Infinite Horizon Optimality Criteria for Equipment Replacement under Technological Change*

Irwin E. Schochetman
Mathematics and Statistics
Oakland University
Rochester, Michigan 48309

and

Robert L. Smith
Industrial and Operations Engineering
University of Michigan
Ann Arbor, Michigan 48109

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Abstract

We consider the problem of optimally acquiring and retiring assets that are undergoing technological change over an unbounded horizon. The allowance for technological change prevents one from adopting the classic repeated-plant assumption that future replacements will be identical in costs to current asset contenders, thus confronting one with a non-stationary infinite horizon problem to solve. In this paper, we explore several variants of optimality criteria in this case where total costs diverge over the infinite horizon. In particular, we prove that under a mild condition, efficient solutions (i.e. replacement schedules that are least-cost to every replacement decision epoch) exist and are also average-cost optimal. Thus, the short-term optimality characteristics of efficient solutions are accompanied by the long-term prospect of being average-cost optimal as well. Moreover, in the case where discounted future per-period acquisition and maintenance costs go to zero, while total discounted costs diverge to infinity, we show that efficient solutions are overtaking optimal and, in particular, an overtaking optimal solution exists.

Key Words: planning horizon, overtaking and average optimal, reachability

AMS Classifications (2000): Primary 90C20, Secondary 49A99

1 Introduction and Background

We consider the problem of optimally acquiring and retiring equipment over an infinite horizon in the presence of technological change ([2,3, 6]). This equipment replacement problem, in the presence of assets that can improve or degrade over time, is a challenging problem in several respects.

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One challenge is that the problem is, in principle, defined by a potentially infinite amount of data describing the time-varying character of the anticipated technological changes that can unfold. Another is that a replacement schedule that cannot be expected to exhibit any recurring patterns, also requires a potentially infinite amount of information to define. Both of these dilemmas are addressed by a planning horizon approach that attempts to recursively define an optimal replacement schedule by solving a sequence of finite horizon truncations of the problem in a rolling horizon fashion (see for example [3]).
In this paper, we explore the challenge of formulating a notion of infinite horizon optimality in the case where total costs diverge. In this case of non-stationary costs where one must seek in general non-stationary strategies, an simply average cost optimal strategy can perform arbitrarily poorly in the short run. In this paper, we establish conditions for the existence of replacement strategies which are short-run optimal, in that they are total cost minimizing to each of their replacement epochs, as well as long-run optimal as measured, for example, by average costs. The key property we establish for equipment replacement problems is a bounded time reachability property among the states of a dynamic programming formulation of the problem. In the case of discounted costs where the rate of interest is insufficiently strong to render total costs finite, but is sufficient to drive per-period costs to zero, we prove that efficient solutions (which always exist) are also overtaking optimal, i.e. any efficient solution is eventually of lower cost than any competing replacement strategy. Algorithms for computing efficient solutions are provided in [10].
We begin in section 2 with the mathematical model of the equipment replacement problem. In section 3, we discuss several notions of state reachability, and establish conditions on our problem data under which these reachability properties hold. Finally, in section 4, we establish conditions for the existence of optimal replacement strategies which are strongly, weakly overtaking, overtaking, efficient and average optimal.

2 The Mathematical Model

Suppose that we have a single piece of equipment in place at the beginning of period 1. At the beginning of each period j (end of period $j-1$), we have the option of keeping the equipment in place or replacing it by (a new piece of) one of m_j alternative equipment types, $\forall j = 1, 2, \ldots$. We denote this set of possible decisions available at the beginning of period j by $Y_j = \{0, 1, 2, \ldots, m_j\}, \quad \forall j = 1, 2, \ldots$, where 0 denotes the decision to retain the current piece of equipment, and $1 \leq k \leq m_j$ denotes the decision to replace the current piece of equipment by one of technology type k. Hence, action spaces are finite. Note that a piece of equipment of type i in period j may *not* be the same type as a piece of equipment of type i in period $k \neq j$. For convenience, we assume that there exists an upper bound on the number of different technologies, i.e., $m \equiv \sup_j m_j < \infty$, so that $m \geq 1$. It will also be convenient to let $Y_0 = \{1\}$ and $m_0 = 1$, corresponding to the equipment type in place at the beginning of period 1. The set Y of all infinite replacement schedules (feasible or not) is the product of the Y_j, i.e., $Y = \prod_{j=1}^{\infty} Y_j$. If $y \in Y$, then y_j represents the type of equipment in place at the start of period j, or the type of equipment installed at the start of period j. This is a compact topological space which is metrizable.
Associated with each equipment type k acquired in period j is its physical life $l_j(k)$, which represents its maximum useful economic life, so that for each $j = 0, 1, 2, \ldots$, we have a function

$$l_j : Y_j \backslash \{0\} \to \mathbb{N}, \qquad \text{where} \qquad Y_j \backslash \{0\} = \{1, \ldots, m_j\}.$$

In particular, $l_0(1)$ is the maximum life of the initial piece of equipment. We assume that

$$\sup_{1 \leq j < \infty} \max_{1 \leq k \leq m_j} l_j(k) < \infty,$$

i.e., any type of equipment in any period has a maximum useful life.
For each $j = 1, 2, \ldots$, define the replacement function $r_j : Y \to \{0, 1, \ldots, j\}$ by

$$r_j(y) = \begin{cases} \max\{i \leq j : y_i > 0\}, & \text{if there exists } 1 \leq i \leq j \text{ for which } y_i > 0, \\ 0, & \text{if } y_i = 0,\ \forall i = 1, \ldots, j, \end{cases}$$

for each $y \in Y$, so that $r_j(y)$ depends only on $y_1, \ldots, y_j$. Thus, following strategy y, when $r_j(y)$ is positive, it represents the period (at or before period j) at the start of which we last replaced equipment. If it is zero, the equipment in place in period j is the original equipment we started with at the beginning of the problem.
We turn next to describing the *feasible* replacement schedules X, i.e., those replacement schedules in Y which can be implemented. Fix a replacement schedule $y \in Y$. Note that $\ell_{r_j(y)}(y_{r_j(y)})$ represents the the maximum life, following strategy y, of the equipment type. Then y is feasible if, in every period j, the age $j - r_j(y)$ of the current piece of equipment does not exceed the maximum life $\ell_{r_j(y)}(y_{r_j(y)})$ of that piece of equipment. Thus, we have:

Lemma 1 *The (infinite horizon) feasible region X is given by*

$$X = \{y \in Y : j - r_j(y) \leq \ell_{r_j(y)}(y_{r_j(y)}),\quad \forall j = 1, 2, \ldots\}.$$

For each $N = 1, 2, \ldots,$ the finite horizon feasible region X_N is determined analogously, $\forall j = 1, \ldots, N$. Thus, $X_{N+1} \subseteq X_N$ and $X = \cap_N^\infty X_N$. In particular, $X \neq \emptyset$.

Proof Note that $X \neq \emptyset$ since $(1, 1, \ldots) \in X$. QED

We next introduce the state spaces. At time 0, the age of the original piece of equipment is (for convenience and without loss of generality) defined to be 1. Thus, the *residual* life of the original piece of equipment at time 0 is $l_0(1) - 1$. In period N, the state $s = (k, h)$ will correspond to being at the beginning of period N with a piece of equipment of type k that was acquired h periods ago, so that, in particular, $s_0 = (1, 1)$. Thus, for each $N = 1, 2, \ldots$, define

$$S_N = \{(k, h) \in \mathbb{N} \times \mathbb{N} : \quad 1 \leq h \leq \ell_{N-h+1}(k), \quad 1 \leq k \leq m_{N-h+1}\}.$$

Hence, state spaces are also finite. For convenience, let $S_0 = \{s_0\} = \{(1, 1)\}$. Note that $\ell_{N-h+1}(k)$ denotes the physical life of a type k piece of equipment acquired in period $N - h + 1$. If $y \in X_N$, then

$$s_N(y) = (y_{r_N(y)},\ N - r_N(y) + 1), \quad \forall N = 1, 2, \ldots.$$

Necessarily, if $N = 1, 2, \ldots$ and $s = (k, h) \in S_N$, then

$$X_N(k, h) = \{y \in X_N : s_N(y) = (k, h)\},$$

i.e.,

$$X_N(k, h) = \{y \in X_N : \quad r_N(y) = N - h + 1, \quad y_{N-h+1} = k\}.$$

Fix $j \geq 1$ and $s_{j-1} \in S_{j-1}$. Then $s_{j-1} = (k, h)$, where $1 \leq h \leq j, \quad 1 \leq k \leq m_{j-h}$ and $h \leq \ell_{j-h}(k)$. Then define $Y_j(s_{j-1}) = Y_j(k, h)$ as follows:

$$Y_j(k, h) = \begin{cases} Y_j, & \text{if } h < l_{j-1}(k), \\ Y_j \backslash \{0\}, & \text{if } h \geq l_{j-1}(k). \end{cases}$$

Lemma 2 *Let $j \geq 1$, $\eta \in Y_j$ and $(k,h) \in \mathbb{N} \times \mathbb{N}$. Then $((k,h),\eta) \in F_j$ (as in section 2) if and only if* $1 \leq h \leq j, \quad 1 \leq k \leq m_{j-h}, \quad h \leq l_{j-h}(k)$, *and*

$$\begin{cases} 0 \leq \eta \leq m_j, & \text{if } h < l_{j-1}(k), \\ 1 \leq \eta \leq m_j, & \text{if } h \geq l_{j-1}(k). \end{cases}$$

Proof Omitted. QED

Now fix $j \geq 1$. Define the state transition function $f_j : F_j \to S_j$ as follows:

$$f_j((k,h),\eta) = \begin{cases} (k,h+1), & \text{if } \eta = 0, \\ (\eta,1), & \text{if } \eta \neq 0, \end{cases}$$

for each $((k,h),\eta) \in F_j$.

The cost of acquiring and operating a new piece of equipment in period j involves a) retiring the current piece of equipment, b) receiving a salvage value, c) acquiring the replacement equipment at a cost-to-purchase, and d) operating that equipment over the periods it is kept. In particular, for $j = 1, 2, \ldots$, let $I_j = \{0, 1, \ldots, j\}$ for convenience, and assume we are given the following data:

- $p_j(k)$ = the purchase price at the beginning of period j of one unit of a piece of equipment of type k, $\forall k = 1, 2, \ldots, m_j$, so that $p_j : Y_j \backslash \{0\} \to \mathbb{R}$, with

$$\|p_j\| = \max\{p_j(k) : \; k = 1, 2, \ldots, m_j\}, \quad \forall j = 1, 2, \ldots;$$

- $$\omega_j(k,h) = \begin{cases} \text{the } j\text{-th period cost of operating a piece of equipment of type } k \text{ which was} \\ \text{acquired } h \text{ periods ago, } \forall k = 1, \ldots, m_{j-h}, \text{ and } \forall h = 0, \ldots, j-1; \\ \\ \text{the } j\text{-th period cost of operating the original piece of equipment,} \\ \text{for } k = 1 \text{ and } h = j, \end{cases}$$

so that $\omega_j : Y_j \times I_j \to \mathbb{R}$, with

$$\|\omega_j\| = \max\{\omega_j(k,h) : \quad h = 0, \ldots, j; \quad k = 1, \ldots, m_j\}, \quad \forall j = 1, 2, \ldots;$$

and

- $$v_j(k,h) = \begin{cases} \text{the salvage value at the beginning of period } j \text{ of a piece of equipment of type } k \\ \text{which was acquired } h \text{ periods ago, } \; \forall h = 0, \ldots, j-1, \quad \forall k = 1, \ldots, m_{j-h}; \\ \\ \text{the salvage value at the beginning of period } j \text{ of the original piece of equipment,} \\ \text{for } k = 1 \text{ and } h = j, \end{cases}$$

so that $v_j : Y_j \times I_j \to \mathbb{R}$, with

$$\|v_j\| = \max\{v_j(k,h) : \quad h = 0, \ldots, j; \quad k = 1, \ldots, m_j\}, \quad \forall j = 1, 2, \ldots,$$

and $v_j \leq p_j$, for all j, i.e., at any time, the salvage value does not exceed the purchase price. Note that purchase prices, operating costs and salvage values may increase over time.

We turn now to defining the cost functions c_j. Fix $j = 1, 2, \ldots$. For fixed $y \in X_j$, with $s_{j-1}(y) = (k, h)$, let

$$c_j(s_{j-1}(y), y_j) = \begin{cases} p_j(y_j) + \omega_j(y_j, 0) - v_j(s_{j-1}(y)), & \text{if } \ y_j > 0, \\ \omega_j(s_{j-1}(y) + (0,1)), & \text{if } \ y_j = 0, \end{cases}$$

i.e.,

$$c_j((k,h), y_j) = \begin{cases} p_j(y_j) + \omega_j(y_j, 0) - v_j(k,h), & \text{if } \ y_j > 0, \\ \omega_j(k, h+1), & \text{if } \ y_j = 0. \end{cases}$$

For each $x \in X$ and discount factor α, define the N-horizon horizon total cost $C_N(x|\alpha)$ by

$$C_N(x|\alpha) = \sum_{j=1}^{N} \alpha^{j-1} c_j(s_{j-1}(x), x_j)$$

and the infinite horizon total cost $C(x|\alpha)$ by

$$C(x|\alpha) = \sum_{j=1}^{\infty} \alpha^{j-1} c_j(s_{j-1}(x), x_j) = \lim_{N\to\infty} C_N(x|\alpha) = \sup_N C_N(x|\alpha),$$

so that $0 \le C(x|\alpha) \le \infty$, in general. Thus, the function $C(\cdot|\alpha) : X \to [0, \infty]$ is both the pointwise limit and the supremum of the continuous functions $C_N(\cdot|\alpha)$. Hence, $C(\cdot|\alpha)$ is lower semi-continuous on X, for each α. We will write $C(x)$ for $C(x|1)$. Thus,

$$0 \le C(x|\alpha) \le C(x) \le \infty, \quad \forall 0 < \alpha \le 1, \quad \forall x \in X.$$

Consequently, for given $x \in X$, if $C(x) < \infty$, then $C(x|\alpha) < \infty$, for each α. However, for $0 < \alpha < 1$, if $C(x) = \infty$, it's posssible that $C(x|\alpha) < \infty$. Our total cost optimization problem is then formulated as follows:

$$C^*(\alpha) = \inf_{x \in X} C(x|\alpha),$$

so that, in general, $0 \le C^*(\alpha) \le \infty$, and $C^*(\alpha) \le C^*(1), \quad \forall 0 < \alpha \le 1$. Note that $C^*(\alpha) < \infty$ if and only if there exists at least one $x \in X$ for which $C(x|\alpha) < \infty$. If $C^*(\alpha) < \infty$, since X is compact and $C(\cdot|\alpha)$ is lower semi-continuous, it follows that $C^*(\alpha)$ is attained. If $C^*(\alpha) = \infty$, then $C(x|\alpha) = \infty, \quad \forall x \in X$.
As is customary, we define the infinite horizon average cost (per-period) of $x \in X$ to be

$$A(x|\alpha) = \limsup_N A_N(x|\alpha), \quad \forall 0 < \alpha \le 1,$$

where $A_N(x|\alpha) = C_N(x|\alpha)/N, \quad \forall N = 1, 2, \ldots$. Our average cost optimization problem is then:

$$A^*(\alpha) = \inf_{x \in X} A(x|\alpha).$$

As before, $A^*(\alpha) < \infty$ if and only if there exists $x \in X$ such that $A(x|\alpha) < \infty$, and $A^*(\alpha) = \infty$ if and only if $A(x|\alpha) = \infty, \quad \forall x \in X$. In general, $A^*(\alpha)$ need not be attained. In particular, $X^a(\alpha) = \emptyset$ if $A^*(\alpha) = \infty$, i.e., $A(x|\alpha) = \infty, \quad \forall x \in X$, or if $A^*(\alpha) < \infty$ and is not attained.

3 Optimal Replacement Strategies and Reachability Conditions

In this section, we recall several well-known notions of optimal replacement schedule for our infinite horizon equipment problem. These include efficient and average optima and, to a lesser extent, strong, overtaking and weakly overtaking optima. We also consider certain additional state-reachability conditions for our problem which will prove to be useful for comparing our optimality criteria in the case $C^*(\alpha) = \infty$. We refer the reader to [1, 4, 5, 7-9, 12-14] for information regarding these criteria, related inclusions and reachability conditions.

Let $X^s(\alpha)$ denote the set of all **strongly optimal** replacement strategies, i.e.,

$$X^s(\alpha) = \{x \in X : C^*(\alpha) = C(x|\alpha) < \infty, \quad C(x|\alpha) \leq C(y|\alpha), \quad \forall y \in X\}.$$

If $C^*(\alpha) < \infty$, then $X^s(\alpha) \neq \emptyset$. It's possible that $C^*(\alpha) = \infty$, equivalently $X^s(\alpha) = \emptyset$. For our purposes here, this is the case of interest. Let $X^o(\alpha)$ denote the set of all **overtaking** replacement strategies, i.e.,

$$X^o(\alpha) = \{x \in X : \liminf_N [C_N(y|\alpha) - C_N(x|\alpha)] \geq 0, \quad \forall y \in X\}$$

and $X^w(\alpha)$ denote the set of all **weakly overtaking** replacement strategies, i.e.,

$$X^w(\alpha) = \{x \in X : \limsup_N [C_N(y|\alpha) - C_N(x|\alpha)] \geq 0, \quad \forall y \in X\}.$$

Of course, $X^o(\alpha) \subseteq X^w(\alpha)$. Let $X^e(\alpha)$ denote the set of all **efficient** replacement strategies, i.e.,

$$X^e(\alpha) = \{x \in X : C_N(x|\alpha) \leq C_N(y|\alpha), \quad \forall y \in X \text{ such that } s_N(x) = s_N(y)\}.$$

Finally, let $X^e(\alpha)$ denote the set of all **average optimal** replacement strategies, i.e.,

$$X^a(\alpha) = \{x \in X : A^*(\alpha) = A(x|\alpha) < \infty, \quad A(x|\alpha) \leq A(y|\alpha), \quad \forall y \in X\}.$$

In particular, $X^a(\alpha) = \emptyset$ if $A^*(x|\alpha) = \infty, \quad \forall x \in x$, or if $A^*(\alpha) < \infty$ and is not attained.
It is known [13], that, in general,

$$\emptyset \subseteq X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^w(\alpha) \subseteq X^a(\alpha).$$

If, in addition, α is such that $A^*(\alpha) < \infty$, then

$$\emptyset \subseteq X^s(\alpha) \subseteq X^o(\alpha) \subseteq \begin{cases} X^w(\alpha) \subseteq X^e(\alpha), \\ X^a(\alpha). \end{cases}$$

Remarks. Observe that if $C^*(\alpha) < \infty$, then

$$\emptyset \neq X^s(\alpha) = X^o(\alpha) = X^w(\alpha) \subseteq \begin{cases} X^e(\alpha), \\ X^a(\alpha), \end{cases}$$

so that strongly optimal strategies exist and have all the other properties. However, if $C^*(\alpha) = \infty$, then $X^s(\alpha) = \emptyset$, and the remaining optimality criteria become important, particularly efficiency, since such optima exist in our model. Needless to say, the strong emphasis here is on the case $C^*(\alpha) = \infty$.

We next turn to the state-reachability conditions. These conditions are *controllability* notions. A very strong version of such a notion in the literature is *complete reachability*, which requires that the system be able to transition from any state in any period to any state in the very next period. Another strong controllability notion requires that transition from any state at any time to any future state be accomplished by a feasible *stationary* strategy. Our state-reachability conditions are considerably weaker than these.

Recall that our equipment replacement problem has the **Bounded Time Reachability (BTR)** property if there exists a positive integer R such that for each $1 \leq K < \infty$ and each $x, y \in X$, there exists $K \leq L \leq K + R$ and $z \in X_L$ (depending on K, x, y) for which $s_K(z) = s_K(y)$ and $s_L(z) = s_L(x)$. Note that property (BTR) is independent of the cost structure and the discount factor. Consequently, we introduce two other notions of state-reachability which do depend on these data. Our problem has the **Total Cost Reachability (TCR|$\boldsymbol{\alpha}$)** if, for all $x, y \in X$, $0 < \alpha \leq 1$ and $\epsilon > 0$, there exists a positive integer M (depending on ϵ), such that for all $N \geq M$, there exists $0 \leq K \leq N$ and $z \in X$ (depending on N) such that $s_K(z) = s_K(y)$, $s_N(z) = s_N(x)$ and $C_N(z|\alpha) - C_K(z|\alpha) < \epsilon$. Thus, given $\epsilon > 0$, for sufficiently large N, there exists an earlier period K and a strategy z which steers state $s_K(y)$ at time K to state $s_N(x)$ at time N with cost less than ϵ. Our problem has the **Average Cost Reachability (ACR|$\boldsymbol{\alpha}$)** if for all $x, y \in X$ and $\epsilon > 0$, there exists a positive integer M such that for all $N \geq M$, there exists $0 \leq K \leq N$ and $z \in X$ such that $s_K(z) = s_K(y)$, $s_N(z) = s_N(x)$ and $C_N(z|\alpha) - C_K(z|\alpha) < N\epsilon$. If, in addition, $A^*(\alpha) < \infty$, then efficient implies average optimal, i.e., [13]

$$X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^w(\alpha) \subseteq X^e(\alpha) \subseteq X^a(\alpha).$$

Note that (TCR|α) implies (ACR|α), $\forall 0 < \alpha \leq 1$. The converse is false, in general. For fixed α, if property (TCR|α) is satisfied, then every efficient strategy is overtaking optimal, i.e.,

$$X^s(\alpha) \subseteq X^o(\alpha) = X^w(\alpha) = X^e(\alpha).$$

If, in addition, $A^*(\alpha) < \infty$, then [13]

$$X^s(\alpha) \subseteq X^o(\alpha) = X^w(\alpha) = X^e(\alpha) \subseteq X^a(\alpha).$$

4 Reachability Properties and Optimality

Next, we establish sufficient conditions for the previous reachability conditions to hold in our equipment replacement problem. As consequences, we obtain additional inclusions for our optimality criteria. The following is our first main result.

Theorem 1 *We have the following reachability results.*
(i) For the previous problem data, property (BTR) holds without any additional assumptions.

(ii) If α is such that

$$\lim_{j \to \infty} \frac{\alpha^{j-1} \|p_j\|}{j} = \lim_{j \to \infty} \frac{\alpha^{j-1} \|w_j\|}{j} = \lim_{j \to \infty} \frac{\alpha^{j-1} \|v_j\|}{j} = 0,$$

then property (ACR|α) holds. In particular, if

$$\lim_{j \to \infty} \frac{\|p_j\|}{j} = \lim_{j \to \infty} \frac{\|w_j\|}{j} = \lim_{j \to \infty} \frac{\|v_j\|}{j} = 0,$$

then property (ACR|α) holds for all $0 < \alpha \leq 1$*.*

(iii) If α *is such that*

$$\lim_{j\to\infty} \alpha^{j-1}\|p_j\| = \lim_{j\to\infty} \alpha^{j-1}\|w_j\| = \lim_{j\to\infty} \alpha^{j-1}\|v_j\| = 0,$$

then property (TCR|α) holds.

Proof (i) Let R be a positive integer such that

$$R \geq \sup_{1\leq j<\infty} \max_{1\leq k\leq m_j} \ell_j(k)$$

(which we have assumed to be finite). Let $1 \leq K < \infty$ and $x, y \in X$. Denote $s_K(y) = (k, h)$ and $s_N(x) = (u_N, t_N)$, $\forall N = 1, 2, \ldots$. Let $L = K + R$. To choose $z \in X_L$, observe that $1 \leq t_{K+R} \leq R$, so that

$$K + R - t_{K+R} + 1 \geq K + 1,$$

where the quantity on the left is the time at which the piece of type u_{K+R} equipment (in place in period $K + R$) was installed. Define $z \in Y$ as follows:

$$z_j = \begin{cases} y_j, & j = 1, \ldots, K, \\ 1, & j = K+1, \ldots, K+R-t_{K+R}, \\ u_{K+R}, & j = K+R-t_{K+R}+1, \\ 0, & j = K+R-t_{K+R}+2, \ldots, K+R, \\ arb., & j = K+R+1, \ldots, \infty. \end{cases}$$

Then $z \in X_L$ because we have chosen a piece of equipment of the type available and its physical life has not been exceeded. Moreover, it is clear that $s_K(z) = (k, h) = s_K(y)$, and

$$s_L(z) = (u_{K+R}, t_{K+R}) = s_L(x) = s_{K+R}(x).$$

To complete the proof of part (i), see [12].
For part (ii), observe that

$$\|c_j\| \leq \|p_j\| + \|w_j\| + \|v_j\|, \quad \forall j \geq 1.$$

By hypothesis,

$$\lim_{j\to\infty} \frac{\alpha^{j-1}\|c_j\|}{j} = 0.$$

Thus, given $x, y \in X$ and $\epsilon > 0$, let J be sufficiently large such that

$$\frac{\alpha^{j-1}\|c_j\|}{j} < \epsilon/R, \quad \forall j \geq J.$$

Let $M = J + R$ and $N \geq M$. Set $K = N - R \geq J$. Since property (BTR) holds (part (i)), there exists L such that

$$N - R = K \leq L \leq K + R = N,$$

and $w \in X_L$ such that $s_K(w) = s_K(y)$ and $s_L(w) = s_L(x)$. Let $z = (w\ |_L\ x)$ so that $s_K(z) = s_K(w) = s_K(y)$ and $s_N(z) = s_N(x)$. Also,

$$\begin{aligned} C_N(z|\alpha) - C_K(z|\alpha) &= \textstyle\sum_{j=K+1}^N \alpha^{j-1} c_j(s_{j-1}(z), z_j) \\ &\leq \textstyle\sum_{j=K+1}^N \alpha^{j-1}\|c_j\| \\ &\leq \epsilon/R \textstyle\sum_{j=K+1}^N j \\ &\leq N(N-K)\epsilon/R \\ &= \epsilon N. \end{aligned}$$

Thus, property (ACR$|\alpha$) holds. The proof of part (iii) is similar and is left to the interested reader. QED

It's worth noting that, for each $0 < \alpha \leq 1$, it can happen that the hyptheses of Theorem 4.1 hold, together with the property that $C^*(\alpha) = \infty$. (For example, if $\alpha = 1$, suppose $\|p_j\| = B/3j, \ \ \|w_j\| = B/3j$, and $\|v_j\| = B/3j$). The following is our second main result.

Theorem 2 *We have the following additional optima inclusions for our equipment replacement problem.*

(i) Efficient optimal replacement strategies exist, i.e., $X^e(\alpha) \neq \emptyset$.

(ii) If $A^*(\alpha) < \infty$ *and the functions* p_j, w_j, v_j *satisfy part (ii) of Theorem 4.1, then*

$$X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^w(\alpha) \subseteq X^e(\alpha) \subseteq X^a(\alpha),$$

and, in particular, there exist efficient optima which are also average optimal.

(iii) If $A^*(\alpha) < \infty$ *and the functions* p_j, w_j, v_j *satisfy part (iii) of Theorem 4.1, then*

$$X^s(\alpha) \subseteq X^o(\alpha) = X^w(\alpha) = X^e(\alpha) \subseteq X^a(\alpha),$$

and, in particular, there exist efficient optima which are also overtaking, weakly overtaking and average optimal.

Proof This follows from Theorem 4.1, reference [12] and the previous discussion. QED

Theorem 4.2 assures us that in the case where discounted future per-period acquisition and maintenance costs go to zero (a condition one would normally expect) while total discounted costs diverge to infinity (so that the discounted cost criterion is completely underselective), an overtaking optimal solution exists and is in fact efficient. This is a very strong notion of optimality and moreover since it is an efficient solution as well, we can find it by a forward dynamic programming procedure developed in [12].

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