EXISTENCE OF EFFICIENT SOLUTIONS IN INFINITE HORIZON OPTIMIZATION UNDER CONTINUOUS AND DISCRETE CONTROLS

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ABSTRACT. We consider a general deterministic infinite horizon optimization problem over discrete time with time-varying, i.e., non-stationary, data. Our formulation requires only that action spaces be compact, including both continuous and discrete controls. In the event that all total costs diverge, i.e., no least total cost optimum exists, we investigate the existence of efficient optima. (An infinite horizon feasible solution is *efficient* if it is optimal to each of the states through which it passes.) We show that the mapping from controls to states (i.e. state transition function) being open is a sufficient condition for existence of efficient solutions. In this event, we also give a necessary and sufficient condition for there to exist a *unique* efficient optimum. Our results are then applied to an infinite horizon production planning problem with no backlogging.

1. Introduction

We consider a general infinite horizon optimization problem, formulated as a dynamic programming problem over discrete time, with deterministic, time-varying data. It is clear that even in the presence of discounting, the total cost of the infinite streams of cost flows associated with feasible decision sequences may all be infinite, i.e., it may be that no least total cost optimal solution exists. In such cases, we require an optimality criterion other than minimum total cost and there are many such criteria; see, for example, [3] and [11]. In recent papers, [13] have considered the notion of optimality called *efficiency* or finite optimality [5]). A feasible solution is efficient if it is least-cost optimal to each of the states through which it passes. In this paper, we give a sufficient condition for the existence of efficient solutions in the presence of compact action spaces. Hence, continuous as well as discrete action spaces are allowable. If an efficient solution exists, then we give sufficient conditions for it to be unique.

It is worth noting that we will *not* make any reachability, differentiability, or convexity assumptions here, as is often the case. Moreover, by the familiar device of replacing decisions by policies to construct a deterministic equivalent, stochastic infinite horizon problems can be modeled within our framework as well. Our modeling framework includes for example production planning under non-stationary demand, parallel and serial equipment replacement under technological change, capacity planning under nonlinear demand, and optimal search in a time-varying environment.

In section 2, we formulate the state-transition and cost structures for our problem. In section 3, we present our main results on the existence and uniqueness of efficient solutions. Finally, in section 4, we apply our main results to a general problem in production planning.

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2. Problem Formulation

Consider the problem of making a sequence of decisions, where each decision is made at the beginning of each of a series of equal time periods, indexed by $j = 1, 2, \ldots$. The set of all feasible decisions available in period j is contained in Y_j. The feasibility of a decision depends on the past decisions made. We assume that each Y_j is a compact, non-empty metric space.

Our dynamic system is governed by the state transition equation $s_j = f_j(s_{j-1}, y_j), \quad \forall j = 1, 2, \ldots$, where we assume that

- s_0 is the fixed and given *initial state* of the system (beginning period 1),
- s_j is the *state* of the system at the end of period j, i.e., beginning period $j + 1$,
- y_j is the feasible *action* (or *control*) selected in period j with knowledge of the state s_{j-1},
- S_j is the compact metric space of all *feasible states* ending period j (with $S_0 = \{s_0\}$), so that $s_j \in S_j$,
- $Y_j(s_{j-1})$ is the given closed, non-empty subset of Y_j consisting of the feasible actions available in period j when the beginning state is $s_{j-1} \in S_{j-1}$ (so that $y_j \in Y_j(s_{j-1}) \subseteq Y_j$), with $Y_1(s_0) = Y_1$,
- D_j is the *graph* of the compact-valued set mapping $s_{j-1} \to Y_j(s_{j-1})$ in the compact space $S_{j-1} \times Y_j$, i.e.,

$$D_j \;=\; \{(s_{j-1}, y_j) \in S_{j-1} \times Y_j : y_j \in Y_j(s_{j-1})\}, \qquad \forall j = 1, 2, \ldots, \quad \text{and}$$

- f_j is the given continuous *state transition function* in period j, where $f_j : D_j \to S_j$.

Note that the non-emptiness of $Y_j(s_{j-1})$, for $s_{j-1} \in S_{j-1}$, is equivalent to the assumption that all finite horizon feasible solutions can be feasibly continued from state s_{j-1} in period j. We assume that our problem has the following *closed graph property*: for each j, if $s_{j-1}^n \to s_{j-1}$ in S_{j-1}, and $y_j^n \to y_j$ in Y_j, as $n \to \infty$, where $y_j^n \in Y_j(s_{j-1}^n), \ \forall n$, then $y_j \in Y_j(s_{j-1})$. Then each graph D_j is closed (hence, compact) in $S_{j-1} \times Y_j$. We also require that $S_j = f_j(D_j), \quad \forall j = 1, 2, \ldots$, so that, in particular, $S_1 = f_1(D_1)$, where $D_1 = \{s_0\} \times Y_1$. Thus, each S_j is precisely the set of *all* feasible, i.e., attainable, states in period j.

For each j, consider the set-valued mapping $s_{j-1} \to Y_j(s_{j-1})$ of S_{j-1} into the compact, non-empty subsets of Y_j. Let

$$Y_j(S_{j-1}) \;=\; \cup\{Y_j(s_{j-1}) : s_{j-1} \in S_{j-1}\},$$

so that $Y_j(S_{j-1}) \subseteq Y_j, \ \forall j$. Note that the actions $Y_j \backslash Y_j(S_{j-1})$ (set difference) will never be used. Moreover, by the closed graph property, this set-valued mapping is upper semi-continuous [10, p.61]. Thus, $Y_j(S_{j-1})$ is compact by [2, p.110]. Consequently, without loss of generality, we may assume that $Y_j(S_{j-1}) = Y_j, \ \forall j$.

The product set $Y = \prod_{j=1}^{\infty} Y_j$ of all potential decision sequences, or infinite horizon *strategies*, is then a compact topological space relative to the product topology, i.e., the topology of component-wise convergence. The product topology on Y is well-known to be metrizable.

Now fix a positive integer N and let $(y_1, \ldots, y_N) \in Y_1 \times \cdots \times Y_N$. Then $(y_1, \ldots, y_N)$ is *feasible through period* N if $y_j \in Y_j(s_{j-1})$, where $s_j = f_j(s_{j-1}, y_j)$, for all $j = 1, 2, \ldots, N$. Denote all such finite horizon strategies by F_N, which is thus a closed, compact, non-empty subset of $Y_1 \times \cdots \times Y_N$. In particular, $F_1 = Y_1$. Note that if $(y_1, \ldots, y_N)$ is feasible through period N, then $(y_1, \ldots, y_{N-1})$ is feasible through period $N-1$, i.e., $F_N \subseteq F_{N-1} \times Y_N$. Moreover, $y \in Y$ is a *feasible* strategy if $(y_1, \ldots, y_N)$ is feasible through period N, for each $N = 1, 2, \ldots$. We define the *feasible region* X to be the subset of Y consisting of all those $y \in Y$ which are feasible through *each* period N, i.e., $(y_1, \ldots, y_N) \in F_N, \ \forall N$. We define X_N to be the set of all arbitrary extensions of F_N in Y, i.e.,

$$X_N \;=\; F_N \times \prod_{j=N+1}^{\infty} Y_j.$$

Then the non-empty, compact sets X_N are decreasing subsets of Y and $X = \cap_{N=1}^{\infty} X_N$. This set is closed, compact in Y and non-empty, since $Y_j(s_{j-1})$ is assumed to be non-empty, for all j , and all $s_{j-1} \in S_{j-1}$. In fact, as a consequence of this assumption, if $(y_1, \ldots, y_N)$ is feasible through a given period N, then it may be feasibly extended over all remaining periods.

For $(y_1, \ldots y_N) \in F_N$, we may define $\sigma_N : F_N \to S_N$ by

$$\begin{aligned}
\sigma_1(y_1) &= f_1(s_0, y_1), \\
\sigma_2(y_1, y_2) &= f_2(\sigma_1(y_1), y_2), \\
&\vdots \\
\sigma_N(y_1, \ldots, y_N) &= f_N(\sigma_{N-1}(y_1, \ldots, y_{N-1}),\ y_N),
\end{aligned}$$

so that $\sigma_N(y_1, \ldots, y_N) \in S_N$. We will refer to each such $\sigma_N(y_1, \ldots, y_N)$ as *the state which* $(y_1, \ldots, y_N)$ *attains at the end of period* N. Thus, for each N, the mapping $\sigma_N : F_N \to S_N$ is *onto* since S_N consists of all feasible states. Consequently, F_N is partitioned into equivalence classes of the form $\sigma_N^{-1}(s)$, for $s \in S_N$.

2.1 Lemma. *For each N, the mapping σ_N of F_N onto S_N is continuous and closed. Hence, the topology of S_N is contained in the quotient topology on the equivalence classes $F_N/\sigma_N = \{\sigma_N^{-1}(s) : s \in S_N\}$ in F_N defined by σ_N.*

Proof. The continuity of the σ_N follows from the continuity of the f_j. Since σ_N is continuous, the topology of S_N is contained in the quotient topology of S_N [7, p. 95]. The remaining property follows from the compactness of the Y_j, as well as Theorem 8 of [7, p.95]. □

2.2 Lemma. *For each N, the quotient topological space F_N/σ_N is homeomorphic to S_N.*

Proof. The resulting quotient mapping $\overline{\sigma}_N : F_N/\sigma_N \to S_N$ is continuous, one-to-one and onto. Since F_N/σ_N is compact, $\overline{\sigma}_N$ is also open. □

Turning to the objective function, we allow the cost of a decision made in period j to also depend (indirectly) on the sequence of previous decisions, or more directly, on the state resulting from these decisions. Specifically, we let $c_j(s_{j-1}, y_j)$ be the cost of decision y_j in period j, when s_{j-1} is the state beginning period j. We thus obtain cost functions $c_j : D_j \to \mathbb{R}$ which we require to be continuous. Thus, each c_j attains its extrema. We assume that any discount factor has been absorbed into the period costs.

For each positive integer N and $(x_1, \ldots, x_N) \in F_N$, we define the associated total N-horizon cost by

$$C_N(x_1, \ldots, x_N) \;=\; \sum_{j=1}^{N} c_j(\sigma_{j-1}(x_1, \ldots, x_{j-1}),\ x_j).$$

Thus, $C_N : F_N \to \mathbb{R}$ is a continuous function, for each N. For each $x \in X$, also define

$$C(x) \;=\; \sum_{j=1}^{\infty} c_j(\sigma_{j-1}(x_1, \ldots, x_{j-1}),\ x_j),$$

so that the function $x \to C(x)$ is extended-real valued in general. The classical least-total-cost optimization problem is then given by $\min_{x \in X} C(x)$ which may have no optimal solutions, i.e., $C(x) = \infty, \ \forall x \in X$. In this event, our main objective is to ensure the existence of a feasible strategy which is efficient.

3. Existence of Efficient Optima

The state-space construction introduced above associated a unique state at the end of each time period with every finite horizon feasible strategy. Feasible strategies $x \in X$ which have the property of *optimally* reaching each of the states $\sigma_N(x_1, \ldots, x_N)$ through which they pass have been called *efficient strategies.* (See [11,12,13] for early introductions of this concept.) This efficiency of movement through the state space suggests efficient solutions as candidates for optimality.

Efficiency (Finite Optimality): Let $x \in X$. Then x is *efficient* if, for each N, and each $(y_1, \ldots, y_N) \in F_N$ such that $\sigma_N(y_1, \ldots, y_N) = \sigma_N(x_1, \ldots, x_N)$, we have $C_N(x_1, \ldots, x_N) \le C_N(y_1, \ldots, y_N)$. Also known as *finite optimality,* this criterion was originally introduced in a special case by Halkin in [5], who called it *finite horizon clamped end-point optimality.*

Let X^e denote the subset of X consisting of efficient strategies. It was shown in [13, Lemma 3.5] that efficient strategies exist in our context, provided each of the spaces Y_j and S_{j-1} is *discrete.* We next show that efficient solutions exist for our problem under the more general assumption that the period state mappings σ_N are open, thus allowing for the presence of *continuous* action and state spaces Y_j and S_j.

Fix N, and for each $s \in S_N$, let $\Phi_N(s)$ denote the set of N-horizon feasible strategies which attain state s at the end of period N, i.e.,

$$\Phi_N(s) \;=\; \sigma_N^{-1}(s) \;=\; \{(x_1, \ldots, x_N) \in F_N \;:\; \sigma_N(x_1, \ldots, x_N) = s\}.$$

(The collection $\{\Phi_N(s); s \in S_N\}$ is a partition of F_N.) Since σ_N is continuous, we thus obtain a sequence of set-valued mappings Φ_N of S_N into F_N with compact, non-empty values. For each N, let $\mathcal{K}_0(F_N)$ denote the collection of all compact, non-empty subsets of F_N. Then Φ_N is a mapping of S_N into $\mathcal{K}_0(F_N)$.

Now, for each N and $s \in S_N$, consider the least-total-cost optimization problem

$$\min\{C_N(x_1, \ldots, x_N) \;:\; (x_1, \ldots, x_N) \in \Phi_N(s)\}.$$

If we let $\Phi_N^*(s)$ denote the set of optimal solutions to this problem, then this set is a closed, compact non-empty subset of F_N. We thus obtain another compact-valued set mapping of $\Phi_N^* : S_N \to \mathcal{K}_0(F_N)$. If we define

$$F_N^e = \bigcup_{s \in S_N} \Phi_N^*(s), \qquad \text{and} \qquad X_N^e = F_N^e \times \prod_{j=N+1}^{\infty} Y_j,$$

then the X_N^e are non-empty, nested downward and $X^e = \bigcap_{N=1}^{\infty} X_N^e$.

Next we give a Dynamic Programming formulation of our problem and a corresponding inductive description of the F_N^e. For each N, define $q_N(s,t)$ to be the minimum cost of transitioning from state s at the start of period $N-1$ to state t at the start of period N, if this is possible, so that

$$q_N(s,t) = \min\Big\{c_N(s,y_N) : y_N \in Y_N(s) \text{ and } t = f_N(s,y_N)\Big\}, \qquad \forall s \in S_{N-1}, \qquad \forall t \in S_N.$$

Otherwise, define $q_N(s,t)=\infty$. Also define $Q_N(s)$ to be the minimum cost of transitioning from state s_0 to state s ending period N, i.e.,

$$\begin{aligned} Q_N(s) &= \min\Big\{C_N(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \in F_N \text{ and } \sigma_N(x_1,\ldots,x_N) = s\Big\} \\ &= \min\Big\{C_N(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \in \Phi_N(s) = \sigma_N^{-1}(s)\Big\}, \qquad \forall s \in S_N. \end{aligned}$$

By the Principle of Optimality, we have the following forward recursion:

$$Q_N(t) = \min_{s \in S_{N-1}} \Big(Q_{N-1}(s) + q_N(s,t)\Big), \qquad \forall t \in S_N,$$

with $Q_0(s_0)=0$. Consequently, F_N^e may be determined inductively as follows:

$$F_1^e = \Big\{x_1 \in F_1 \ : \ C_1(x_1) = c_1(s_0,x_1) = Q_1(\sigma_1(x_1))\Big\},$$

$$F_N^e = \Big\{(x_1,\ldots,x_N) \in F_N \ : \ (x_1,\ldots,x_{N-1}) \in F_{N-1}^e \text{ and } C_N(x_1,\ldots x_N) = Q_N(\sigma_N(x_1,\ldots,x_N))\Big\},$$

for $N \geqslant 2$.

For each N and feasible strategy $(x_1,\ldots,x_N) \in F_N$, let $\Gamma_N(x_1,\ldots,x_N)$ be the set of all N-horizon feasible strategies that attain the same state at the end of period N as $(x_1,\ldots,x_N)$, i.e. $\Gamma_N : F_N \to \mathcal{K}_0(F_N)$ where

$$\Gamma_N(x_1,\ldots,x_N) = \Phi_N(\sigma(x_1,\ldots,x_N)) = \sigma_N^{-1}(\sigma_N(x_1,\ldots,x_N).$$

Let $A \subseteq F_N$. Define the *weak saturation* of A in F_N [6, p.22] to be

$$\Gamma_N^w(A) = \Big\{(x_1,\ldots,x_N) \in F_N \ : \ \Gamma_N(x_1,\ldots,x_N) \cap A \neq \emptyset\Big\} = \sigma_N^{-1}(\sigma_N(A)),$$

so that, in particular, $\Gamma_N(x_1,\ldots,x_N)$ is the weak saturation of $(x_1,\ldots,x_N)$. Note that $\Gamma_N^w(A)$ is the union of those classes which *intersect* A. Also define the *strong saturation* of A in F_N [6, p.22] to be the complement in F_N of the weak saturation of $F_N \backslash A$ (set difference), i.e.,

$$\Gamma_N^s(A) = \Big\{(x_1,\ldots,x_N) \in F_N : \Gamma_N(x_1,\ldots,x_N) \subseteq A\Big\} = F_N \backslash (\sigma_N^{-1}(\sigma_N(F_N \backslash A))).$$

Note that $\Gamma_N^s(A)$ is the union of those classes which are *contained in* A, and $\Gamma_N^s(A) \subseteq A \subseteq \Gamma_N^w(A)$, in general.

3.1 Lemma. *For each N, and each open subset A of F_N, the strong saturation $\Gamma_N^s(A)$ of A is open in F_N. The weak saturation $\Gamma_N^w(A)$ of A is open in F_N if and only if the mapping σ_N is open.*

Proof. This follows from Theorem 10 of [7, p. 97], together with the fact that each σ_N is a closed mapping. □

For various notions of continuity for set-valued mappings, see [1,2,6,8,9,10]. In particular, observe that the mapping $\Gamma_N : F_N \to \mathcal{K}_0(F_N)$ is:

- *lower semi-continuous* if and only if, for each open $A \subseteq F_N$, the weak saturation $\Gamma_N^w(A)$ of A is open in F_N.
- *upper semi-continuous* if and only if, for each open $A \subseteq F_N$, the strong saturation $\Gamma_N^s(A)$ of A is open in F_N.
- *continuous* if and only if it is both upper and lower semi-continuous.

3.2 Lemma. *For each N, the set mapping $\Gamma_N : F_N \to \mathcal{K}_0(F_N)$ is upper semi-continuous. Thus, the mapping Γ_N is continuous if and only if it is lower semi-continuous, i.e., the mapping σ_N is open. This holds, for example, if S_N is discrete.*

Proof. The result follows from the definitions and Lemma 3.1, as well as, for example, Theorems 7.1.4 and 7.1.7 of [8, pp. 74-75]. □

For each N, we have the continuous function $C_N : F_N \to \mathbb{R}$ and the upper semi-continuous set mapping $\Gamma_N : F_N \to \mathcal{K}_0(F_N)$. Let $C_N^*(x_1, \dots, x_N)$ denote the (attained) minimum value of C_N on $\Gamma_N(x_1, \dots, x_N)$ and let $\Gamma_N^*(x_1, \dots, x_N)$ denote the set of $(y_1, \dots, y_N) \in \Gamma_N(x_1, \dots, x_N)$ which attain this minimum value. This set is compact and non-empty. We thus obtain mappings $C_N^* : F_N \to \mathbb{R}$ and $\Gamma_N^* : F_N \to \mathcal{K}_0(F_N)$. Note that these mappings are constant on equivalence classes, i.e., they may be viewed as mappings defined on S_N. In particular, we obtain the set mapping $\Phi_N^* \circ \sigma_N$ which satisfies

$$\Phi_N^*(\sigma_N(x_1, \dots, x_N)) = \Gamma_N^*(x_1, \dots, x_N), \qquad \forall (x_1, \dots, x_N) \in F_N.$$

3.3 Lemma. *Suppose the set mapping Γ_N is lower semi-continuous. Then the function C_N^* is continuous and the set mapping Γ_N^* is upper semi-continuous.*

Proof. These properties follow immediately from the previous lemma and the (minimum version of the) Maximum Theorem of [2, p. 116]. See also Corollaries 9.2.6 and 9.2.7 of [8]. □

3.4 Lemma. *Suppose the set mapping Γ_N is lower semi-continuous. Then the subset F_N^e of F_N is compact and non-empty.*

Proof. By the previous lemma, Γ_N^* is upper semi-continuous. From Theorem 3 of [2, p.110], we have that the subset $\Gamma_N^*(F_N)$ of F_N given by

$$\Gamma_N^*(F_N) = \cup \left\{ \Gamma_N^*(x_1, \dots, x_N) : (x_1, \dots, x_N) \in F_N \right\} = \bigcup_{s \in S_N} \Phi_N^*(s)$$

is compact. It is also non-empty. But $F_N^e = \Gamma_N^*(F_N), \quad \forall N$. This completes the proof. □

The following is our first main result. It generalizes Lemma 3.5 of [13] and in particular includes continuous action spaces.

3.5 Theorem. *Suppose that there exists a subsequence $\{\Gamma_{N_k}\}_{k=1}^{\infty}$ of the sequence $\{\Gamma_N\}_{N=1}^{\infty}$ for which each Γ_{N_k} is lower semi-continuous, i.e., σ_{N_k} is open. Then efficient solutions exist for our problem, i.e., the set X^e is a non-empty, compact subset of X.*

Proof. Since

$$X_{N_k}^e = F_{N_k}^e \times \prod_{j=N+1} Y_j,$$

it follows that the $X_{N_k}^e$ are also compact and non-empty in Y. But they are also monotonically decreasing. Hence, their intersection is non-empty, i.e.,

$$X^e = \bigcap_{N=1}^{\infty} X_N^e = \bigcap_{k=1}^{\infty} X_{N_k}^e$$

is compact and non-empty. □

3.6 Lemma. *Suppose that for each j, we have that $Y_j(s_{j-1}) = Y_j, \quad \forall s_{j-1} \in S_{j-1}$. Then*

$$F_j = F_{j-1} \times Y_j = \prod_{i=1}^{j} Y_i, \qquad \textit{and} \qquad D_j = S_{j-1} \times Y_j, \qquad \forall j.$$

Proof. Proceed by induction. □

The following is our second main result. It shows that X^e is non-empty in an important special case.

3.7 Theorem. *Suppose that, for each j, $Y_j(s_{j-1}) = Y_j, \quad \forall s_{j-1} \in S_{j-1}$. If each f_j is open on D_j, then each σ_j is open on F_j, and the set X^e is a non-empty, compact subset of X.*

Proof. For $j = 1$, we have that $\sigma_1(x_1) = f_1(s_0, x_1)$, so that σ_1 is open on $F_1 = Y_1$, since f_1 is open. Now suppose σ_{j-1} is open. By Lemma 3.6, $F_j = F_{j-1} \times Y_j, \quad \forall j$. In this event, σ_j is the composition of $\sigma_{j-1} \times 1_j$ followed by f_j, where 1_j is the identity map on Y_j. If U is an arbitrary open subset of $Y_1 \times \cdots \times Y_{j-1}$, and V is an arbitrary open subset of Y_j, then $U \times V$ is an arbitrary basic open subset of $Y_1 \times \cdots \times Y_j$, and

$$F_j \cap (U \times V) \quad = \quad (F_{j-1} \times Y_j) \cap (U \times V) \quad = \quad (F_{j-1} \cap U) \times (Y_j \cap V) \quad = \quad (F_{j-1} \cap U) \times V$$

is an arbitrary basic open subset of F_j. Moreover,

$$(\sigma_{j-1} \times 1_j)\big((F_{j-1} \times Y_j) \cap (U \times V)\big) \quad = \quad \sigma_{j-1}(F_{j-1} \cap U) \times V,$$

where $F_{j-1} \cap U$ is a typical open subset of F_{j-1}. Since σ_{j-1} is an open mapping on F_{j-1}, we have that $\sigma_{j-1}(F_{j-1} \cap U)$ is an open subset of S_{j-1}. Hence, $D_j \cap (\sigma_{j-1}(F_{j-1} \cap U) \times V)$ is an open subset of D_j. Since f_j is an open mapping on D_j by hypothesis, it follows that σ_j is also open. This completes the proof by induction. □

Remark. Note that if $Y_j(s_{j-1}) = G_j, \quad \forall s_{j-1} \in S_{j-1}$, where G_j is a closed subset of Y_j, then there is no loss of generality in assuming $G_j = Y_j$, since the decisions $Y_j \backslash G_j$ will not be used.

In view of the previous discussion, and the needs of what follows, it is desirable to have a general sufficient condition for an onto mapping to be open - for example, each σ_j. Let V and W be first-countable topological spaces and $g : V \to W$ an onto mapping. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of non-empty subsets of V. Define $\limsup_n A_n$ to be the (closed) subset of V which is the set of all cluster points of the A_n, i.e., $v \in \limsup_n A_n$ if and only if there exists a subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ of $\{A_n\}_{n=1}^{\infty}$, and a corresponding sequence $\{v_k\}_{k=1}^{\infty}$ in V such that $v_k \in A_{n_k} \quad \forall k$, and $\lim_{k\to\infty} v_k = v$. Analogously, define $\liminf_n A_n$ to be the (closed) subset of V which is the set of all limit points of the A_n, i.e., $v \in \limsup_n A_n$ if and only if there exists a sequence $\{v_n\}_{=1}^{\infty}$ in V such that $v_n \in A_n, \quad \forall n$, and $\lim_{n\to\infty} v_n = v$. In general, $\liminf_n A_n \subseteq \limsup_n A_n$. We write

$$\lim_n A_n \;=\; A \subseteq V \quad \text{if} \quad \liminf_n A_n \;=\; \limsup_n A_n \;=\; A, \quad \text{i.e.,} \quad \limsup_n A_n \subseteq \liminf_n A_n \;=\; A$$

[9,10].

3.8 Theorem. *The mapping g is open if, for each convergent sequence $\lim_{n\to\infty} w_n = w$ in W, we have $g^{-1}(w) \subseteq \limsup_n g^{-1}(w_n)$.*

Proof. Suppose g is not open. Then there exists an open subset U of V for which $g(U)$ is not open in W, i.e., $W\backslash g(U)$ is not closed in W. Then there exists a convergent sequence $\lim_n w_n = w$ in W such that $w_n \notin g(U), \quad \forall n$, while $w \in g(U)$. Since $w \in g(U)$, there exists $v \in U$ such that $g(v) = w$, i.e. $v \in g^{-1}(w)$. By hypothesis, $g^{-1}(w) \subseteq \limsup_n g^{-1}(w_n)$. Thus, there exists a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$, and corresponding sequence $\{v_k\}_{k=1}^{\infty}$ in V such that $v_k \in g^{-1}(w_{n_k}), \quad \forall k$, and $\lim_k v_k = v$. Then $v_k \notin U, \quad \forall k$; if not, $v_k \in U$ implies $w_{n_k} = g(v_k) \in g(U)$, which is a contradiction. Since $V\backslash U$ is closed and $v_k \in V\backslash U, \quad \forall k$, it follows that $v \in V\backslash U$, i.e., $v \notin U$, a contradiction. □

Next, we turn to the question of uniqueness in the presence of existence. Since each Y_j is a compact Hausdorff space, it is metrizable. Similarly for each S_j. Let $d_j \leq 1$ denote such a metric, for each j, and let $0 < \beta_j < 1$ be such that $\sum_{j=1}^{\infty} \beta_j < \infty$, with $\beta = (\beta_1, \beta_2, \dots)$ and $\beta^N = \min_{1\leq j\leq N} \beta_i$. Then the product space admits the metric d given by

$$d_\beta(x, y) \;=\; \sum_{j=1}^{\infty} \beta_j d_j(x_j, y_j).$$

Note that, for each N, the induced metric d_β^N on $Y_1 \times \cdots \times Y_N$ is given by

$$d_\beta^N\big((x_1, \dots, x_N), (y_1, \dots, y_N)\big) \;=\; \sum_{j=1}^{N} \beta_j d_j(x_j, y_j),$$

which is equivalent to the metric d^N on $Y_1 \times \cdots \times Y_N$ given by

$$d^N\big((x_1, \dots, x_N), (y_1, \dots, y_N)\big) \;=\; \sum_{j=1}^{N} d_j(x_j, y_j),$$

since $\beta^N d^N \leq d_\beta^N \leq d^N$. Furthermore, let $\operatorname{diam}_\beta(A)$ denote the diameter of subset $A \subseteq Y$ with respect to d_β. If $A \subseteq Y_1 \times \cdots \times Y_N$, let $\operatorname{diam}^N(A)$ denote the diameter of A with respect to d^N, and let $\operatorname{diam}_\beta^N(A)$ the diameter with respect to d_β^N.

3.9 Lemma. *If* $\lim_{N\to\infty} diam^N(F_N^e) = 0$*, then* $\lim_{N\to\infty} diam_\beta(X_N^e) = 0$.

Proof. Since

$$X_N^e \;=\; F_N^e \times \prod_{N+1}^{\infty} Y_j,$$

we have that

$$\mathrm{diam}_\beta(X_N^e) \;=\; \mathrm{diam}_\beta^N(F_N^e) + \sum_{j=N+1}^{\infty} \beta_j, \qquad \forall N,$$

which completes the proof, since $\lim_{N\to\infty} diam_\beta^N(F_N^e) = 0$, if this is the case for $diam^N(F_N^e)$. □

3.10 Theorem. *Suppose* $X^e \neq \emptyset$*. Then there exists a unique efficient optimum for our problem, i.e.,* X^e *is a singleton, if* $\lim_{N\to\infty} diam^N(F_N^e) = 0$.

Proof. Suppose the condition of the theorem holds. Then, by Lemma 3.9 and [6, p. 14], X^e is a singleton. □

We next give an algorithmic procedure for constructing the unique efficient strategy, if such is the case. Let $\mathcal{K}_0(Y)$ the set of non-empty, compact subsets of Y. Since Y is compact with metric d_β, the corresponding Hausdorff metric D_β is defined on $\mathcal{K}_0(Y)$, which is compact in the resulting metric topology [1, Theorem 3.2.4]. Moreover, metric convergence in $\mathcal{K}_0(Y)$ is equivalent to Kuratowski set convergence [10, p. 49]. Since the $X_N,\ X_N^e \in \mathcal{K}_0(Y)$ are descending with intersections equal to $X,\ X^e \in \mathcal{K}_0(Y)$ respectively, it follows that $\lim_{N\to\infty} X_N = X$ and $\lim_{N\to\infty} X_N^e = X^e$ in the sense of Kuratowski [9], i.e.,

$$\limsup_N X_N \;=\; \liminf_N X_N \;=\; X \qquad \text{and} \qquad \limsup_N X_N^e \;=\; \liminf_N X_N^e \;=\; X^e.$$

Consequently, $\lim_{N\to\infty} D_\beta(X_N^e, X^e) = 0$. Thus, every element z of X^e is the componentwise limit of *some* sequence chosen from the X_N^e. In particular, if there is a unique efficient strategy, i.e., $X^e = \{z\}$, then z is the componentwise limit of *every* sequence $\{z_N\}_{N=1}^\infty$ chosen from the X_N^e, for all j, i.e., $z_j^N \to z_j$ in Y_j, as $N \to \infty$.

Recall that

$$X_N^e \;=\; F_N^e \times Y_{N+1} \times Y_{N+2} \times \cdots, \qquad \forall N,$$

and the F_N^e can be determined by the DP procedure discussed above. For each N, let $(z_1^N, \ldots, z_N^N)$ be any element of F_N^e. Then, for each j, z_j is the limit of the sequence $\{z_j^N\}_{N=j}^\infty$. In this way, we may successively *approximate* $z_1, z_2, \ldots$.

Finally in this section, under certain additional hypotheses, we obtain a measure of the rate of convergence of X_N to X. By definition, the Hausdorff metric D_β is given by

$$D_\beta(A,B) \;=\; \max\left(\max_{x\in A} d_\beta(x,B),\ \max_{x\in B} d_\beta(x,A)\right), \qquad \forall A,B \in \mathcal{K}_0(Y).$$

Since $X \subseteq X_N,\ \ \forall N$, it follows that $d_\beta(x, X_N) = 0,\ \ \forall x \in X$. Thus,

$$D_\beta(X_N, X) \;=\; \max_{x\in X_N} d_\beta(x, X)), \qquad \forall N.$$

Now let $x \in X_N$, so that $x = (x_1, \ldots, x_N, x_{N+1}, \ldots)$, with $(x_1, \ldots, x_N) \in F_N$. Since all finite horizon strategies are infinitely feasibly extendable, there exists $x^N \in X$ such that $(x_1^N, \ldots, x_N^N) = (x_1, \ldots, x_N)$. This implies that

$$d_\beta(x,X) \;=\; \min_{y\in X} d_\beta(x,y) \;\leq\; d_\beta(x,x^N) \;=\; \sum_{j=N+1}^{\infty} \beta_j d_j(x_j, x_j^N), \qquad \forall x \in X_N.$$

Recall that

(i) for each j, $\ Y_j = \cup\{Y_j(s_{j-1} : s_{j-1} \in S_{j-1}\}$, i.e., every available decision in period j is feasible for some feasible state ending period $j-1$.

For the remainder of this section, we make the following additional assumptions:

(ii) for each j, and $s_{j-1} \in S_{j-1}$, the mapping $f_j(s_{j-1}, \cdot) : Y_j(s_{j-1}) \to S_j$ is one-to-one with range given by some $S_j(s_{j-1}) \subseteq S_j$, and inverse mapping $f_j(s_{j-1}, \cdot)^{-1}$;

(iii) for each j, the mappings $\{f_j(s_{j-1},\cdot)^{-1} : \sigma_{j-1} \in S_{j-1}\}$ satisfy a uniform Lipschitz condition of the form

$$d_j\big(f_j(s_{j-1},\cdot)^{-1}(s_j),\ f_j(s'_{j-1},\cdot)^{-1}(s'_j)\big) \quad \leq \quad \lambda_j(\rho_{j-1}(s_{j-1},s'_{j-1}) + \rho_j(s_j,s'_j)\big),$$

for all $s_{j-1},\ s'_{j-1} \in S_{j-1},\ \ \forall s_j,\ s'_j \in S_j$, where $\lambda_j > 0,\ \rho_j$ is the metric on S_j and the right hand side defines the corresponding metric $\rho_{j-1} \times \rho_j$ on $S_{j-1} \times S_j$.

Since, $x^N \in X$, it follows that $s^N_{j-1} = \sigma_{j-1}(x^N_1, \ldots, x^N_{j-1}) \in S_{j-1}, \quad \forall j \geqslant N+1$. However, in general, x_j is just an arbitrary element of $Y_j, \quad \forall j \geqslant N+1$. By property (i), there exists $s_{j-1} \in S_{j-1}$ such that $s_{j-1} = \sigma_{j-1}(x_1, \ldots, x_j), \quad \forall j \geqslant N+1$. Hence, $(s_{j-1}, x_j),\ (s^N_{j-1}, x^N_j) \in D_j$, with

$$s_j = f_j(s_{j-1}, x_j) \in S_j \qquad \text{and} \qquad s^N_j = f_j(s^N_{j-1}, x^N_j) \in S_j, \qquad \forall j \geqslant N+1.$$

By property (ii),

$$x_j = f_j(s_{j-1},\cdot)^{-1}(s_j) \qquad \text{and} \qquad x^N_j = f_j(s^N_{j-1},\cdot)^{-1}(s^N_j), \qquad \forall j \geqslant N+1,$$

and, by property (iii),

$$\begin{aligned} d_j(x_j, x^N_j) &= d_j\big(f_j(s_{j-1},\cdot)^{-1}(s_j), \quad f_j(s^N_{j-1},\cdot)^{-1}(s^N_j)\big) \\ &\leq \lambda_j\big(\rho_{j-1}(s_{j-1}, s^N_{j-1}) + \rho_j(s_j, s^N_j)\big) \\ &\leq \lambda_j\big(\mathrm{diam}(S_{j-1}) + \mathrm{diam}(S_j)\big). \end{aligned}$$

Hence,

$$d_\beta(x, x^N) \ \leq \sum_{j=N+1}^{\infty} \beta_j\lambda_j\big(\mathrm{diam}(S_{j-1}) + \mathrm{diam}(S_j)\big) \ \leq \sum_{j=N+1}^{\infty} \beta_j\lambda_j\ \big(\mathrm{diam}(S_{j-1}) + \mathrm{diam}(S_j)\big), \qquad \forall N.$$

Consequently,

$$\begin{aligned} D_\beta(X_N, X) &= \max_{x \in X_N} d_\beta(x, X) \\ &\leq \max_{x \in X_N} \big(\min_{y \in X} d_\beta(x, y)\big) \\ &\leq \max_{x \in X_N} d_\beta(x, x^N) \\ &\leq \sum_{j=N+1}^{\infty} \beta_j\lambda_j\big(\mathrm{diam}(S_{j-1}) + \mathrm{diam}(S_j)\big), \qquad \forall N. \end{aligned}$$

Since the β_j are arbitrary, while the λ_j and the $\mathrm{diam}(S_j)$ are problem data, we may choose the β_j such that

$$\sum_{j=1}^{\infty} \beta_j\lambda_j\big(\mathrm{diam}(S_{j-1}) + \mathrm{diam}(S_j)\big) \ < \ \infty.$$

In particular, if $\sup_j \mathrm{diam}(S_j) < \infty$, then choose the β_j such that $\sum_{j=1}^{\infty} \beta_j\lambda_j < \infty$. Finally, if $\sup_j \lambda_j < \infty$, then simply let $\beta_j = r^j$, for any choice of $0 < r < 1, \ \ \forall j$.

Obviously, all the previous results hold for any problem in which the decision spaces are all finite and discrete, in which case the mappings σ_N are automatically open, so that $X^e \neq \emptyset$. If it is also a singleton $z = (z_j)_{j=1}^{\infty}$, we may successively construct the components z_j *precisely*. In the next section, we apply our results to a case where the action spaces are non-discrete.

4. Application to Production Planning

Consider a production planning problem involving a single product [4]. Suppose:

- there is no imposed maximum possible production level in each period j;
- there is no backlogging permitted in each period j;
- $0 \leq d_j \in \mathbb{R}$ denotes the *deterministic* product demand level in period j;
- $0 \leq x_j$ denotes the production decision level in period j;
- $0 \leq s_j \in \mathbb{R}$ denotes the resulting inventory level ending period j;
- the maximum allowable inventory in period j is $b_j > 0$, so that $0 \leq s_j \leq b_j, \ \ \forall j$;

• there is zero inventory starting period 1;
• $p_j : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the production cost function in period j;
• $h_j : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the inventory holding cost function in period j;

We will now employ the theory developed in the previous sections of this paper to establish the existence of an efficient solution for this problem. Since efficient solutions are also average optimal under mild regularity conditions [13], they enjoy strong properties of optimality for undiscounted problems. This existence proof for an efficient solution to this production planning problem under continuous controls is the first we are aware of for this model.

Clearly, for each j, the state space $S_j = [0, b_j], \quad \forall j$, with state transition function given by

$$f_j(s_{j-1}, y_j) \quad = \quad s_{j-1} + y_j - d_j.$$

Given inventory $0 \le s_{j-1} \le b_j$ ending period $j-1$, decision $y_j \geqslant 0$ is feasible for s_{j-1} if and only if the resulting inventory $s_{j-1} + y_j - d_j$ satisfies $0 \le s_{j-1} + y_j - d_j \le b_j$. Thus, for $0 \le s_{j-1} \le b_j$, the set $Y_j(s_{j-1})$ of feasible decisions for s_{j-1} is the set of all y_j belonging to the compact interval

$$\big[\max\{0, d_j - s_{j-1}\},\ b_j + d_j - s_{j-1}\big],$$

and consequently, the set Y_j of all feasible decisions in period j is the set of all y_j belonging to the compact interval

$$\big[\max\{0, d_j - b_j\},\ b_j + d_j\big].$$

For each $(y_1, \dots, y_j)$, the resulting ending state is given by

$$\sigma_j(x_1, \dots, x_j) \quad = \quad \sum_{i=1}^{j} x_i - \sum_{i=1}^{j} d_i,$$

and $(x_1, \dots, x_j)$ is equivalent to $(y_1, \dots, y_j)$ if and only if

$$\sum_{i=1}^{j} x_i = \sum_{i=1}^{j} y_i.$$

It follows that F_j is the set of all $(y_1, \dots, y_j)$ satisfying

$$\max\{0, d_j - \sigma_{i-1}(y_1, \dots, y_{i-1})\} \ \le\ y_i \ \le\ b_i + d_i - \sigma_{i-1}(y_1, \dots, y_{i-1}), \qquad \forall 1 \le i \le j,$$

with $\sigma_{i-1}(y_1, \dots, y_{i-1}) = 0$, for $i = 1$. Alternately, since

$$\sigma_i(y_1, \dots, y_i) = \sigma_{i-1}(y_1, \dots, y_{i-1}) + y_i - d_i, \qquad \forall 2 \le i \le j,$$

and $\sigma_1(y_1) = y_1 - d_1$, the feasible region F_j is the set of all $(y_1, \dots, y_j)$ such that $y_i \geqslant 0$ and

$$0 \le \sigma_i(y_1, \dots, y_i) \le b_i, \qquad \forall 1 \le i \le j.$$

The feasible region F is determined analogously. Since the functions $x \to \max\{0, b_j - x\}$ and $x \to b_j + d - x$ are continuous, it follows that our production planning model has the closed graph property. The infinite horizon optimization problem is then given by:

$$\max \sum_{j=1}^{\infty} \big[p_j(x_j) + h_j(\sigma_{j-1}(x_1, \dots, x_{j-1}))\big] \qquad \text{subject to} \qquad (x_1, x_2, \dots) \in F.$$

4.1 Theorem. *For each j, the mapping $\sigma_j : F_j \to S_j$ is open. Hence, there exists an efficient solution for our production planning problem.*

Proof. We apply Theorem 3.8. Fix j and let $\lim_{k\to\infty} s_j^k = s_j$ be a convergent sequence in $S_j = [0, b_j]$. Then $\sigma_j^{-1}(s_j^k) \subseteq F_j, \quad \forall k$, and $\sigma_j^{-1}(s_j) \subseteq F_j$. We next show that $\sigma_j^{-1}(s_j) \subseteq \limsup_k \sigma_j^{-1}(s_j^k)$. Passing to a subsequence if necessary, we may assume that $s_j^k \downarrow s_j$ or $s_j^k \uparrow s_j$ monotonically, as $k \to \infty$.

Let $(y_1, \dots, y_j) \in \sigma_j^{-1}(s_j)$. Then $\sigma_j(y_1, \dots, y_j) = s_j$ and

$$0 \quad \le \quad \sigma_i(y_1, \dots, y_i) \quad = \quad \sum_{n=1}^{i} y_n - \sum_{n=1}^{i} d_n \quad \le \quad b_j, \qquad \forall 1 \le i \le j.$$

Suppose $s_j^k \downarrow s_j$, as $k \to \infty$. If $s_j = b_j$, then necessarily $s_j^k = b_j$, $\forall k$, so that $\sigma_j^{-1}(s_j^k) = \sigma_j^{-1}(s_j)$, $\forall k$, and $\sigma_j^{-1}(s_j) = \limsup_k \sigma_j^{-1}(s_j^k) = \sigma_j^{-1}(s_j^k)$, $\forall k$.

Now suppose $0 \leq s_j < b_j$. If there exists a subsequence of $\{s_j^k\}_{k=1}^{\infty}$ which is equal to s_j, then proceed as in the previous case. Thus, we may also assume that $s_j^k > s_j$, $\forall k$. Define $y_j^k = y_j + s_j^k - s_j$, $\forall k$. Then

$$\max\{0, d_j - \sigma_{i-1}(y_1, \ldots, y_{j-1})\} \;\leq\; y_j^k \;\leq\; b_j + d_j - \sigma_{i-1}(y_1, \ldots, y_{j-1}) \qquad \text{and}$$

$$\sigma_j(y_1, \ldots, y_{j-1}, y_j^k) \;=\; \sum_{i=1}^{j-1} y_i + y_j^k - \sum_{i=1}^{j} d_i \;=\; \sum_{i=1}^{j} y_i + s_j^k - s_j - \sum_{i=1}^{j} d_i \;=\; s_j^k, \quad \forall k,$$

where $0 \leq s_j \leq s_j^k \leq b_j$. Since

$$0 \;\leq\; \sigma_i(y_1, \ldots, y_i) \;=\; \sum_{n=1}^{i} y_n - \sum_{n=1}^{i} d_n \;\leq\; b_j, \qquad \forall 1 \leq i \leq j-1,$$

also, it follows that $(y_1, \ldots, y_{j-1}, y_j^k) \in \sigma_j^{-1}(s_j^k)$, $\forall k$. Clearly, $\lim_{k\to\infty}(y_1, \ldots, y_{j-1}, y_j^k) = (y_1, \ldots, y_{j-1}, y_j)$, for the resulting subsequence, so that $(y_1, \ldots, y_{j-1}, y_j) \in \limsup_k \sigma_j^{-1}(s_j^k)$.

Next suppose $s_j^k \uparrow s_j$, as $k \to \infty$. The proof here is similar to that of the previous case. We leave the details to the interested reader.

□

Theorem 4.1 assures the existence of an efficient solution to the production planning problem under very general conditions, including essentially arbitrary cost and demand profiles.

Finally, observe that, for this case, the Lipschitz constants $\lambda_j = 1$, $\forall j$, so that

$$D_\beta(X_N, X) \leq \sum_{i=N+1}^{\infty} r^i, \qquad \forall 0 < r < 1.$$

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