feature article

Crumpling, buckling, and cracking: Elasticity of thin sheets

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Deforming thin surfaces creates a rich set of structural and dynamical problems combining physics and geometry.

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Take a thin sheet of paper, plastic, or rubber. Roll, crumple, stretch, or tear it. Sometimes the sheet can spring right back to its original form, as with a roll of paper, while other times it is permanently changed, as with torn plastic. Much can be learned from such everyday acts. The subtle mathematics of differential geometry is needed to make sense of the deformed sheets, and along the way it offers insights into issues such as the shapes of flowers and the speed of earthquakes.

History

Start the study of surfaces with Carl Friedrich Gauss. Chiefly known today for his abstract mathematics, Gauss devoted years of his life to practical pursuits, from contributing to the invention of the telegraph to mapping the Kingdom of Hanover.[1] His most famous geographical measurement was of a triangle formed by the shortest paths between the three mountain peaks shown in figure 1. Adrien-Marie Legendre had previously established that for triangles drawn on the surface of a sphere, the sum of the interior angles exceeds π by A/R^2, where A is the area of the triangle and R is the radius of the sphere. In 1827 Gauss completed 40 pages of stupendous calculations generalizing Legendre's result to arbitrary curved surfaces, and found a formula giving the specific amounts to subtract from the angles of a triangle on a curved surface to bring the sum of the angles to π.

For a triangle on a perfect sphere, Gauss's formula says that the same amount ($A/3R^2$) should be subtracted from each angle. Earth is not quite a perfect sphere, but it is close enough that Gauss's correction to Legendre's result for each of the three mountain-straddling angles was less than one thousandth of an arcsecond. Gauss dryly noted at the time

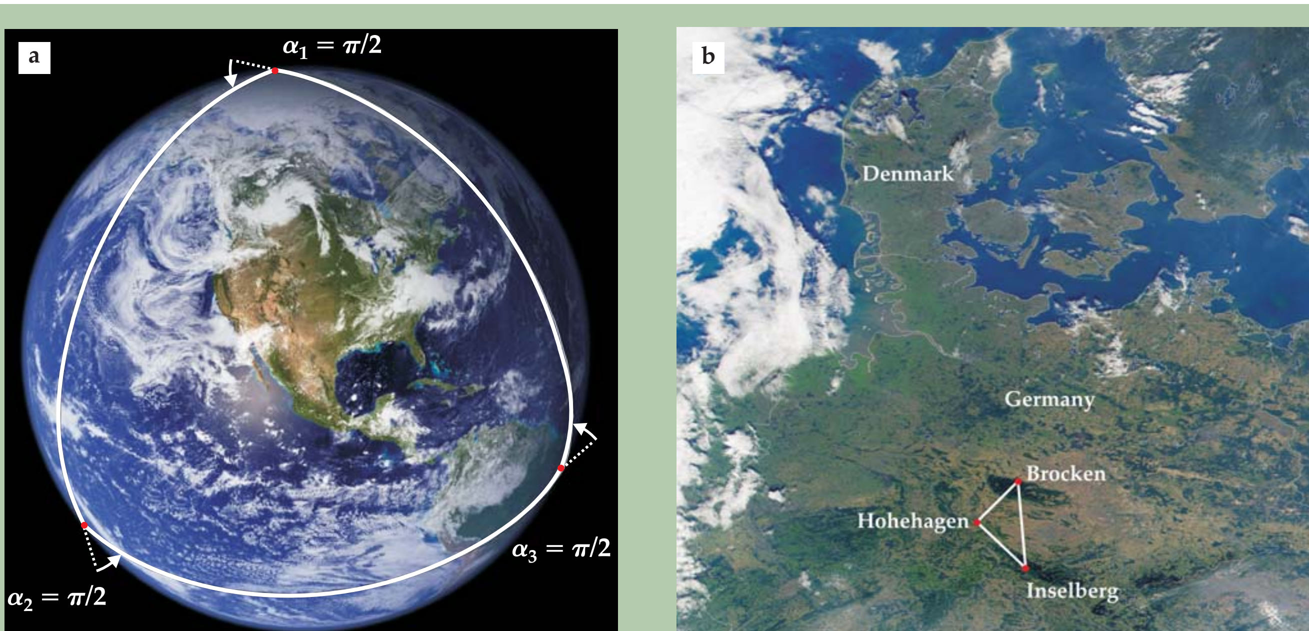


Figure 1. Triangles on spheres have angles that add up to more than π radians. **(a)** In the large triangle, the sum of angles is $3\pi/2$ radians. **(b)** The largest triangle on Earth's surface measured by Gauss; the sum of interior angles exceeds π by 7.2×10^{-5} radians. (Earth images courtesy of NASA.)

Figure 2. Patterns and dynamics of thin sheets. Each experiment can easily be reproduced at home or with students. **(a)** A plastic sheet ripping apart is viewed with polarized light. **(b)** A buckling pattern forms in the wake of the tear. **(c)** A plastic sheet placed in a cup and poked with a pencil creates a *d*-cone singularity, as pointed out by Enrique Cerda and L. Mahadevan.[6] **(d)** Crumpled mylar contains an ensemble of *d*-cone singularities, connected by ridges. (From T. Witten, *Rev. Mod. Phys.* **79**, in press.) **(e)** A multiple-exposure image captures a balloon breaking. The crack follows an oscillating path and **(f)** leaves behind a wavy pattern.

that the difference was “insensible,” but that did not stop him from creating the ideas of the metric and Gaussian curvature—two concepts at the core of modern differential geometry—and proving the celebrated Theorema Egregium. (Some believe that Gauss performed his mountaintop measurement to check whether three-dimensional space itself is Euclidean, but in the paper he published at the time he did the work, he made no reference to any such question.)

Most physicists first learn about metrics and curved spaces in the context of general relativity, but differential geometry has long been deeply entwined with the theory of elasticity. The bending of space is quite a practical topic when the space is a 2D material surface. Applications of differential geometry include the engineering of structures, such as airplanes, built from thin surfaces. For example, when Alexander Alexandrov and Alexei Pogorelov proved in the 1940s that closed convex surfaces are uniquely specified by their metrics, the result explained why ping-pong balls and airplane fuselages are not floppy. And when August Föppl and Theodore von Kármán determined the energies of nonlinearly deformed plates, they helped to figure out how much airplane wings should flex in flight.

Many of the physics experiments on bending and stretching of thin surfaces are quite recent, although not for technical reasons; the apparatus needed to do the experiments is usually modest. What has developed is partly a matter of style: The recent work has a whimsical character, exploring patterns and dynamics of thin surfaces for their own sake. Maybe it is becoming increasingly acceptable just to be curious.

Buckling and crumpling

Thin elastic sheets are special. David Nelson was one of the first physicists to see why. Like thin elastic rods, they are floppy and can bend into hosts of different shapes, but bending and stretching them create more complex patterns than are possible with rods. Nelson initiated studies of the statistical mechanics of flexible sheets as a generalization of polymer physics; among many other things, he found a phase transition between flat and highly crumpled surfaces.[2]

Sheets find many tricks to play when changes to their internal structure force them to buckle out of the plane, a phenomenon similar to the wrinkling of leaves, flowers, and human skin.[3] We noticed some of their strange properties by looking at ripped cookie wrappers,[4] and our experiment is easily repeatable at home. Take any thin sheet of pliable plastic—a garbage bag will do—and cut out a square a few centimeters on a side. Make a thin initial slit with scissors, and then tear the sheet apart. Notice how the plastic looks smooth and featureless at the tip of the tear (figure 2a), but away from the tip where the stress is relaxed, buckles become visible (figure 2b). In very thin plastic, as many as six generations of buckles upon buckles can appear. The shapes formed in ripped plastic bear a great resemblance to the edges of leaves and flowers. The similarity is not accidental. There is an underlying geometrical explanation. In fact, ripping is just the means to a geometrical end.

When a thin piece of plastic is ripped, the material close to the new edge is stretched irreversibly, while far away from the edge it remains undeformed. If the material were to remain confined to the plane, it would have to undergo tremendous strain by compressing and expanding to differing degrees across the sheet. It is far more energetically favorable for the sheet to buckle out of the plane. One way to understand why is to pretend that the plastic is made up of a network of masses and springs, as shown in figure 3. Initially, all the springs are the same length and are in equilibrium. Imagine that all the horizontal springs in the bottom row of the network are permanently deformed and acquire a new equilibrium length that is 50% greater than before the deformation. The horizontal springs one row up are also deformed, but not as much; their new equilibrium length is 40% greater than before the deformation. The next row of springs is deformed even less, and so on up to the top of the network, which is almost completely unchanged.

A long strip of material deformed in that way is essentially guaranteed to buckle. It is favorable for each material point to lie at a specific distance from each of its horizontal and vertical neighbors. If the sheet remains flat, adjacent horizontal rows must slide past one another, stretching the vertical connecting springs more and more for longer and longer sheets. Something has to give, and what gives is the planar constraint of an unbuckled structure.

From a formal point of view, assigning a new collection of equilibrium distances to nearby material points is equivalent to specifying a new target metric; see the details in box 1. In the target metric tensor for the network shown in figure 3, only the horizontal component g_{xx} is different from 1, and it depends only on the vertical position: $g_{xx} = g_{xx}(y)$. We often assume that once a sheet relaxes to equilibrium, its actual metric is equal to its target metric, to a first approximation.

For almost any decreasing functional form of the target metric component $g_{xx}(y)$ of a long sheet, the sheet will spontaneously form the ramified structures appearing in figure 2b. A way to show that buckled structures are necessary is to employ the Theorema Egregium, the most famous result from Gauss's 1827 paper, which expresses the Gaussian curvature K of a surface in terms of the metric. In our case,

$$K(y) = -\frac{1}{\sqrt{g_{xx}}}\frac{d^2}{dy^2}\sqrt{g_{xx}}\,. \tag{1}$$

If $\sqrt{g_{xx}}$ decreases in a convex fashion, its second derivative is positive, so the Gaussian curvature must be negative, which means that at every point the surface resembles a saddle, as shown in box 2. The only way that every part of a surface can look like a saddle is if the surface buckles.

Sheets can form fascinating patterns even when they are flat almost everywhere. Origami provides one set of examples, but even if you lack the dexterity to fold a Kawasaki rose, you can still do some interesting home experiments by taking sheets of paper and simply crumpling them. Martine Ben Amar and Yves Pomeau realized that a fundamental singularity of crumpled paper, called a d-cone, is generated by taking an elastic plate and applying forces to its boundary.[5] The same type of singularity causes body panels to crumple and form sharp creases during car accidents.

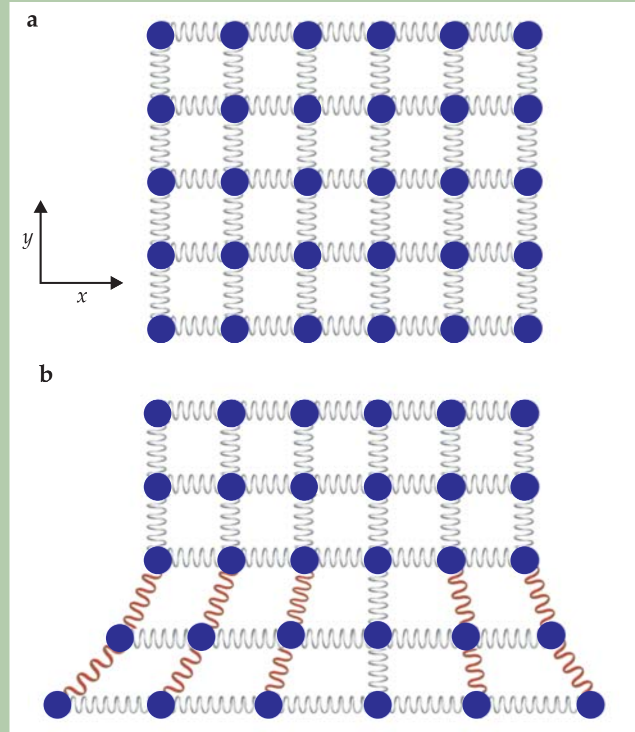


Figure 3. (a) Elastic network in equilibrium with all masses in the reference state. **(b)** The equilibrium lengths of springs in successive rows are increased, but vertical springs are not changed. In the configuration shown, the red springs are under tension and are not at their equilibrium lengths. All configurations of the masses in the plane have high energy, so the structure will buckle.

Box 1. Metrics

Defining a metric on a surface means comparing the surface in two different states. First, think of a flat sheet of material—the material in its reference state. Draw a grid of closely spaced perpendicular lines to form a coordinate system with the variables x and y. The distance between adjacent lines is dx along x and dy along y. Now deform the sheet, stretching or compressing it to change the distances between the lines. Let the new position in space of a point originally at (x, y) be called $\mathbf{r}(x, y)$. The square of the distance between two points originally separated by (dx, dy) becomes

$$|\mathbf{r}(x+dx, y+dy) - \mathbf{r}(x,y)|^2 = \frac{\partial \mathbf{r}}{\partial x}\cdot\frac{\partial \mathbf{r}}{\partial x}dx^2 + 2\frac{\partial \mathbf{r}}{\partial x}\cdot\frac{\partial \mathbf{r}}{\partial y}dxdy + \frac{\partial \mathbf{r}}{\partial y}\cdot\frac{\partial \mathbf{r}}{\partial y}dy^2. \tag{1}$$

The above computation motivates the definition of the metric tensor

$$g_{\alpha\beta} = \frac{\partial \mathbf{r}}{\partial \alpha}\cdot\frac{\partial \mathbf{r}}{\partial \beta}, \tag{2}$$

where α and β can adopt values x and y.

When discussing physical sheets, two different metric tensors are important. One, the target metric, is derived from the shape the sheet would take if all neighboring material points were located at the equilibrium distances preferred by the imaginary springs of figure 3. The second, the actual metric, is obtained from the real configuration of the material. The difference between the two tensors describes how much the material is strained and is the starting point of the theory of nonlinear elasticity. For example, the simplest theory for the energy per volume U of stretched rubber is that it is proportional to the trace of the actual metric tensor $g(x,y)$ minus the target metric (a unit tensor):[18]

$$U = (G/2)(g_{xx} + g_{yy} - 2), \tag{3}$$

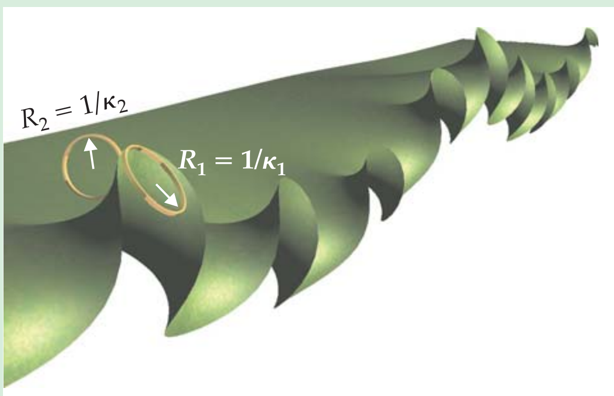
where G is the shear modulus of the material.

Box 2. The Gauss–Bonnet theorem

The Gauss–Bonnet theorem connects in an extraordinary way four quantities relating to a closed path on a surface:

$$2\pi\chi = \int K dA + \int \kappa ds + \sum_i \alpha_i. \tag{1}$$

The **Euler characteristic** χ equals the number of vertices V minus the number of edges E of the path, plus the number of faces F the path encloses: $\chi = V - E + F$. An inimitable dialog inspired by the history of Leonhard Euler's formula as it is usually applied to polyhedra is *Proofs and Refutations: The Logic of Mathematical Discovery* by Imre Lakatos (Cambridge University Press, 1976), probably the only mathematics book to mention a "sick mind, twisting in pain." The **Gaussian curvature** K is integrated over the area enclosed by the path. For each point, find the radius of curvature R in every direction, as shown in the left figure. If $\kappa_1 = 1/R_1$ is the maximum curvature and $\kappa_2 = 1/R_2$ is the minimum curvature, then the Gaussian curvature is $K = \kappa_1\kappa_2$. In the left figure, the two circles point in opposite directions, so the Gaussian curvature is negative.



The **geodesic curvature** κ is integrated along the path. Think of a mountain path connecting two cities. When a traveler on the path notices it turning sideways left or right, the reciprocal of the radius of the sideways turn gives the geodesic curvature. The **turning angles** α_i are the exterior angles at the vertices of the path, where the path turns abruptly.

One application of the Gauss–Bonnet theorem is to the legs of the triangle traveled by Gauss and formed by the peaks of Hohehagen, Brocken, and Inselberg. Assume that Earth is a perfect sphere with radius $R = 6.4 \times 10^6$ m. The Euler characteristic of the triangle is 1. The Gaussian curvature is $1/R^2$ everywhere on the surface of the sphere. The geodesic curvature is zero along paths of minimal distance, and the sum of the turning angles is 3π minus the sum of the interior angles. The Gauss–Bonnet theorem therefore gives Legendre's result that $\Sigma_i\,\theta_i = \pi + A/R^2$. Each distance between the mountain peaks was about 100 km, giving an area of approximately 5×10^9 m^2, and $A/R^2 \approx 10^{-4}$. Gauss, who did the measurement more precisely, got 7.2×10^{-5} radians.

A second application of the theorem is to the buckling instabilities of flowers. Consider a sheet wrapped into a cylindrically symmetric shape of radius $R(y) = \sqrt{g_{xx}(y)}$, where $R(y) \to 1$ as $y \to -\infty$. The Euler characteristic χ is zero, and there are no abrupt turns, so

$$\int \kappa ds = -\int K dA. \tag{2}$$

Because of the cylindrical symmetry, κ is constant for any value of y and cannot be greater in absolute value than $1/R(y)$. Thus, at the top of the cylinder at $y = 0$,

$$-2\pi \le \int \kappa ds \le 2\pi. \tag{3}$$

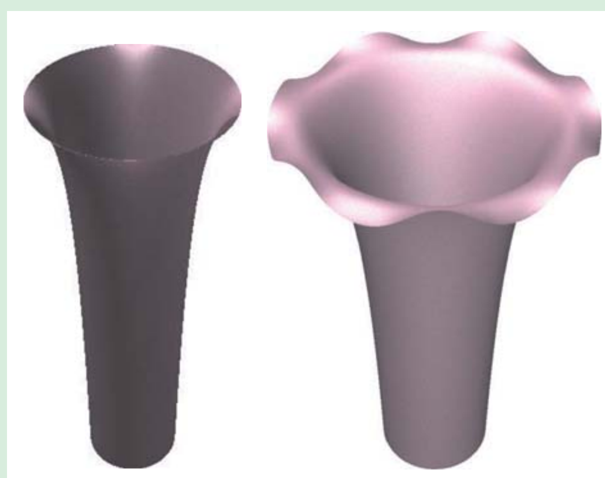
The right-hand side of equation 2 can be computed exactly in cylindrical coordinates, using equation 1 from the main text:

$$\int K dA = -2\pi \left.\frac{dR}{dy}\right|_{y=0}. \tag{4}$$

Combining equations 2, 3, and 4 gives the condition

$$-1 \le \left.\frac{dR}{dy}\right|_{y=0} \le 1. \tag{5}$$

When the radius of a flower increases more rapidly than allowed by the above bound, axial symmetry can no longer be maintained, and the flower must buckle, as shown in the right figure.



Enrique Cerda and L. Mahadevan found a special case in which the mathematical buckling of paper can be analyzed with particular completeness and elegance.[6] Lay a sheet of material on top of a drinking glass and press in the center with a pencil, as shown in figure 2c. The paper hugs the edge of the glass around part of the circumference, and then jumps off at a specific location and definite angle. Now look at a complete piece of crumpled material. It contains an ensemble of such singularities connected by ridges (see figure 2d). Thomas Witten and Alexander Lobkovsky have shown that the ensemble of singularities explains the energy needed to squash paper into a ball.[7] In a big sheet of crumpled paper, the ridges connecting singular d-cones contain much more energy than do the d-cones. So a sheet of crumpled paper is unstretched almost everywhere, and its resistance to compression comes almost entirely from ridges stretching between the rare points where the paper buckles into sharp corners.

Cracking

Thin elastic sheets not only form static patterns, they also have interesting dynamical behavior, particularly when they are stretched to the point of failure—that is, when they break. The notion that balloons pop like soap bubbles is deeply ingrained, but they don't: They crack, as shown in figure 2e. One day in our lab, Stefan Luding showed us that the fragments of a broken balloon have wavy edges, like rows of shark fins, as shown in figure 2f. All you have to do to see that for yourself is inflate a balloon and pop it. The waviness of the edge means that the crack tip spontaneously changes direction as the crack propagates around the balloon. The size and shape of the waves depend on the degree of inflation of the balloon, the history of inflation, and even the age of the rubber. The size of the waves changes rapidly as the rupture propagates, and it is usually between one and several millimeters.

Delightful as they are,[8] balloons make a bad setting for controlled studies of rupture. Their curved surfaces are inconvenient for making precise measurements, and the rapid drop in pressure when the balloon pops creates complicated time-dependent conditions. Working with Paul Petersan and Harry Swinney, we created a controlled experiment based on a machine that could stretch flat sheets of rubber by any amount along two axes.[9]

Before long we completed an experimental phase diagram showing that ruptures are straight when the rubber is stretched mainly along one axis and wavy when it is stretched nearly evenly along both axes. But obtaining a theory for when cracks become wavy has been challenging, a reflection of the general difficulty of predicting the direction of motion of cracks. Slowly moving cracks seem to obey the principle of local symmetry, so called for reasons that no one seems quite to remember, but probably due to Grigory Barenblatt. The principle gives a recipe for the advancement of a crack. Draw a circle centered on the crack tip. The crack moves toward the point at which tensions along the circle are greatest.

The principle of local symmetry has been checked most carefully for oscillating crack paths that develop when hot slabs of glass are dunked into cool water baths. Recent experiments by Benoît Roman, Pedro Reis, and Basile Audoly, in which a blunt cutting tool is slowly dragged through a thin plastic sheet, have shown another oscillating instability that can also be explained (figure 4).[10]

Figure 4. A wavy crack pattern is created by dragging a blunt cutting tool through a thin plastic sheet gripped along its sides.[10] (Courtesy of Benoît Roman.)

However, for quickly moving cracks like the ones in rupturing rubber, remarkably little consensus has been reached on what equation determines the direction of propagation, or even on the general form such an equation should take. Thus, despite promising first attempts by Hervé Henry and Herbert Levine,[11] explaining when such cracks begin to oscillate remains out of reach.

When we examined high-speed photographs of the propagating rupture, such as figure 2e, we noticed something else puzzling. The leading edge was sharp and pointed, like a sonic boom. That was supposed to be impossible. The theory of dynamic fracture[12] says that when an object breaks under tension, the cracks that run through it must have rounded parabolic tips and must travel slower than any sound waves. Yet here in rupturing rubber was something that looked supersonic. And on measuring the speeds of sound in the rubber ahead of the tip, we found that the rupture was indeed moving faster than some sound waves.[13] Shear, or transverse, sound waves in solids travel slower than the more familiar longitudinal sound waves. The cracks we saw traveled faster than shear waves but slower than longitudinal waves, which made them technically "intersonic," not supersonic, but that didn't make our observation any less surprising.

The problem has a detailed analytical solution using equation 3 from box 1 to describe the energy of rubber.[14] The energy density of rubber is given by the trace of the metric tensor; in other words, geometry and energy are the same.

Here is what is going on. Rubber pops when extended to several hundred percent of its original size. The elastic energy stored in it is thousands of times what is needed to sever the polymers and cut the rubber in two. In the customary theory of fracture, the energy needed to make a crack propagate must be transferred from the farthest reaches of the stretched solid. The energy arrives no faster than the speed of sound, so the crack can travel no faster. For popping rubber, however, enough energy to make the rupture run can be found within a few microns of the tip, and conventional speed limits do not apply. Behind the tip of the rupture, the rubber contracts like the end of a rubber band, freely snapping back at around the speed of sound. The angular tip is indeed a Mach cone resembling a sonic boom.

Ruptures traveling faster than sound are known in only one other context: earthquakes. In 1976 Dudley Andrews found, using computer simulations, that materials torn apart by shear forces, as when two of Earth's plates slide against each other, can have cracks that travel at $\sqrt{2}$ times the shear wave speed. Ares Rosakis has found cracks of that sort in laboratory experiments, and Huajian Gao, Young Huang, and Farid Abraham have seen them in molecular dynamics simulations.[15] Attempts to deduce the velocity of the 1999 Izmit earthquake in Turkey from data collected at seismic stations suggested that it traveled faster than the shear wave speed for a distance of hundreds of kilometers.[16] That earthquakes might travel faster than sound had been doubted. Tensile cracks propagating faster than shear waves seemed even less possible, but now we know they happen in the cracking of balloons. If a thin sheet can be stretched far enough before it fails, and if a single propagating failure remains stable, the crack speed blasts to the shear sound speed and beyond.

Physics and geometry

The experiments described here are simple and accessible. Yet the problems they raise are quite challenging, and the mathematical ideal of an infinitely thin surface is not enough to explain them. Gauss anticipated that too. In 1830 he wrote in a letter to Wilhelm Bessel:[17]

> We must in humility admit that if number is *merely* a product of our minds, space has a reality outside our minds whose laws we cannot a priori state.

Whether in the buckles of a flower or leaf, the folds of crumpled paper, or the crack tips in a popping balloon, thin sheets naturally develop singularities. In the regions around the singularities, detailed features of the material become important. Many unanswered questions remain. For buckled plastic, the target metric desired by the internal springs, as in figure 3, seems always to differ slightly from the actual metric of the lowest-energy structure, and we do not understand what determines the difference. The scale of the crescent-shaped core in the corners of crumpled paper is still controversial. In cracking rubber, the stresses in the wake of the tip are actually greater than the stresses right at the tip where material gives way. We do not fully understand how that is possible. Many unsolved problems thus stem from the interaction of small and large features of surfaces. Geometry and other large mathematical ideas are not enough to solve those problems; they must have help from the underlying physics.

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References

1. E. Breitenberger, *Arch. Hist. Exact Sci.* **31**, 273 (1984).
2. D. Nelson, T. Piran, S. Weinberg, eds., *Statistical Mechanics of Membranes and Surfaces,* 2nd ed., World Scientific, River Edge, NJ, (2004).
3. J. Genzer, J. Groenwold, *Soft Matter* **2**, 310 (2006); S. Nechaev, R. Voituriez, *J. Phys. A* **34**, 11069 (2001).
4. E. Sharon, B. Roman, M. Marder, G.-S. Shin, H. L. Swinney, *Nature* **419**, 579 (2002).
5. M. Ben Amar, Y. Pomeau, *Proc. R. Soc. A* **453**, 729 (1997).
6. E. Cerda, L. Mahadevan, *Proc. R. Soc. A* **461**, 671 (2005).
7. A. E. Lobkovsky, T. A. Witten, *Phys. Rev. E* **55**, 1577 (1997).
8. I. Müller, P. Strehlow, *Rubber and Rubber Balloons: Paradigms of Thermodynamics,* Springer, Berlin, Germany (2004).
9. R. D. Deegan, P. Petersan, M. Marder, H. L. Swinney, *Phys. Rev. Lett.* **88**, 014304 (2002).
10. A. Yuse, M. Sano, *Nature* **362**, 329 (1993); B. Audoly, P. Reis, B. Roman, *Phys. Rev. Lett.* **94**, 129601 (2005.
11. H. Henry, H. Levine, *Phys. Rev. Lett.* **93**, 105504 (2004).
12. K. B. Broberg, *Cracks and Fracture*, Academic Press, San Diego, CA (1999).
13. P. J. Petersan, R. D. Deegan, M. Marder, H. L. Swinney, *Phys. Rev. Lett.* **93**, 015504 (2004).
14. M. Marder, *J. Mech. Phys. Solids* **54**, 491 (2006).
15. D. J. Andrews, *J. Geophys. Res.* **81**, 5679 (1976); A. J. Rosakis, *Adv. Phys.* **51**, 1189 (2002); H. Gao, Y. Huang, F. F. Abraham, *J. Mech. Phys. Solids* **49**, 2113 (2001).
16. M. Bouchon, M.-P. Bouin, H. Karabulut, M. N. Toksöz, M. Dietrich, A. J. Rosakis, *Geophys. Res. Lett.* **28**, 2723 (2001).
17. C. F. Gauss, letter to Wilhelm Bessel (1830), available at http://www.math.uwaterloo.ca/~snburris/htdocs/geometry.pdf, p. 14.
18. L. R. G. Treloar, *The Physics of Rubber Elasticity,* 3rd ed., Oxford University Press, New York (2005).