A Paradox in Equipment Replacement under Technological Improvement *

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Abstract

It is commonly assumed that an acceleration in technological improvement should result in a more rapid introduction of new technology. Under a simple technological change model of a constant factor improvement in equipment costs per period, we show that, paradoxically, the effect is to optimally delay the introduction of new technology.

Key Words and Phrases

Equipment replacement, technological change, adoption of new technology, infinite horizon optimization

1 Introduction

US competitiveness was perceived as in a weakened condition in the early 80's, resulting in considerable effort to determine the reasons for this predicament. One explanation offered at the time was the failure of US Industry to take the long view, thus failing to justify the large upfront costs of replacing old technology by new. A report commissioned by the National Association of Accountants and a consortium of high-technology manufacturing companies

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concluded that among the reasons responsible for the failure of American manufacturing companies to update their factories was their reliance "on old tools to deal with changing, new and very different manufacturing environments" (International Herald Tribune, March 29, 1987). Among the problems cited was the insistence of many companies on short payback periods for investment in technology. To excluce the possibility of distortions by such end-of-study effects, we explore in this note a simple but intuitively appealing model of technological improvement in the context of an infinite horizon planning model. Nonetheless, as shall see, our analysis suggests that a reluctance to introduce new technology may not be attributable to shorter planning horizons alone.

The current literature on equipment replacement models in the presence of technological change is predominantly directed toward developing algorithms for determining optimal replacement decisions. Much of the effort is directed toward establishing the existence and discovery of so-called forecast horizons that are sufficiently long that the current replacement decision is unaffected by what transpires past this horizon. See for example Bean, Lohmann and Smith [1, 2], Sethi and Chand [8], Chand and Sethi [3], Goldstein et al [5]. Since then, by construction, the current replacement decision is in agreement with an infinite horizon optimal replacement decision, these nominally finite horizon models also have the potential to uncover the effect of technological improvement on the rapidity of acquiring new and better technology. For example, Niar and Hopp [7] take the forecast horizon approach within the context of a random lead time to introduction of a new technology. They show that as the probability increases that the new technology will appear, it is more likely we will keep our existing machine. However, Hopp and Niar [6] introduced an alternative model of technological breakthroughs where the tendency can be to replace the current machine earlier as the probability of a future technological breakthrough increases. Both of these models are limited to a single type of technological breakthrough and offer no guidance as to which model is more realistic.

An example of an infinite horizon model of technological improvement is offered in Elton and Gruber [4] who assume a linear model of technological change. They establish that an equal life replacement policy is optimal despite the assumption that technology is improving over time. However they do not explore the effect on this cycle time of an acceleration in technological change.

We adopt a geometric model of technological change in this paper. Specifically, we assume that the costs of equipment acquisition as well as maintenance and operating costs drop by a constant factor after each time period. For example, the costs of computer memory chips follow the so-called Moore Law of dropping by a factor of 2 every year. Our focus within this simplest model of technological change is to determine the effect of an acceleration in technological improvement on the frequency of equipment replacement. We establish the paradoxical result that *we replace less often as technology improves.* An important and perhaps unsettling implication is that rapid technological improvement may not and indeed should not necessarily lead to more rapid replacement of old technology.

2 Model Formulation

We begin by modeling the costs of acquiring and maintaining a machine in the absence of technological improvement. Let $\bar{C}_i$ represent the undiscounted cost of acquiring, operating, maintaining, and salvaging a machine kept for i periods, $i = 1, 2, \ldots$. Let

$$C_i = \bar{C}_i - \bar{C}_{i-1}$$

represent the incremental cost of keeping a machine for i periods, $i = 1, 2, \ldots$ where $\bar{C}_0 = 0$. We assume that it is never less costly to keep a machine for an additional period, i.e. that $C_i \geq 0, i = 1, 2, \ldots$. Setting $\tilde{C}_i(\gamma)$ to be the discounted cost of keeping a machine for the first i periods when the discount factor is γ, we get

$$\tilde{C}_i(\gamma) = \sum_{k=1}^{i} \gamma^{k-1} C_k.$$

Note that we do not attempt to correct for when a cost may be incurred within a period.

Now for our model of technological improvement. We assume that all costs decrease by a constant factor λ over each period of time where $0 < \lambda < 1$. Hence the incremental cost C_i of keeping a machine for the first i periods becomes $\lambda^{i-1} C_i$. Then letting $\tilde{C}_i(\gamma, \lambda)$ be the discounted cost of keeping a machine for the first i periods, we have

$$\tilde{C}_i(\gamma, \lambda) = \sum_{k=1}^{i} \gamma^{k-1} \lambda^{k-1} C_k = \sum_{k=1}^{i} (\gamma\lambda)^{k-1} C_k = \tilde{C}_i(\alpha)$$

where $\alpha = \gamma\lambda$. That is , *the effect of technological improvement is to reduce the discount factor within a stationary model without technological change.* Therefore we assume henceforth a time stationary model of equipment costs where the effect of technological improvement becomes the effect of dropping the prevailing discount factor.

3 The Effect of Technological Improvement on the Rate of Acquisition of New Technology

Since costs are stationary, our optimal replacement strategy is to keep equipment for a time $\tau^*(\alpha)$ where

$$\tau^*(\alpha) = \text{argmin}_{i=1,2,\ldots} \frac{\tilde{C}_i(\alpha)}{1 - \alpha^i}.$$

Equivalently, we may find $\tau^*(\alpha)$ as that replacement cycle i that minimizes the equated annual charge $A_i(\alpha)$ where

$$A_i(\alpha) = \frac{(1 - \alpha)\tilde{C}_i(\alpha)}{1 - \alpha^i}.$$

We now explore how $A_i(\alpha)$ changes as α decreases. Our ultimate goal is to see how $\tau^*(\alpha)$ changes as α decreases.

We begin with an assumption that we make henceforth.

Assumption 1 There exists a positive integer m such that $C_i \geq C_{i+1}$ for $1 \leq i < m$ and $C_i \leq C_{i+1}$ for $i \geq m$.

Assumption 1 requires that the undiscounted marginal cost of keeping the equipment initially drops and then eventually increases as the equipment ages. For example, the assumption is met with $m = 1$ if total undiscounted cost $\bar{C}_i$ is convex in i. This would correspond to the case where the rate of decline in undiscounted salvage value is exceeded by a rise in undiscounted maintainance and operating costs.

To compare equated annual charges when the discount factor drops, let A_i be the equated annual charge for keeping the machine from period 1 through i when the discounting factor is α and let B_i be the equated annual charge of keeping the machine from period 1 through i when the discounting factor is β where $\alpha < \beta$. That is, $\sum_{k=1}^{i} \alpha^{k-1} A_i = \sum_{k=1}^{i} \alpha^{k-1} C_k$ where $0 < \alpha < 1$ and $\sum_{k=1}^{i} \beta^{k-1} B_i = \sum_{k=1}^{i} \beta^{k-1} C_k$ and $0 < \alpha < \beta < 1$.

Lemma 1 *If $A_{i-1} > C_i$ then $A_{i-1} > A_i > C_i, i \geq 2$. (This is also true with $>$ replaced by $\geq$ throughout.)*

Proof : Since $A_{i-1} > C_i$,

$$\sum_{k=1}^{i} \alpha^{k-1} A_{i-1} \quad > \quad \sum_{k=1}^{i-1} \alpha^{k-1} A_{i-1} + \alpha^{i-1} C_i \quad > \quad \sum_{k=1}^{i} \alpha^{k-1} C_i. \tag{1}$$

By definition of A_i,

$$\begin{aligned} \sum_{k=1}^{i} \alpha^{k-1} A_i &= \sum_{k=1}^{i} \alpha^{k-1} C_k \\ &= \sum_{k=1}^{i-1} \alpha^{k-1} C_k + \alpha^{i-1} C_i. \end{aligned}$$

By definition of A_{i-1},

$$\sum_{k=1}^{i} \alpha^{k-1} A_i \quad = \quad \sum_{k=1}^{i-1} \alpha^{k-1} A_{i-1} + \alpha^{i-1} C_i. \tag{2}$$

From (1) and (2), we have

$$\sum_{k=1}^{i} \alpha^{k-1} A_{i-1} \quad > \quad \sum_{k=1}^{i} \alpha^{k-1} A_i \quad > \quad \sum_{k=1}^{i} \alpha^{k-1} C_i.$$

Therefore, $A_{i-1} > A_i > C_i$. Note that is argument is also true with $>$ replaced by $\geq$. □

Lemma 2 *It is optimal to keep the machine at least m periods.*

PROOF : By definition, $A_1 = C_1$. For $i < m$, $C_i \geq C_{i+1}$. By Lemma 1 for $i = 2$, we get since $A_1 = C_1 \geq C_2$ that $A_1 \geq A_2 \geq C_2 \geq C_3$ if $2 < m$. Again, by Lemma 1 for $i = 3$, we get since $A_2 \geq C_3$ that $A_2 \geq A_3 \geq C_3 \geq C_4$ if $3 < m$. In general, for all $i < m$, we have $A_i \geq A_{i+1}$. Hence, the discounted cost of keeping the machine for $i+1$ periods is less than keeping it for i periods for $i < m$. Thus, it is optimal to keep the machine at least m periods.□

Let $A'_\gamma(k, A, j)$ be such that paying $A'_\gamma(k, A, j)$ equally from period 1 through k is equivalent to paying A equally from period 1 through j, $1 \leq j < k$, when the discounting factor is γ. Note that $A'(k, A, j) \leq A$ for $k > j$.

Lemma 3 *If $\beta > \alpha$ and $k > j \geq 1$, then $A'_\alpha(k, A, j) \geq A'_\beta(k, A, j)$.*

PROOF : By definition of $A'_\gamma(k, A, j)$,

$$\sum_{i=1}^{j} \alpha^{i-1} A = \sum_{i=1}^{k} \alpha^{i-1} A'_\alpha(k, A, j) \quad \text{and} \quad \sum_{i=1}^{j} \beta^{i-1} A = \sum_{i=1}^{k} \beta^{i-1} A'_\beta(k, A, j).$$

Since incremental costs are assumed to be nonnegative, one of the following two cases can happen. Case I, $A = 0$. Then, $A'_\alpha(k, A, j) = A'_\beta(k, A, j) = 0$. Case II, $A > 0$. Then, we have

$$A'_\alpha(k, A, j) \left(1 + \frac{\sum_{i=j+1}^{k} \alpha^{i-1}}{\sum_{i=1}^{j} \alpha^{i-1}}\right) \quad = \quad A \quad = \quad A'_\beta(k, A, j) \left(1 + \frac{\sum_{i=j+1}^{k} \beta^{i-1}}{\sum_{i=1}^{j} \beta^{i-1}}\right).$$

Hence,

$$A'_\alpha(k, A, j) \left(1 + \frac{\sum_{i=j+1}^{k} \alpha^{i-1}}{\sum_{i=1}^{j} \alpha^{i-1}} \cdot \frac{\alpha^{-j}}{\alpha^{-j}}\right) \quad = \quad A'_\beta(k, A, j) \left(1 + \frac{\sum_{i=j+1}^{k} \beta^{i-1}}{\sum_{i=1}^{j} \beta^{i-1}} \cdot \frac{\beta^{-j}}{\beta^{-j}}\right).$$

Therefore,

$$A'_\alpha(k, A, j) \left(1 + \frac{\sum_{i=1}^{k-j} \alpha^{i-1}}{\sum_{i=1}^{j} \alpha^{i-j-1}}\right) \quad = \quad A'_\beta(k, A, j) \left(1 + \frac{\sum_{i=1}^{k-j} \beta^{i-1}}{\sum_{i=1}^{j} \beta^{i-j-1}}\right).$$

But $0 < \alpha < \beta$ implies $\sum_{i=1}^{k-j} \alpha^{i-1} < \sum_{i=1}^{k-j} \beta^{i-1}$, and $i - j - 1 < 0$ for all $i \leq j$ implies $\sum_{i=1}^{j} \alpha^{i-j-1} > \sum_{i=1}^{j} \beta^{i-j-1}$. Therefore, if $A > 0$, then $A'_\alpha(k, A, j) > A'_\beta(k, A, j)$. Therefore, $A'_\alpha(k, A, j) \geq A'_\beta(k, A, j)$. □

Lemma 4 $A_m \geq B_m$.

Proof : By rearranging the summation, we get

$$\sum_{i=1}^{m} \alpha^{i-1} C_i = \sum_{k=1}^{m-1} \sum_{i=1}^{k} \alpha^{i-1}(C_k - C_{k+1}) + \sum_{i=1}^{m} \alpha^{i-1} C_m. \tag{3}$$

Since $C_k \geq C_{k+1}$ for all $k < m$, $C_k - C_{k+1} \geq 0$ for all $k < m$. Let $A_{m,k} = A'_\alpha(m, C_k - C_{k+1}, k)$, which means paying $A_{m,k}$ equally from period 1 through m is equivalent to paying $C_k - C_{k+1}$ equally from period 1 through k, $k < m$, when the discount factor is α, i.e.,

$$\sum_{i=1}^{m} \alpha^{i-1} A_{m,k} = \sum_{i=1}^{k} \alpha^{i-1}(C_k - C_{k+1}).$$

From (3), we have

$$\sum_{i=1}^{m} \alpha^{i-1} C_i = \sum_{k=1}^{m-1} \sum_{i=1}^{m} \alpha^{i-1} A_{m,k} + \sum_{i=1}^{m} \alpha^{i-1} C_m.$$

By again rearranging the summation, we have

$$\begin{aligned} \sum_{i=1}^{m} \alpha^{i-1} C_i &= \sum_{i=1}^{m} \alpha^{i-1} \sum_{k=1}^{m-1} A_{m,k} + \sum_{i=1}^{m} \alpha^{i-1} C_m \\ &= \sum_{i=1}^{m} \alpha^{i-1} \left(C_m + \sum_{k=1}^{m-1} A_{m,k} \right). \end{aligned}$$

By definition of A_m, $\sum_{i=1}^{m} \alpha^{i-1} A_m = \sum_{i=1}^{m} \alpha^{i-1} C_i$. Thus,

$$\sum_{i=1}^{m} \alpha^{i-1} A_m = \sum_{i=1}^{m} \alpha^{i-1} \left(C_m + \sum_{k=1}^{m-1} A_{m,k} \right).$$

Therefore,

$$A_m = C_m + \sum_{k=1}^{m-1} A_{m,k}.$$

Now apply these arguments to the case where the discount factor is β. Let $B_{m,k} = A'_\beta(m, C_k - C_{k+1}, k)$, then we also have

$$B_m = C_m + \sum_{k=1}^{m-1} B_{m,k}.$$

By Lemma 3 and by definition of $A_{m,k}$ and $B_{m,k}$, we have for all $k < m$,

$$A_{m,k} = A'_\alpha(m, C_k - C_{k+1}, k) \geq A'_\beta(m, C_k - C_{k+1}, k) = B_{m,k}.$$

We then have $A_m \geq B_m$. □

Lemma 5 *If $A_{i-1} < C_i$ for some $i > m$, then $A_{i-1} < A_j$ for all $j \geq i$. (This is also true if we replace $<$ by $\leq$ throughout.)*

PROOF : Proof by induction. If for some $k > m$, $A_{k-1} < C_k$,

$$\sum_{j=1}^{k} \alpha^{j-1} A_{k-1} \quad < \quad \sum_{j=1}^{k-1} \alpha^{j-1} A_{k-1} + \alpha^{k-1} C_k \quad < \quad \sum_{j=1}^{k} \alpha^{j-1} C_k. \tag{4}$$

By definition of A_k,

$$\begin{aligned} \sum_{j=1}^{k} \alpha^{j-1} A_k &= \sum_{j=1}^{k} \alpha^{j-1} C_j \\ &= \sum_{j=1}^{k-1} \alpha^{j-1} C_j + \alpha^{k-1} C_k. \end{aligned}$$

By definition of A_{k-1},

$$\sum_{j=1}^{k} \alpha^{j-1} A_k \quad = \quad \sum_{j=1}^{k-1} \alpha^{j-1} A_{k-1} + \alpha^{k-1} C_k. \tag{5}$$

From (4) and (5), we have

$$\sum_{j=1}^{k} \alpha^{j-1} A_{k-1} \quad < \quad \sum_{j=1}^{k} \alpha^{j-1} A_k \quad < \quad \sum_{j=1}^{k} \alpha^{j-1} C_k.$$

Thus, $A_{k-1} < A_k < C_k$. By assumption, $C_k \leq C_{k+1}$ for all $k \geq m$. Thus, $A_k < C_k \leq C_{k+1}$. In conclusion, if $A_{k-1} < C_k$ for some $k > m$, then $A_{k-1} < A_k$ and $A_k < C_{k+1}$. Since we have assumed that $A_{i-1} < C_i$ for some $i > m$, the result follows by mathematical induction. □

Lemma 6 *If $\alpha < \beta$ and $A_{j-1} \geq B_{j-1} \geq C_j$ for some j, $2 \leq j$, then $A_j \geq B_j$.*

PROOF : By definition of A_i and A_{i-1}, and since $A_{j-1} \geq B_{j-1} \geq C_j$,

$$\begin{aligned} \sum_{i=1}^{j} \alpha^{i-1} A_j &= \sum_{i=1}^{j-1} \alpha^{i-1} A_{j-1} + \alpha^{j-1} C_j \\ &\geq \sum_{i=1}^{j-1} \alpha^{i-1} B_{j-1} + \alpha^{j-1} C_j \\ &= \sum_{i=1}^{j} \alpha^{i-1} C_j + \sum_{i=1}^{j-1} \alpha^{i-1} (B_{j-1} - C_j). \end{aligned} \tag{6}$$

Let $A'' = A'_\alpha(j, B_{j-1} - C_j, j-1)$. That is, paying A'' equally from period 1 through j is equivalent to paying $B_{j-1} - C_j$ equally from period 1 through $j-1$ when the discount factor is α, i.e.,

$$\sum_{i=1}^{j} \alpha^{i-1} A'' \quad = \quad \sum_{i=1}^{j-1} \alpha^{i-1}(B_{j-1} - C_j). \tag{7}$$

From (6) and (7), we have

$$\sum_{i=1}^{j} \alpha^{i-1} A_j \quad \geq \quad \sum_{i=1}^{j} \alpha^{i-1} C_j + \sum_{i=1}^{j} \alpha^{i-1} A'' \quad = \quad \sum_{i=1}^{j} \alpha^{i-1}(C_j + A'').$$

Thus, $A_j \geq C_j + A''$. By definition of B_i and B_{i-1}, and since $B_{j-1} \geq C_j$,

$$\begin{aligned} \sum_{i=1}^{j} \beta^{i-1} B_j \quad &= \quad \sum_{i=1}^{j-1} \beta^{i-1} B_{j-1} + \beta^{j-1} C_j \\ &= \quad \sum_{i=1}^{j} \beta^{i-1} C_j + \sum_{i=1}^{j-1} \beta^{i-1}(B_{j-1} - C_j). \end{aligned} \tag{8}$$

Let $B'' = A'_\beta(j, B_{j-1} - C_j, j-1)$. Then we also have,

$$\sum_{i=1}^{j} \beta^{i-1} B'' \quad = \quad \sum_{i=1}^{j-1} \beta^{i-1}(B_{j-1} - C_j). \tag{9}$$

From (8) and (9), we have

$$\sum_{i=1}^{j} \beta^{i-1} B_j \quad = \quad \sum_{i=1}^{j} \beta^{i-1} C_j + \sum_{i=1}^{j} \beta^{i-1} B'' \quad = \quad \sum_{i=1}^{j} \beta^{i-1}(C_j + B'').$$

Thus, $B_j = C_j + B''$. By Lemma 3 and by definition of A'' and B'',

$$A'' \quad = \quad A'_\alpha(j, B_{j-1} - C_j, j-1) \quad \geq \quad A'_\beta(j, B_{j-1} - C_j, j-1) \quad = \quad B''.$$

Thus, $A_j \geq C_j + A'' \geq C_j + B'' = B_j$. □

We finally come to our main result.

Theorem 1 *For $\alpha \leq \beta, \tau^*(\alpha) \geq \tau^*(\beta)$, i.e. the optimal time between equipment replacements increases as technology improves.*

PROOF : Since we can model the effect of technological improvement by adopting a smaller discounting factor, it is sufficient to show that the optimal time for keeping the machine increases as the discounting factor decreases.

From Lemma 2, it is optimal to keep the machine at least m periods. If $A_k \geq B_k$, for some k, $m \leq k$, then one of the following three cases occurs.

Case I: $C_{k+1} \geq A_k \geq B_k$. Then it is optimal in period k to replace in both cases, by Lemma 5.

Case II: $A_k \geq B_k \geq C_{k+1}$. By Lemma 1, it is not optimal to replace at k in both cases. Morever, by Lemma 6, we have $A_{k+1} \geq B_{k+1}$.

Case III: $A_k > C_{k+1} > B_k$. By Lemma 1, it is optimal to keep the machine in period k in the α-case. By Lemma 5 (replace α with β and A with B), it is optimal to replace the machine in period k in the β-case.

By Lemma 4 $A_m \geq B_m$. In addition, $A_k \geq B_k$, for some k, $m \leq k$, either implies a) we never keep the machine in period k in the β-case while replacing in the α-case, or b) $A_{k+1} \geq B_{k+1}$ and we inductively conclude the same result for subsequent periods. Therefore, the optimal time for keeping the machine increases as the discounting factor decreases. □

From the above theorem, we conclude that the optimal response to an acceleration of technological improvement is to keep our current equipment longer, thus delaying introduction of the new technology.

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