistently support the visual inspections and in a full agreement with the log-likelihood selection criterion. Our final selection gives to Gauss, *t* and Gumbel for years 2004, 2006 and 2009, respectively.

### 7.2. Analysis of losses and allocated loss adjustment expenses

Now, we apply the proposed tests to a well-known insurance dataset on losses and allocated loss adjustment expenses (ALAE), which are collected by the US Insurance Service Office. Such data has been previously analyzed by many authors, including Frees and Valdez (1998), Genest et al. (1998), Klugman and Parsa (1999), Denuit and Scaillet (2004), Scaillet (2005), Chen and Fan (2005), Denuit et al. (2006) and Giacomini and Rossi (2009), among others.

The dataset consists of 1500 general liability claims, among which 34 claims are censored due to late settlement lags. Each claim consists of an indemnity payment (i.e. the loss) and an allocated loss adjustment expense (ALAE). Here we determine a dependence model using the 1466 complete data. We run the proposed goodness-of-fit tests on four families of copulas, including Gaussian copula, Student's *t* copula, Gumbel copula and Clayton copula. For each copula, we estimate the dependence parameter by the PMLE approach described in Section 2.

Table 6 reports the results of PMLE, test statistics and *p*-values. The estimated degree of freedom of Student's *t* copula is 11.11. From Table 6, we find that Gumbel copula appears to be the most adequate and Gaussian copula is least suitable among the four copula models. This result is also supported by the maximum value of the log-likelihood function reported in the last row of the table.

Our findings obtained by the hybrid tests are consistent with the model selection results reported by Frees and Valdez (1998), Genest et al. (1998), Chen and Fan (2005) and Denuit et al. (2006). Frees and Valdez (1998) and Denuit et al. (2006) point out a positive upper-tail dependence between loss and ALAE, implying that large losses tend to be associated to large ALAEs. This is because expensive claims usually take some time to be settled and induce extra costs for the insurance company. Thus, it is reasonable to observe a positive upper-tail dependence. On the other hand, no lower tail dependence is detected. Among the four copula models, Gumbel copula exhibits a strong upper-tail dependence, which properly reflects the relationship between loss and ALAE. The other copula models do not have similar features of upper tail dependence.

## 8. Discussion

In this paper, we focus on goodness-of-fit tests for specification of semiparametric copula dependence models. We propose a new method based on pseudo likelihood of cross-validation

**Table 6**
Summary of data analysis results obtained from the four copulas: Gaussian, Student's $t$, Clayton and Gumbel, including dependence parameter estimates $\hat{\theta}$ with the standard errors in the parentheses and $p$-values with test statistics in the parentheses. The maximum value of the log-likelihood is reported at the last line.

| Statistic | Copula | | | |
|---|---|---|---|---|
| | Clayton | Gumbel | Gauss | $t$ |
| $\hat{\theta}$ | 0.511 (0.043) | 1.428 (0.029) | 0.456 (0.019) | 0.466 (0.020) |
| $T_n(1)$ | 0.000 (1.316) | 0.370 (0.954) | 0.000 (1.223) | 1.000 (0.998) |
| $R_n$ | 0.000 (1.323) | 0.315 (0.959) | 0.000 (1.274) | 1.000 (1.654) |
| $S_n$ | 0.000 (0.407) | 0.006 (0.072) | 0.000 (0.118) | 0.000 (0.163) |
| $J_n$ | 0.000 (0.095) | 0.789 (0.023) | 0.041 (0.038) | 0.296 (0.033) |
| $ST_n(1)$ | 0.000 | 0.012 | 0.000 | 0.000 |
| $SR_n$ | 0.000 | 0.012 | 0.000 | 0.000 |
| $JT_n(1)$ | 0.000 | 0.740 | 0.000 | 0.592 |
| $JR_n$ | 0.000 | 0.630 | 0.000 | 0.592 |
| $SJT_n(1)$ | 0.000 | 0.018 | 0.000 | 0.000 |
| $SJR_n$ | 0.000 | 0.018 | 0.000 | 0.000 |
| loglik | 89.95 | 191.4 | 171.2 | 177.9 |

leading to the construction of a test statistic by comparing the "in-sample" pseudo-likelihood and "out-of-sample" pseudo-likelihood. As shown in the theory and numerical examples, the proposed comparison of pseudo likelihoods over different datasets has provided a highly competitive performance to indicate how well a copula model fits the data. To mitigate the computational burden of the proposed $T_n(m)$ test, we introduce the $R_n$ test, which show similar performance to $T_n(m)$ test. We establish the large sample properties for both $T_n(m)$ and $R_n$ tests, develop the asymptotic local theory in the Pitman sense. In comparison to the blanket tests considered in Genest et al. (2009) and Scaillet (2007), all of which are rank-based tests, the proposed test enables us to avoid using any probability integral transformations.

In addition, we extend the PIOS test to the case of SCOMDY model. To take into account uncertainty in estimating the dynamic parameters in finite sample, we propose a new bootstrap procedure, in which time series data is resampled and the dynamic parameters are re-estimated in each bootstrap sample case. This bootstrap procedure is shown to control type I error better, compared to the commonly used bootstrap based on resampling of the estimated innovation processes.

By means of Bonferroni correction for multiple testing, we propose a hybrid test to combine several different test statistics for a common hypothesis problem. In terms of average performance, the hybrid test is clearly superior to any of the individual tests used in the combination. An important property is that if there is at least one consistent test in the combination, then the hybrid test is consistent. This hybrid strategy is particularly appealing when there is no *a priori* knowledge which test might be the top performer at a given occasion.

We conduct extensive simulation experiments to investigate and compare the finite-sample performances between our proposed tests and the $S_n$ test (Genest et al., 2009) and the $J_n$ test (Scaillet, 2007). The results of Monte Carlo simulations show that the proposed tests perform satisfactorily in type I error control and that they are very comparable to the existing best performer $S_n$ and $J_n$ tests. In particular, when the data are generated from Student's $t$ copula, the proposed tests are more powerful than $S_n$ test and $J_n$ test in differentiating $t$ and Gaussian copulas. Also, the proposed hybrid tests have shown a superior performance in all the cases, regardless of the choice of the copula family or the choice of the dependency strength, and hence they are highly recommended as a desirable method to be applied in practice. In our simulation studies we found, that all the tests, including proposed $R_n$ and $T_n(m)$ are not uniformly powerful in all situations, like the situation when Gumbel is the true copula but $t$ is under $H_0$. These situations are worth for further investigation in order to improve our proposed test.

As suggested by a referee, combining test statistics may be alternatively done by a "Portmanteau" type test statistic, namely the maximum of the involved test statistics. This approach is meaningful when the test statistics are comparable with the same type of probability distribution. By normalizing the test statistics to have a common null distribution, the resulting Portmanteau test would be equivalent to the minimum of their corresponding $p$-values. Note that, the method of $p$-values based combination is more general as it allows the test statistics to have different types of distributions, which is the choice of method in this paper to form the hybrid test.

In this paper, we focus all numerical illustrations only on the occasion of 2-dimensional copula families, and it is of great interest to evaluate these tests to multi-dimensional copulas, such as vine copulas. It is also interesting to explore in general the effect of block size on the test power.

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## Appendix A

This appendix is devoted to the proofs of the theorems given in Section 4.

**Proof of Theorem 1.** Define the rescaled empirical copula of $X_1, \ldots, X_n$ by

$$\tilde{C}(u) = \frac{1}{n+1}\sum_{t=1}^{n} I\left\{\tilde{F}(X_t) \le u\right\}$$
$$= \frac{1}{n+1}\sum_{t=1}^{n} I\left\{\tilde{F}_1(X_{t1}) \le u_1, \ldots, \tilde{F}_d(X_{td}) \le u_d\right\}.$$

For any $\theta \in \Theta$, we can rewrite $S(\theta)$, $\hat{S}(\theta)$, $V(\theta)$ and $\hat{V}(\theta)$ as follows:

$$S(\theta) = -\int_{u\in[0,1]^d} l_{\theta\theta}(u;\theta)dC_0(u);$$
$$\hat{S}(\theta) = -\frac{n+1}{n}\int_{u\in[0,1]^d} l_{\theta\theta}(u;\theta)d\tilde{C}(u),$$

and

$$V(\theta) = \int_{u\in[0,1]^d} l_\theta(u)l_\theta^T(u)dC_0(u);$$
$$\hat{V}(\theta) = \frac{n+1}{n}\int_{u\in[0,1]^d} l_\theta(u)l_\theta^T(u)d\tilde{C}(u),$$

where $C_0(\cdot)$ and $\tilde{C}(\cdot)$ are the true copula and the rescaled empirical copula.

By condition (A1), applying Lemma 1(c) in Chen and Fan (2005) (see also Fermanian et al. (2004)), we have

$$\sup_{\theta\in\mathcal{N}(\theta^*)}\left\|\hat{S}(\theta)-S(\theta)\right\| = \sup_{\theta\in\mathcal{N}(\theta^*)}\left\|\int_{u\in[0,1]^d} l_{\theta\theta}(u;\theta)d\left\{\frac{n+1}{n}\tilde{C}(u)-C_0(u)\right\}\right\| \xrightarrow{pr} 0,$$

as $n\to\infty$.

Hence, using the two facts $\left\|\hat{S}(\hat\theta)-S(\theta^*)\right\| \le \left\|\hat{S}(\hat\theta)-S(\hat\theta)\right\| + \left\|S(\hat\theta)-S(\theta^*)\right\|$, and $\hat\theta \xrightarrow{pr} \theta^*$,we obtained $\hat{S}(\hat\theta) \xrightarrow{pr} S(\theta^*)$.

Applying the same arguments above, we can show $\hat{V}(\hat\theta) \xrightarrow{pr} V(\theta^*)$.

Furthermore, by condition (A2) and Slutsky's theorem, we have

$$R_n = \mathrm{tr}\left\{\hat{S}(\hat\theta)^{-1}\hat{V}(\hat\theta)\right\} \xrightarrow{pr} \mathrm{tr}\left\{S(\theta^*)^{-1}V(\theta^*)\right\}.$$

**Proof of Theorem 2(i).** First note that, $\hat\theta$ solves the equation $\sum_{t=1}^n l_\theta\{\tilde{F}(X_t);\hat\theta\} = 0$. Applying the mean-value theorem, we have

$$0=\sum_{t=1}^n l_\theta\{\tilde{F}(X_t);\theta^*\}+\sum_{t=1}^n l_{\theta\theta}\{\tilde{F}(X_t);\tilde\theta\}(\hat\theta-\theta^*),$$

where $\tilde\theta$ lies between $\theta^*$ and $\hat\theta$. Thus

$$\hat\theta-\theta^* = -\left[\frac{1}{n}\sum_{t=1}^n l_{\theta\theta}\{\tilde{F}(X_t);\tilde\theta\}\right]^{-1}\frac{1}{n}\sum_{t=1}^n l_\theta\{\tilde{F}(X_t);\theta^*\}.$$

For any $1\le i,j\le p$, expanding $l_{\theta\theta}\{\tilde{F}(X_t);\hat\theta\}_{ij}$ around $\theta^*$ leads to

$$\begin{aligned}\hat{S}(\hat\theta)_{ij} &= \frac{1}{n}\sum_{t=1}^n l_{\theta\theta}\{\tilde{F}(X_t);\hat\theta\}_{ij}\\ &= \frac{1}{n}\sum_{t=1}^n l_{\theta\theta}\{\tilde{F}(X_t);\theta^*\}_{ij}+\frac{1}{n}\sum_{t=1}^n\frac{\partial l_{\theta\theta}\{\tilde{F}(X_t);\check\theta\}_{ij}}{\partial\theta^T}(\hat\theta-\theta^*)\\ &= \frac{1}{n}\sum_{t=1}^n l_{\theta\theta}\{\tilde{F}(X_t);\theta^*\}_{ij}-\frac{1}{n}\sum_{t=1}^n\frac{\partial l_{\theta\theta}\{\tilde{F}(X_t);\check\theta\}_{ij}}{\partial\theta^T}\\ &\quad\times\left[\frac{1}{n}\sum_{t=1}^n l_{\theta\theta}\{\tilde{F}(X_t);\tilde\theta\}\right]^{-1}\frac{1}{n}\sum_{t=1}^n l_\theta\{\tilde{F}(X_t);\theta^*\},\end{aligned}$$

where $\check\theta$ lies between $\theta^*$ and $\hat\theta$.

By condition (B3), applying again Lemma 1(c) in Chen and Fan (2005), we obtain

$$\frac{1}{n}\sum_{t=1}^n\frac{\partial l_{\theta\theta}\{\tilde{F}(X_t);\check\theta\}_{ij}}{\partial\theta^T}\xrightarrow{pr}\mathbb{E}_0\left[\frac{\partial l_{\theta\theta}\{F(X_1);\theta^*\}_{ij}}{\partial\theta^T}\right].$$

Also, we know $\frac{1}{n}\sum_{t=1}^n l_{\theta\theta}\{\tilde{F}(X_t);\tilde\theta\} \xrightarrow{pr} S(\theta^*)$ as $n\to\infty$. Therefore

$$\begin{aligned}\hat{S}(\hat\theta)_{ij} &= \frac{1}{n}\sum_{t=1}^n\left[l_{\theta\theta}\{\tilde{F}(X_t);\theta^*\}_{ij}+M_1^{ij}S^{-1}(\theta^*)l_\theta\{\tilde{F}(X_t);\theta^*\}\right]\\ &\quad+o_p(1)\\ &\triangleq \frac{1}{n}\sum_{t=1}^n h_S\{\tilde{F}(X_t);\theta^*\}_{ij}+o_p(1),\end{aligned}\tag{14}$$

where, $M_1^{ij}\triangleq\mathbb{E}_0\left[\frac{\partial l_{\theta\theta}\{F(X_1);\theta^*\}_{ij}}{\partial\theta^T}\right]$ is a $1\times p$ vector, $h_S$ is a $p\times p$ matrix with element $h_S\{\tilde{F}(X_t);\theta^*\}_{ij}$.

Employing the same arguments above, we have

$$\begin{aligned}\hat{V}(\hat\theta)_{ij} &= \frac{1}{n}\sum_{t=1}^n\Big[l_\theta\{\tilde{F}(X_t);\theta^*\}_i l_\theta\{\tilde{F}(X_t);\theta^*\}_j\\ &\quad+M_2^{ij}S^{-1}(\theta^*)l_\theta\{\tilde{F}(X_t);\theta^*\}\Big]+o_p(1)\\ &\triangleq \frac{1}{n}\sum_{t=1}^n h_V\{\tilde{F}(X_t);\theta^*\}_{ij}+o_p(1),\end{aligned}\tag{15}$$

where $M_2^{ij}=\mathbb{E}_0[\frac{\partial l_\theta\{F(X_1);\theta^*\}_i}{\partial\theta^T}l_\theta\{F(X_1);\theta^*\}_j+\frac{\partial l_\theta\{F(X_1);\theta^*\}_j}{\partial\theta^T}l_\theta\{F(X_1);\theta^*\}_i]$ and $h_V$ is a $p\times p$ matrix with element $h_V\{\tilde{F}(X_t);\theta^*\}_{ij}$.

Under the null hypothesis of the copula model being correctly specified, by Bartlett identity, we have $S(\theta^*)=V(\theta^*)$, moreover the test statistic $R_n$ given in (5) can be represented as follows:

$$\begin{aligned}\sqrt{n}(R_n-p) &= \sqrt{n}\mathrm{tr}\left\{\hat{S}^{-1}(\hat\theta)\hat{V}(\hat\theta)-I_p\right\}\\ &= \sqrt{n}\mathrm{tr}\left\{\hat{S}^{-1}(\hat\theta)\hat{V}(\hat\theta)-S^{-1}(\theta^*)V(\theta^*)\right\}\\ &= \mathrm{tr}\left[S^{-1}(\theta^*)\sqrt{n}\left\{\hat{V}(\hat\theta)-V(\theta^*)\right\}\right]\\ &\quad+\mathrm{tr}\left[S^{-1}(\theta^*)\hat{V}(\hat\theta)S^{-1}(\theta^*)\sqrt{n}\left\{S(\theta^*)-\hat{S}(\hat\theta)\right\}\right]\\ &\quad+\mathrm{tr}\left[\hat{S}^{-1}(\hat\theta)\hat{V}(\hat\theta)S^{-2}(\theta^*)\sqrt{n}\left\{S(\theta^*)-\hat{S}(\hat\theta)\right\}^2\right].\end{aligned}$$

Utilizing the asymptotic expansion in (14) and (15), we have

$$\begin{aligned}&\sqrt{n}\left\{\hat{S}(\hat\theta)-S(\theta^*)\right\}\\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^n\left[h_S\{\tilde{F}(X_t);\theta^*\}-S(\theta^*)\right]+o_p(1)\\ &= \sqrt{n}\int_{u\in(0,1)^d}h_S(u;\theta^*)d\left\{\frac{n+1}{n}\tilde{C}(u)-C_0(u)\right\}+o_p(1),\end{aligned}$$

and

$$\begin{aligned}&\sqrt{n}\left\{\hat{V}(\hat\theta)-V(\theta^*)\right\}\\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^n\left[h_V\{\tilde{F}(X_t);\theta^*\}-V(\theta^*)\right]+o_p(1)\\ &= \sqrt{n}\int_{u\in(0,1)^d}h_V(u;\theta^*)d\left\{\frac{n+1}{n}\tilde{C}(u)-C_0(u)\right\}+o_p(1).\end{aligned}$$

By conditions (B1) and (B2), employing Lemma 2 in Chen and Fan (2005) (see also Ruymgaart et al. (1972), Ruymgaart (1974) or Genest et al. (1995)), we have $\|\hat{S}(\hat\theta)-S(\theta^*)\|=O_p(n^{-1/2})$ and $\|\hat{V}(\hat\theta)-V(\theta^*)\|=O_p(n^{-1/2})$. In addition, giving these facts: $\sqrt{n}\|\hat{S}(\hat\theta)-S(\theta^*)\|^2=o_p(1)$, $\hat{S}(\hat\theta)\xrightarrow{pr}S(\theta^*)$ and $\hat{V}(\hat\theta)\xrightarrow{pr}V(\theta^*)$, we reach the following expression:

$$\begin{aligned}\sqrt{n}(R_n-p) &= \sqrt{n}\int_{u\in[0,1]^d}h_R(u;\theta^*)d\left\{\frac{n+1}{n}\tilde{C}(u)-C_0(u)\right\}\\ &\quad+o_p(1),\end{aligned}$$

where

$$h_R(u;\theta^*)=\sum_{i,j=1}^p S^{-1}(\theta^*)_{ij}\left\{h_S(u;\theta^*)_{ji}+h_V(u;\theta^*)_{ji}\right\}.$$

Again, applying Lemma 2 in Chen and Fan (2005), we have

$$\sqrt{n}(R_n-p)\xrightarrow{d}N(0,\sigma_R^2),$$

where

$$\sigma_R^2 = var_0\left[h_R(u;\theta^*) + D\{F(X_1);\theta^*\}\right], \quad (16)$$

and

$$D\{F(X_1);\theta^*\} = \sum_{j=1}^{d}\int_{u\in[0,1]^d}\frac{\partial h_R(u;\theta^*)}{\partial u_j}I\{F_j(X_{1j})\le u_j\}dC_0(u).$$

Note that the additional term $D\{F(X_1);\theta^*\}$ comes from the uncertainty of the estimator for the marginal distribution function $F(X_1) = \{F_1(x_1),\ldots,F_d(x_d)\}$. It vanishes when $F(\cdot)$ is known.

The asymptotic variance $\sigma_R^2$ may be consistently estimated by

$$\hat\sigma_R^2 = \frac{1}{n}\sum_{t=1}^{n}\left[h_R\{\tilde F(X_t);\hat\theta\} - \sum_{i,j=1}^{p}\hat S(\hat\theta)_{ij}^{-1}\hat V(\hat\theta)_{ji} + D[\tilde F(X_t);\hat\theta\}\right]^2. \quad (17)$$

To prove Theorem 2(ii), we need the following lemma.

**Lemma 1.** *Under the conditions* (A1) *and* (C1), *we have*

$$\sup_{1\le b\le B}\|\hat\theta - \hat\theta_{-b}\| = o_p(n^{-\frac{3}{4}}). \quad (18)$$

**Proof of Lemma 1.** By Eq. (3), $\hat\theta_{-b}$ solves the following equation

$$0 = \sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b'});\hat\theta_{-b}\}.$$

Expanding $l_\theta\{\tilde F(X_i^{b'});\hat\theta_{-b}\}$ around $\hat\theta$ leads to

$$\begin{aligned}0 &= \sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b'});\hat\theta_{-b}\}\\ &= -\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\} + \sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde F(X_i^{b'});\tilde\theta_{-b}\}(\hat\theta_{-b}-\hat\theta),\end{aligned}$$

where $\tilde\theta_{-b}$ lies between $\hat\theta$ and $\hat\theta_{-b}$. It follows that

$$\hat\theta_{-b}-\hat\theta = \left[\frac{1}{n}\sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde F(X_i^{b'});\tilde\theta_{-b}\}\right]^{-1}\frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\}.$$

By conditions (A1) and (C1),

$$\begin{aligned}\sup_{1\le b\le B}\frac{1}{n}\left\|\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\}\right\| &\le \frac{m}{n}\sup_{1\le b\le B}\sup_{\theta\in\mathcal N(\theta^*)}\left\|l_\theta\{\tilde F(X_i^{b});\theta\}\right\|\\ &= o_p(n^{-\frac{3}{4}})O_p(1)\\ &= o_p(n^{-\frac{3}{4}}).\end{aligned}$$

In addition, by condition (A1), using the similar arguments in the proof of Theorem 1, we can show

$$\frac{1}{n}\sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde F(X_i^{b'});\tilde\theta_{-b}\} \xrightarrow{pr} S(\theta^*).$$

Moreover,

$$\begin{aligned}\sup_{1\le b\le B}\|\hat\theta_{-b}-\hat\theta\| &\le \sup_{1\le b\le B}\left\|\left[\frac{1}{n}\sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde F(X_i^{b'});\tilde\theta_{-b}\}\right]^{-1}\right\|\\ &\quad\times\sup_{1\le b\le B}\left\|\frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\}\right\|\\ &= o_p(n^{-\frac{3}{4}}).\end{aligned}$$

**Proof of Theorem 2(ii).** By definition

$$T_n(m) = \sum_{b=1}^{B}\sum_{i=1}^{m} l\{\tilde F(X_i^{b});\hat\theta\} - \sum_{b=1}^{B}\sum_{i=1}^{m} l\{\tilde F(X_i^{b});\hat\theta_{-b}\},$$

expanding $l\{\tilde F(X_i^{b});\hat\theta_{-b}\}$ around $\hat\theta$ leads to

$$\begin{aligned}T_n(m) &= -\sum_{b=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\}(\hat\theta-\hat\theta_{-b})\\ &\quad-\frac{1}{2}\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat\theta-\hat\theta_{-b})^T l_{\theta\theta}\{\tilde F(X_i^{b});\tilde\theta_{-b}\}(\hat\theta-\hat\theta_{-b})\\ &= -\sum_{b=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\}\left[\frac{1}{n}\sum_{b'=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde F(X_i^{b'});\hat\theta\}\right.\\ &\quad\left.+e_{1b}+e_{2b}\right]^{-1}\frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\}\\ &\quad-\frac{1}{2}\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat\theta-\hat\theta_{-b})^T l_{\theta\theta}\{\tilde F(X_i^{b});\tilde\theta_{-b}\}(\hat\theta-\hat\theta_{-b})\\ &\triangleq R_n - W_1 - W_2,\end{aligned}$$

where

$$W_1 = \frac{1}{2}\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat\theta-\hat\theta_{-b})^T l_{\theta\theta}\{\tilde F(X_i^{b});\tilde\theta_{-b}\}(\hat\theta-\hat\theta_{-b}),$$

and

$$\begin{aligned}W_2 &= \sum_{b=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\}\left[\frac{1}{n}\sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b'});\tilde\theta_{-b}\}\right]^{-1}\\ &\quad\times(e_{1b}+e_{2b})\\ &\quad\times\left[\frac{1}{n}\sum_{b'=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b'});\hat\theta\}\right]^{-1}\frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\hat\theta\},\end{aligned}$$

with

$$e_{1b} = \frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde F(X_i^{b});\tilde\theta_{-b}\},$$

and

$$e_{2b} = \frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde F(X_i^{b});\tilde\theta_{-b}\} - \frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde F(X_i^{b});\hat\theta\}.$$

Lemma 1 implies that

$$\begin{aligned}\sup_{1\le b\le B}\|W_1\| &= \sup_{1\le b\le B}\left\|\sum_{b=1}^{B}\sum_{i=1}^{m}(\hat\theta-\hat\theta_{-b})^T l_\theta\{\tilde F(X_i^{b});\hat\theta\}(\hat\theta-\hat\theta_{-b})\right\|\\ &= o_p(n^{-\frac{1}{2}})\sup_{1\le b\le B}\sup_{\theta\in\Theta}\|l_\theta\{\tilde F(X_i^{b});\theta\}\|\\ &= o_p(n^{-\frac{1}{2}})O_p(1)\\ &= o_p(n^{-\frac{1}{2}}).\end{aligned}$$

We now prove that $W_2 = o_p(n^{-\frac{1}{2}})$. By conditions (A1) and (C1),

$$\begin{aligned}\sup_{1\le b\le B}\|e_{1b}\| &\le \frac{m}{n}\sup_{1\le b\le B}\sup_{\theta\in\mathcal N(\theta^*)}\left\|l_\theta\{\tilde F(X_i^{b});\theta\}\right\| = o_p(n^{-\frac{3}{4}})O_p(1)\\ &= o_p(n^{-\frac{3}{4}}).\end{aligned}$$

Expanding $l_{\theta\theta}\{\tilde{F}(X_i^b);\tilde{\theta}_{-b}\}$ around $\hat{\theta}$ leads to, under condition (B3),

$$\begin{aligned}\|e_{2b}\| &= \left\|\frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^b);\tilde{\theta}_{-b}\} - \frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m} l_{\theta\theta}\{\tilde{F}(X_i^b);\hat{\theta}\}\right\| \\ &\le \left\|\frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m}\sum_{k,l=1}^{p}\frac{\partial}{\partial\theta} l_{\theta\theta}\{\tilde{F}(X_i^b);\tilde{\tilde{\theta}}_{-b}\}_{kl}\right\| \sup_{1\le b\le B}\|\hat{\theta}-\tilde{\theta}_{-b}\| \\ &= o_p(n^{-\frac{3}{4}})O_p(1) = o_p(n^{-\frac{3}{4}})\end{aligned}$$

where $\tilde{\tilde{\theta}}_{-b}$ lies between $\hat{\theta}$ and $\hat{\theta}_{-b}$. Therefore,

$$\begin{aligned}&\sup_{1\le b\le B}\|W_2\| \\ &\le n\left\|\frac{1}{n}\sum_{b=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\}\right\| \\ &\quad\times \sup_{1\le b\le B}\left\|\left[\frac{1}{n}\sum_{b'=1,b'\ne b}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^{b'});\tilde{\theta}_{-b}\}\right]^{-1}\right\| \\ &\quad\times \sup_{1\le b\le B}\|(e_{1b}+e_{2b})\|\left\|\left[\frac{1}{n}\sum_{b'=1}^{B}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^{q});\hat{\theta}\}\right]^{-1}\right\| \\ &\quad\times \sup_{1\le b\le B}\left\|\frac{1}{n}\sum_{i=1}^{m} l_\theta\{\tilde{F}(X_i^b);\hat{\theta}\}\right\| \\ &= n\times O_p(1)O_p(1)o_p(n^{-\frac{3}{4}})O_p(1)o_p(n^{-\frac{3}{4}}) \\ &= o_p(n^{-\frac{1}{2}}).\end{aligned}$$

In summary, we prove that

$$T_n(m) - R_n = o_p(n^{-\frac{1}{2}}).$$

**Proof of Theorem 3.** Under assumption (D1), Lemma 3.10.11 of van der Vaart and Wellner (1996) implies that the likelihood ratio process of $P_n^{C_1,\delta}$ over $P_0$ has the following asymptotic representation:

$$\begin{aligned}\Lambda_n &= \sum_{t=1}^{n}\log\frac{dP_n^{C_1,\delta}}{dP_0}(X_t) \\ &= \frac{\delta}{\sqrt{n}}\sum_{t=1}^{n} g(U_t) - \frac{\delta^2}{2n}\sum_{t=1}^{n} g(U_t)^2 + o_p(1), \qquad (19)\end{aligned}$$

where $g(U_t) = \frac{c_1(U_t)-c_0(U_t;\theta_0)}{c_0(U_t;\theta_0)}$ and $U_t = F(X_t)$.

Thus, under assumption (B1)–(B3) and (E1), the likelihood ratio process $\Lambda_n$ converges to a normal distribution with mean $-\frac{\delta^2}{2}\Gamma_0$ and variance $\delta^2\sigma_g^2$, where $\Gamma_0 = \mathbb{E}_{c_0}\left[g\{F(X_1)\}^2\right]$ and $\sigma_g^2 = Var_{c_0}\left[g\{F(X_1)\}\right]$.

According to Chen and Fan (2005), the PMLE $\hat{\theta}$ has the following asymptotic representation:

$$\begin{aligned}\hat{\theta}-\theta_0 &= \frac{1}{n}\sum_{t=1}^{n} S^{-1}\left\{l_\theta(U_t;\theta_0) + \sum_{j=1}^{d} D_j(U_t;\theta_0)\right\} + o_p(n^{-1/2}) \\ &\triangleq \frac{1}{n}\sum_{t=1}^{n}\phi_\theta(U_t;\theta_0) + o_p(n^{-1/2}), \qquad (20)\end{aligned}$$

where $S = \mathbb{E}_{c_0}\left[l_{\theta\theta}\{\tilde{F}(X_t);\tilde{\theta}\}\right]$ and $D_j(U_t;\theta_0) = \mathbb{E}_{c_0}[l_{\theta,j}(U_t;\theta_0)\{I\left(U_{tj}\le U_{sj}\right) - U_{sj}\}|U_{tj}]$.

By condition (B1), expanding $l_{\theta\theta}\left\{\tilde{F}(X_t);\hat{\theta}\right\}$ around $\theta_0$ leads to

$$\begin{aligned}\hat{S}(\hat{\theta}) &= -\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left\{\tilde{F}(X_t);\hat{\theta}\right\} \\ &= -\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left\{\tilde{F}(X_t);\theta_0\right\} - \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{p}\frac{\partial l_{\theta\theta}\{\tilde{F}(X_t);\theta_0\}}{\partial\theta_j} \\ &\quad\times(\hat{\theta}_j - [\theta_0]_j) + o_p(n^{-1/2}).\end{aligned}$$

By condition (B3) and Eq. (20), we have

$$\begin{aligned}&\frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d}\frac{\partial l_{\theta\theta}\{\tilde{F}(X_t);\theta_0\}}{\partial\theta_j}(\hat{\theta}_j - [\theta_0]_j) \\ &= \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} M_j^s\phi_\theta(U_t;\theta_0)_j + o_p(n^{-1/2}),\end{aligned}$$

where

$$M_j^s = \mathbb{E}_{c_0}\left[\frac{\partial}{\partial\theta_j} l_{\theta\theta}\left\{\tilde{F}(X_t);\theta_0\right\}\right], \quad j=1,\ldots,p.$$

By condition (B2), applying functional Taylor expansion in the direction of $dF = \tilde{F} - F$ (van der Vaart, 2000), we have

$$\begin{aligned}&\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left\{\tilde{F}(X_t);\theta_0\right\} \\ &= \frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}(U_t;\theta_0) + \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}(U_t;\theta_0)\left\{\tilde{F}_j(X_{tj}) - F_j(X_{tj})\right\} \\ &\quad + o_p(n^{-1/2}) \\ &= \frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}(U_t;\theta_0) + \frac{1}{n(n+1)}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}(U_t;\theta_0) \\ &\quad\times\left\{I\left(X_{sj}\le X_{tj}\right) - F_j(X_{tj})\right\} + o_p(n^{-1/2}).\end{aligned}$$

Applying the standard arguments in the theorem of $U$-statistics (Serfling, 2009), we have

$$\begin{aligned}&\frac{1}{n(n+1)}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}(U_t;\theta_0)\left\{I\left(X_{sj}\le X_{tj}\right) - F_j(X_{tj})\right\} \\ &= \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d}\mathbb{E}_{c_0}\left[l_{\theta\theta,j}(U_t;\theta_0)\left\{I\left(U_{sj}\le U_{tj}\right) - U_{tj}\right\}|U_{tj}\right] \\ &\quad + o_p(n^{-1/2}).\end{aligned}$$

Combining the arguments above, we have

$$\begin{aligned}&\hat{S}(\hat{\theta}) \\ &= -\frac{1}{n}\sum_{t=1}^{n}\left[l_{\theta\theta}\{U_t;\theta_0\} + \sum_{j=1}^{d} M_j^s\phi_\theta(U_t;\theta_0)_j\right. \\ &\quad\left. + \sum_{j=1}^{d}\mathbb{E}_{c_0}\left\{l_{\theta\theta,j}\{U_t;\theta_0\}\left[I\left\{U_{sj}\le U_{tj}\right\} - U_{tj}\right]|U_{tj}\right\}\right] \\ &\quad + o_p(n^{-1/2}) \\ &\triangleq -\frac{1}{n}\sum_{t=1}^{n}\psi^S(U_t;\theta) + o_p(n^{-1/2}).\end{aligned}$$

By conditions (B1)–(B3), using the above similar arguments, we also can expand $\hat{V}(\hat{\theta})$ as follows:

$$\begin{aligned}\hat{V}(\hat{\theta}) &= \frac{1}{n}\sum_{t=1}^{n} l_\theta(U_t;\theta_0)\, l_\theta(U_t;\theta_0)^T + \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{p} M_j^v\phi_\theta(U_t;\theta_0)_j \\ &\quad + \frac{2}{n}\sum_{t=1}^{n}\sum_{j=1}^{d}\mathbb{E}_{c_0}\left[l_\theta(U_t;\theta_0)\, l_{\theta,j}(U_t;\theta_0)^T\right.\end{aligned}$$

$$\times \left\{I\left(U_{sj} \le U_{tj}\right) - U_{tj}\right\} |U_{tj}\Big] + o_p(n^{-1/2})$$
$$\triangleq \frac{1}{n}\sum_{t=1}^{n} \psi^{v}(U_t;\theta) + o_p(n^{-1/2}),$$

where

$$M_j^v = \mathbb{E}_{c_0}\left[l_\theta\left\{\tilde F(X_t);\theta_0\right\}\frac{\partial}{\partial\theta_j} l_\theta\left\{\tilde F(X_t);\theta_0\right\}^T\right], \quad j = 1,\ldots,p.$$

Therefore, employing similar arguments to those in the proof of Theorem 2(i), we can rewrite $\sqrt{n}\,(R_n - p)$ as follows:

$$\begin{aligned}\sqrt{n}\,(R_n - p) &= \sqrt{n}\hat S(\hat\theta)^{-1}\left(\hat V(\hat\theta) - \hat S(\hat\theta)\right)\\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{i,j=1}^{p}\Big(\left[S^{-1}\right]_{ij}[\psi^{v}\{F(X_t);\theta_0\}\\ &\quad + \psi^{s}\{F(X_t);\theta_0\}]_{ji}\Big) + o_p(1)\\ &\triangleq \frac{1}{\sqrt{n}}\sum_{t=1}^{n} W(X_t) + o_p(1).\end{aligned} \tag{21}$$

Applying the multivariate central limit theorem, the vector $\left\{\sqrt{n}(R_n - p), \Lambda_n\right\}^T$ converges under $P_0$ to a bivariate distribution with mean vector $(0, -\frac{\delta^2}{2}\Gamma_0)^T$ and covariance matrix

$$\begin{pmatrix}\bar\sigma_R^2 & \delta m(c_0,c_1)\\ \delta m(c_0,c_1) & \delta^2\sigma_g^2\end{pmatrix}$$

where

$$\bar\sigma_R^2 = Var_{c_0}\left\{\sum_{i,j=1}^{p}\left[S(\theta_0)^{-1}\right]_{ij}\left[\psi^{v}(U_t;\theta_0) + \psi^{s}(U_t;\theta_0)\right]_{ji}\right\}, \tag{22}$$

and

$$m(c_0,c_1) = \mathbb{E}_{c_0}\left\{W(X_t)g(F(X_t);\theta_0)\right\}.$$

By Le Cam's third lemma (van der Vaart and Wellner, 1996), we have $\sqrt{n}(R_n - p)$ is asymptotically normal with mean $\delta m(c_0,c_1)$ and variance $\bar\sigma_R^2$ under the contiguous sequence $P_n^{C_1,\delta}$.

**Proof of Theorem 4(i).** First, we prove $\tilde S(\hat\theta) \xrightarrow{pr} S(\theta^*)$, since

$$\begin{aligned}\tilde S(\hat\theta) &= -\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left\{\check F(\tilde\epsilon_t);\hat\theta\right\}\\ &= -\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left(U_t;\hat\theta\right)\\ &\quad -\frac{1}{n}\sum_{t=1}^{n}\left[l_{\theta\theta}\left\{\check F(\tilde\epsilon_t);\hat\theta\right\} - l_{\theta\theta}\left(U_t;\hat\theta\right)\right].\end{aligned}$$

By condition (A1), the application of lemma A.1(1) in Chen and Fan (2006), leads to $\frac{1}{n}\sum_{t=1}^{n}\left[l_{\theta\theta}\left\{\check F(\tilde\epsilon_t);\hat\theta\right\} - l_{\theta\theta}\left(U_t;\hat\theta\right)\right] = o_p(1)$. By the uniform law of large numbers, we have $-\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left(U_t;\hat\theta\right) \xrightarrow{pr} S(\theta^*)$. Hence, $\tilde S(\hat\theta) \xrightarrow{pr} S(\theta^*)$.

By the above similar arguments, it is easy to show $\tilde V(\hat\theta) \xrightarrow{pr} V(\theta^*)$. Thus, the result holds by applying condition (A2) and the Slutsky's Theorem.

**Proof of Theorem 4(ii).** We begin with the (functional) Taylor expansion of the negative sensitivity matrix $\tilde S(\hat\theta)$:

$$\begin{aligned}\tilde S(\hat\theta) &= -\frac{1}{n}\sum_{t=1}^{n}\left\{\psi_{S0}\left(U_t;\theta^*\right) + \psi_{S1}\left(U_t;\theta^*\right) + \psi_{S2}\left(U_t;\theta^*\right)\right\}\\ &\quad + o_p(n^{-1/2}),\end{aligned} \tag{23}$$

where $\psi_{S0}\left(U_t;\theta^*\right) = l_{\theta\theta}(U_t;\theta^*)$, $\psi_{S1}\left(U_t;\theta^*\right) = \sum_{j=1}^{d}\mathbb{E}_0\left[l_{\theta\theta,j}\left(U_s;\theta^*\right)\left\{I\left(U_{tj} \le U_{sj}\right) - U_{sj}\right\}|U_{tj}\right]$ and $\psi_{S2}\left(U_t;\theta^*\right) = \sum_{i=1}^{p}\mathbb{E}_0\left\{\frac{\partial l_{\theta\theta}(U_t;\theta^*)}{\partial\theta_i}\right\}\phi_\theta\left(U_t;\theta^*\right)_i$. Note that $\psi_{S1}(\cdot;\cdot)$ and $\psi_{S2}(\cdot;\cdot)$ account for the errors of estimation with respect to parameter $\theta$ distribution function $F(\cdot)$.

Similarly, the variability matrix $\tilde V(\hat\theta)$ can be expanded as

$$\begin{aligned}\tilde V(\hat\theta) &= \frac{1}{n}\sum_{t=1}^{n}\left\{\psi_{V0}\left(U_t;\theta^*\right) + \psi_{V1}\left(U_t;\theta^*\right) + \psi_{V2}\left(U_t;\theta^*\right)\right\}\\ &\quad + o_p(n^{-1/2}),\end{aligned} \tag{24}$$

where $\psi_{V0}\left(U_t;\theta^*\right) = l_\theta(U_t;\theta^*)l_\theta(U_t;\theta^*)^T$, and

$$\begin{aligned}\psi_{V1}\left(U_t;\theta^*\right) &= 2\sum_{j=1}^{d}\mathbb{E}_0\Big[l_\theta\left(U_s;\theta^*\right)l_{\theta,j}\left(U_s;\theta^*\right)^T\\ &\quad\times\left\{I\left(U_{tj} \le U_{sj}\right) - U_{sj}\right\}|U_{tj}\Big],\end{aligned}$$

and $\psi_{V2}$ is a $p \times p$ matrix with the $(i,j)$th element given by

$$2\sum_{k=1}^{p}\mathbb{E}_0\left\{l_\theta\left(U_t;\theta^*\right)_i l_{\theta\theta}\left(U_t;\theta\right)_{jk}\right\}\phi_\theta\left(U_t;\theta^*\right)_k.$$

Also, $\psi_{V1}(\cdot;\cdot)$ and $\psi_{V2}(\cdot;\cdot)$ account for the error of estimation of the finite-dimension parameter $\theta$ and infinite-dimension parameter $F(\cdot)$.

By the asymptotic expansion of $\tilde S(\hat\theta)$ in Eq. (23) and $\tilde V(\hat\theta)$ in Eq. (24), applying similar arguments as that in the proof of Theorem 2(i), we have

$$\sqrt{n}\left(\tilde R_n - p\right) \xrightarrow{d} N(0,\tilde\sigma_R^2), \quad \text{as } n \to \infty$$

where

$$\tilde\sigma_R^2 = Var\left[\sum_{i,j=1}^{p}\left[S(\theta^*)^{-1}\right]_{ij}\left\{\psi_V(U_t;\theta^*) + \psi_S(U_t;\theta)\right\}_{ji}\right], \tag{25}$$

where $\psi_V(\cdot;\cdot) = \sum_{h=0}^{3}\psi_{Vh}(\cdot;\cdot)$, and $\psi_S(\cdot;\cdot) = \sum_{h=0}^{3}\psi_{Sh}(\cdot;\cdot)$.

It is worth pointing out that asymptotically, the estimation error of $\eta$ has no impact on $\tilde S(\theta)$ and $\tilde V(\hat\theta)$, and hence, the behavior of $\tilde R_n$ is the same as if the residuals $\{\tilde\epsilon_t, t = 1,\ldots,n\}$ are observed.

Now we provide the proof of Eq. (23), and the proof of Eq. (24) are similar and hence omitted. By its definition, we have

$$\begin{aligned}\tilde S(\hat\theta) &= -\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left\{\check F(\tilde\epsilon_t);\hat\theta\right\}\\ &= -\frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left\{\check F(\tilde\epsilon_t);\theta^*\right\} - \frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{p}\frac{\partial l_{\theta\theta}\left(U_t;\theta^*\right)}{\partial\theta_i}\\ &\quad\times\left(\hat\theta_i - \theta_i^*\right)\\ &\quad -\frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{p}\left[\frac{\partial l_{\theta\theta}\left\{\check F(\tilde\epsilon_t);\tilde\theta\right\}}{\partial\theta_i} - \frac{\partial l_{\theta\theta}\left(U_t;\theta^*\right)}{\partial\theta_i}\right]\end{aligned}$$

$$\times\left(\hat{\theta}_i-\theta_i^*\right)$$
$$\triangleq -S_I - S_{II} - S_{III},$$

where $\tilde{\theta}$ lies between $\hat{\theta}$ and $\theta^*$.

Consider the first term $S_I$. By conditions (B2) and (E1)–(E4), an application of Lemma A.1(2) in Chen and Fan (2006) leads to

$$S_I = \frac{1}{n}\sum_{t=1}^{n} l_{\theta\theta}\left(U_t;\theta^*\right) + \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{\check{F}_j(\tilde{\epsilon}_{tj}) - F_j(\epsilon_{tj})\right\} + o_p(n^{-1/2}).$$

By conditions (E1)–(E4) and Eq. (12), we have

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{\check{F}_j(\tilde{\epsilon}_{tj}) - F_j(\epsilon_{tj})\right\}\\
&= \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{\tilde{F}_j(\tilde{\epsilon}_{tj}) - F_j(\tilde{\epsilon}_{tj})\right.\\
&\quad \left. + f_j(\tilde{\epsilon}_{tj})C(\tilde{\epsilon}_{tj})(\hat{\eta}-\eta_0) + F_j(\tilde{\epsilon}_{tj}) - F_j(\epsilon_{tj}) + o_p(n^{-1/2})\right\}\\
&= \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{\tilde{F}_j(\epsilon_{tj}) - F_j(\epsilon_{tj})\right\}\\
&\quad + \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right) f_j(\epsilon_{tj})\left[\omega(\epsilon_{tj}) - \left\{\Sigma_{tj}^{-1/2}(\eta^0)\dot{\mu}_{tj}(\eta_1^0)\right.\right.\\
&\quad \left.\left. + \epsilon_{tj}\frac{1}{2}\Sigma_{tj}^{-1}(\eta^0)\dot{\Sigma}_{tj}(\eta^0)\right\}\right]^T(\hat{\eta}-\eta_0)\\
&\quad + o_p(n^{-1/2}).
\end{aligned}$$

Since $\omega(x) = \mathbb{E}_0\left(\omega_1 + \frac{x}{2}\omega_2\right)$, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right) f_j(\epsilon_{tj})\left[\omega(\epsilon_{tj}) - \left\{\Sigma_{tj}^{-1/2}(\eta^0)\dot{\mu}_{tj}(\eta_1^0)\right.\right.\\
&\quad \left.\left. + \frac{\epsilon_{tj}}{2}\Sigma_{tj}^{-1}(\eta^0)\dot{\Sigma}_{tj}(\eta^0)\right\}\right]^T(\hat{\eta}-\eta_0)\\
&= o_p(n^{-1/2}).
\end{aligned}$$

That is, the estimation error of $\eta$ is asymptotically ignorable. Consequently,

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{\check{F}_j(\tilde{\epsilon}_{tj}) - F_j(\epsilon_{tj})\right\}\\
&= \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{\tilde{F}_j(\epsilon_{tj}) - F_j(\epsilon_{tj})\right\} + o_p(n^{-1/2})\\
&= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{I\left(\epsilon_{sj}\le\epsilon_{tj}\right) - F_j(\epsilon_{tj})\right\}\\
&\quad + o_p(n^{-1/2}).
\end{aligned}$$

Under condition (B2), employing the standard arguments in the theorem of $U$-statistics (Chapter 5, Serfling (2009) or Lemma A.2 in Chen and Fan (2006)), we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d} l_{\theta\theta,j}\left(U_t;\theta^*\right)\left\{\check{F}_j(\tilde{\epsilon}_{tj}) - F_j(\epsilon_{tj})\right\}\\
&= \frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{d}\mathbb{E}_0\left[l_{\theta\theta,j}\left(U_s;\theta^*\right)\left\{I\left(U_{tj}\le U_{sj}\right) - U_{sj}\right\}|U_{tj}\right]\\
&\triangleq \frac{1}{n}\sum_{t=1}^{n}\psi_{S1}\left(U_t;\theta^*\right).
\end{aligned}$$

Now consider the second term $S_{II}$, under condition (B3), Eq. (13), applying Lemma A.1(2) in Chen and Fan (2006) gives

$$\frac{1}{n}\sum_{t=1}^{n}\sum_{i=1}^{p}\frac{\partial l_{\theta\theta}\left(U_t;\theta^*\right)}{\partial\theta_i}\left(\hat{\theta}_i-\theta_i^*\right) = \frac{1}{n}\sum_{t=1}^{n}\psi_{S2}\left(U_t;\theta^*\right) + o_p(n^{-1/2})$$

where

$$\psi_{S2}\left(U_t;\theta^*\right) = \sum_{i=1}^{p}\mathbb{E}_0\left\{\frac{\partial l_{\theta\theta}\left(U_t;\theta^*\right)}{\partial\theta_i}\right\}\phi_\theta\left(U_t;\theta^*\right)_i.$$

Therefore, we can rewrite the negative sensitivity matrix $\tilde{S}(\hat{\theta})$ as follows:

$$\begin{aligned}
\tilde{S}(\hat{\theta}) &= -\frac{1}{n}\sum_{t=1}^{n}\left\{\psi_{S0}\left(U_t;\theta^*\right) + \psi_{S1}\left(U_t;\theta^*\right) + \psi_{S2}\left(U_t;\theta^*\right)\right\}\\
&\quad + o_p(n^{-1/2})\\
&\triangleq -\frac{1}{n}\sum_{t=1}^{n}\psi_S\left(U_t;\theta^*\right) + o_p(n^{-1/2}).
\end{aligned}$$

Applying the above similar arguments, an expansion of the variability matrix $\tilde{V}(\hat{\theta})$ may be written as follows:

$$\begin{aligned}
\tilde{V}(\hat{\theta}) &= \frac{1}{n}\sum_{t=1}^{n}\left\{\psi_{V0}\left(U_t;\theta^*\right) + \psi_{V1}\left(U_t;\theta^*\right) + \psi_{V2}\left(U_t;\theta^*\right)\right\}\\
&\quad + o_p(n^{-1/2})\\
&\triangleq \frac{1}{n}\sum_{t=1}^{n}\psi_V\left(U_t;\theta^*\right) + o_p(n^{-1/2}),
\end{aligned}$$

where $\psi_{V0}\left(U_t;\theta^*\right) = l_\theta\left(U_t;\theta\right) l_\theta\left(U_t;\theta\right)^T$, and

$$\begin{aligned}
\psi_{V1}\left(U_t;\theta^*\right) &= 2\sum_{j=1}^{d}\mathbb{E}_0\left[l_\theta\left\{U_s;\theta^*\right\} l_{\theta,j}\left\{U_s;\theta^*\right\}^T\right.\\
&\quad \left.\times\left[I\left\{U_{tj}\le U_{sj}\right\} - U_{sj}\right]|U_{tj}\right],
\end{aligned}$$

and $\psi_{V2}\left(U_t;\theta^*\right)$ is a $p\times p$ matrix with the $(i,j)$th element

$$\begin{aligned}
\left[\psi_{V2}\left(U_t;\theta^*\right)\right]_{ij} &= 2\sum_{k=1}^{p}\mathbb{E}_0\left[l_\theta\left\{F(\epsilon_1);\theta^*\right\}_i l_{\theta\theta}\left\{F(\epsilon_1);\theta\right\}_{jk}\right]\\
&\quad \times\phi_\theta\left\{F\left(\epsilon_t\right);\theta^*\right\}_k.
\end{aligned}$$

**Proof of Theorem 4(iii).** From Proposition 3.2 in Chen and Fan (2006), the asymptotic distribution of $\hat{\theta}$ is not affected by the estimation error of nuisance parameter $\eta$. Thus, under assumption (C1), Lemma 1 still holds with the standardized residuals $\{\tilde{\epsilon}_t, t=1,\ldots,n\}$, which are only dependent on parameter $\eta$. That is, $\sup_{1\le b\le B}\left\|\hat{\theta}-\hat{\theta}_{-b}\right\| = o_p(n^{-3/4})$. Going through similar arguments in the proof of Theorem 1(ii) leads to $\tilde{T}_n(m) - \tilde{R}_n = o_p(n^{-1/2})$.

## Appendix B

This Appendix describes the test proposed by Genest et al. (2009) and Scaillet (2007), termed in short as $S_n$ and $J_n$ tests respectively.

The $S_n$ test statistic is a Cramér–von Mises statistic based on Rosenblatt's transform (Rosenblatt, 1952), defined by

$$S_n = n \int_{[0,1]^d} \{D_n(u) - C_\perp(u)\}^2 \, du$$
$$= n/3^d - 1/2^{d-1} \sum_{t=1}^{n} \prod_{k=1}^{d} \left(1 - E_{tk}^2\right)$$
$$+ 1/n \sum_{t=1}^{n} \sum_{s=1}^{n} \prod_{k=1}^{d} \{1 - \max(E_{tk}, E_{sk})\},$$

where $E_t = (E_{t,1}, \ldots, E_{t,d})^T, t = 1, \ldots, n$, are pseudo observations derived from the following Rosenblatt's transform:

$$E_{tk} = \frac{\partial^{k-1} C(U_{t,1}, \ldots, U_{t,k}, 1, \ldots, 1)/\partial U_{t,1} \ldots \partial U_{t,k-1}}{\partial^{k-1} C(U_{t,1}, \ldots, U_{t,k-1}, 1, \ldots, 1)/\partial U_{t,1} \ldots \partial U_{t,k-1}},$$
$$k = 1, 2, \ldots, d,$$

and $D_n(u) = \frac{1}{n} \sum_{t=1}^{n} I(E_t \leq u)$ is the $d$-dimensional empirical distribution function based on the pseudo observations $E_1, \ldots, E_n$, and $C_\perp(u) = u_1 \times u_2 \times \cdots \times u_d$ is the $d$-dimensional independent copula.

The Scaillet test is a kernel-based goodness-of-fit test with a fixed smoothing parameter. For the copula density $c(u; \theta)$, its kernel estimator is given by

$$\hat{c}(u) = \frac{1}{n} \sum_{t=1}^{n} K_H[u - \{\tilde{F}_1(X_{t1}), \ldots, \tilde{F}_d(X_{td})\}^T]$$

where according to Scaillet (2007) the function $K_H(y) = K(H^{-1}y)/det(H)$ is a $d$-dimensional kernel, and $H$ is a nonsingular, symmetric matrix of smoothing parameters. Following Scaillet (2007), in our simulation study $K_H(y)$ is taken to be a bivariate quadratic product kernel with $H = 2.6073n^{-1/6}\hat{\Sigma}^{1/2}$, specified in terms of the Scott's rule of thumb with $\hat{\Sigma}$ being an estimate of the sample covariance matrix of the vector of the transformed variables $\{\tilde{F}_1(X_{tk}), \ldots, \tilde{F}_d(X_{tk})\}$, $t = 1, \ldots, n$. We also scale the smoothing parameters by $\delta = 0.5$, as the resulting estimator provides on average the best results in Scaillet (2007). Thus, the test statistics used in our simulation study takes the following form:

$$J_n = \int_{[0,1]^d} \{\hat{c}(u) - K_H * c(u; \hat{\theta})\} w(u) du \tag{26}$$

where "$*$" denotes the operation of convolution and $w$ is a certain weight function. Using the computing package from Dr. Scaillet, here we utilized a bivariate Gauss–Legendre quadrature method with $12 \times 12$ grids to compute the integral in (26) numerically.

## Appendix C

This appendix is devoted to the results of the Simulation and Empirical Studies.

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