Infinite Horizon Production Planning in Time-varying Systems with Convex Production and Inventory Costs

Robert L. Smith • Rachel Q. Zhang
Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, Michigan 48109

We consider the planning of production over the infinite horizon in a system with time-varying convex production and inventory holding costs. This production lot size problem is frequently faced in industry where a forecast of future demand must be made and production is to be scheduled based on the forecast. Because forecasts of the future are costly and difficult to validate, a firm would like to minimize the number of periods into the future it needs to forecast in order to make an optimal production decision today. In this paper, we first prove that under very general conditions finite horizon versions of the problem exist that lead to an optimal production level at any decision epoch. In particular, we show it suffices for the first period infinite horizon production decision to solve for a horizon that exceeds the longest time interval over which it can prove profitable to carry inventory. We then develop a closed-form expression for computing such a horizon and provide a simple finite algorithm to recursively compute an infinite horizon optimal production schedule.
(*Forecast Horizon*; *Dynamic Lot Sizing*; *Time-varying Costs*)

1. Introduction

The production lot sizing problem is a model for the control of production over a multiperiod planning horizon (Denardo 1982). It is one of the most frequently used single-item deterministic inventory planning models (Federgruen and Tzur 1991). The objective is to schedule production over the planning horizon so that demand is satisfied at minimum cost. Standard assumptions are that demand is deterministic (i.e., known in advance) and backordering is not allowed (i.e., demand cannot be satisfied by future production).

The fundamental economic tradeoff here is the balance of reductions in cost of production against corresponding increases in costs of carrying inventory. In the presence of economies of scale on the cost of production, it can prove profitable to produce more than the current period's demand and carry inventory forward to satisfy future demand, thereby lowering the average cost of production (cycle stock motive). Even in the absence of economies of scale in production costs, the future cost of production may exceed the cost of current production plus inventory carrying costs again leading to current production that exceeds current demand (speculative motive (Chand and Morton 1986)).

The choice of planning horizon to employ is a difficult issue, because the system being modeled typically has a long but otherwise indefinite lifespan. A resolution of this problem is to utilize an infinite horizon to model the underlying long but unknown finite horizon lifespan of the system. In the general case of time-varying demand and cost, the resulting model presents a challenging problem to solve (the stationary case reduces to the classic economic lot size (ELS) model (Harris 1913).

Early efforts to solve infinite horizon versions of the problem allowed time-varying demand but restricted costs to be either stationary linear (Thompson and Sethi 1980, Morton 1978a), nonstationary linear (Kunreuther and Morton 1973), or stationary convex (Kunreuther

0025-1909/98/4409/1313$05.00
Copyright © 1998, Institute for Operations Research and the Management Sciences

and Morton 1974, Morton 1978b, Modigliani and Hohn 1955, Lee and Orr 1977). Their approaches to establishing the existence of and procedures for discovery of solution and forecast horizons variously used either marginal analysis to bound optimal production levels (Morton 1978; Kunreuther and Morton 1973, 1974; Morton 1978), or Lagrange multipliers to decouple present from future decisions by forcing ending inventory to be zero (Modigliani and Hohn 1955, Lee and Orr 1977). Most of these papers provide forward algorithms together with a stopping rule that, if met, results in discovery of a solution horizon to the underlying infinite horizon problem. However, existence of solution and forecast horizons were only established under additional significantly stronger assumptions on the cost structure of the problem.

The so-called dynamic lot size version of the problem where production costs are fixed-plus-linear and inventory holding costs are linear has been extensively studied in the nonstationary case. Although the recent focus has been on computational breakthroughs in solving finite horizon versions of the problem (see, e.g., Aggarwal and Park 1990; Federgruen and Tzur 1991; and Wagelmans, Van Hoesel, and Kolen 1989), the properties exploited there have, in some cases, been used to establish conditions on finite horizon versions of the infinite horizon problem that guarantee early decision agreement with optimal decisions of the infinite horizon problem. Such a finite horizon is called a solution horizon. When the agreement does not depend on problem data (in this case demand) beyond this solution horizon, it is also called a forecast horizon because only data over this horizon needs to be forecasted to establish infinite horizon optimal early decisions (Bes and Sethi 1988). Although solution and forecast horizons may fail to exist here, Federgruen and Tzur (1991, 1992) provided a stopping rule that is guaranteed to be met whenever they do exist (see also Chand and Morton 1986).

An important property of the dynamic lot size problem is the monotonicity of the last period with production in the planning horizon N. This last property has been extensively exploited to generate forecast horizon existence and discovery results for the dynamic lot size problem and its variations (see, e.g., Wagner and Whitin 1958; Zabel 1964; Eppen et al. 1969; Thomas 1970; Blackburn and Kunreuther 1974; Lundin and Morton 1975; Bensoussan et al. 1983; Chand 1982; Chand, Sethi, and Proth 1990; and Chand, Sethi, and Sorger 1989). See also Heyman and Sobel (1984) for a general review of using policy monotonicity in homogeneous MDP problems.

In this paper, we consider the infinite horizon version of the general lot sizing problem under diseconomies of scale in production and inventory holding costs. This convexity assumption is equivalent to the condition that marginal production and holding costs be nondecreasing. For example, this includes the case where inventory costs are linear and where a firm experiences a higher overtime rate for production exceeding the standard capacity followed by a still higher unit cost for exceeding overtime capacity through outsourcing.

The optimization problem to be solved falls within the class of doubly infinite convex programming problems, because there are both an infinite number of variables (production levels) and constraints (demand satisfaction in each period). There is an extensive literature on solution and forecast horizon approaches to solving such general problems in infinite horizon optimization (see, e.g., Bean and Smith 1984, 1993; Bes and Sethi 1988; and Schochetman and Smith 1989, 1992). However, a key assumption there that guarantees that general purpose algorithms will successfully discover an equivalent finite horizon problem is uniqueness of an infinite horizon optimal solution. Although this condition is believed to be typically met in practice, it is difficult to verify.

In this paper, we explore instead a novel algorithmic approach for finding solution and forecast horizons that systematically exploits monotonicity of optimal *early* decisions in horizon N when production and inventory holding costs are convex. This focus on early decision monotonicity, as opposed to *late* decision monotonicity as in the treatment of the dynamic lot size problem where costs are concave, leads to a closed form expression for a forecast horizon guaranteed to yield optimal early production decisions for the infinite horizon problem. As we will show, the length of the forecast horizon is the longest interval of time over which it can prove profitable to carry inventory.

The paper is organized as follows. In §2, we formulate the infinite horizon model of the problem. In §3, we prove that under very general conditions, solution horizons exist leading to finite horizon versions of the problem that yield optimal solutions to the infinite horizon problem. In §4, we give a closed-form expression for computing a solution (indeed forecast) horizon and a simple recursive procedure for computing an optimal infinite horizon production schedule.

2. Problem Formulation

Consider a single-product firm where a decision for production must be made at the beginning of each period n, $n = 1, 2, \ldots$. We will adopt the following notation wherein $n = 1, 2, \ldots$.

Constants and functions:

D_n = the demand during period n (nonnegative integers)
α = the discount factor for the time value of money ($0 < \alpha < 1$)
I_0 = the inventory on hand at the beginning of period 1 (nonnegative integer)
$c_n(x)$ = the cost of producing x units of the product during period n (nonnegative)
$h_n(x)$ = the cost of holding x units of inventory ending period n (nonnegative)

Decision variables:

P_n = the production level during period n (nonnegative integers)
I_n = the inventory on hand at the end of period n (nonnegative integers)

We will use the superscript (*) to denote optimality.

With the above notation, we can formulate this infinite horizon problem, labeled Q, as

$$(Q) \text{ Minimize: } \sum_{n=1}^{\infty} \alpha^{n-1}[c_n(P_n) + h_n(I_n)] \qquad (1)$$

$$\text{Subject to: } I_{n-1} + P_n - D_n = I_n,\ n = 1, 2, \ldots. \qquad (2)$$

$$P_n \ge 0,\ I_n \ge 0,\ n = 1, 2, \ldots \qquad (3)$$

$$P_n, I_n : \text{integer},\ n = 1, 2, \ldots \qquad (4)$$

where I_0 is given. As we can see from (2), if we know the production levels P_n in all periods, we can determine the inventory levels I_n. Therefore, it suffices to find an optimal production schedule $P_1^*, P_2^*, P_3^*, \ldots$. Note, however, that this is a doubly infinite integer nonlinear programming problem and is therefore a formidable problem to solve.

3. Existence of Solution Horizons

We now investigate conditions under which a finite horizon version of the problem has an optimal first decision that is in agreement with an infinite horizon optimal first decision. If we can find an optimal infinite horizon first decision P_1^* by solving a finite horizon version of the problem, we can roll forward one period and form a new infinite horizon problem with new initial inventory $I_1^* = I_0 + P_1^* - D_1$ to obtain an optimal infinite horizon second decision for the original problem. This rolling horizon procedure can then recursively recover an optimal infinite horizon production schedule.

In this section, we formulate the N-horizon truncated version of the problem and show that, under convex production and inventory holding costs, optimal production levels of the N-horizon problem are increasing in N. We then identify conditions under which an N-horizon optimal nth decision, $1 \le n \le N$, converges as $N \to \infty$ to an infinite horizon optimal nth decision. Finally, we establish existence of a finite horizon version for solving the infinite horizon problem.

3.1. The N-Horizon Problem

We formulate the N-horizon problem, labeled $(Q(N))$, corresponding to the original infinite horizon problem (Q) as:

$$(Q(N)) \text{ Minimize: } \sum_{n=1}^{N} \alpha^{n-1}[c_n(P_n) + h_n(I_n)] \qquad (5)$$

$$\text{subject to: } I_{n-1} + P_n - D_n = I_n,\ n = 1, 2, \ldots, N \qquad (6)$$

$$P_n \ge 0,\ I_n \ge 0,\ n = 1, 2, \ldots, N \qquad (7)$$

$$P_n, I_n : \text{integer},\ n = 1, 2, \ldots, N. \qquad (8)$$

Let $S \subseteq R^{\infty}$ be the set of all feasible production schedules to (Q), $S(N) \subseteq R^N$ the set of feasible production schedules to $(Q(N))$, $P(N)$ any feasible production schedule to $(Q(N))$, and $I(N)$ the ending on hand inventories resulting from the production schedule $P(N)$, $N = 1, 2, \ldots$. We now adopt our first assumption on (Q) and hence $(Q(N))$, i.e., that both production and inventory holding costs are convex:

A0. *Production and inventory holding costs are convex, i.e., $c_n(\cdot)$ and $h_n(\cdot)$ are convex functions with $c_n(0) = h_n(0) = 0$ for all $n = 1, 2, \ldots$.*

The following lemma provides that for the N-horizon problem $(Q(N))$, increasing demand leads to a monotone increase in production.

LEMMA 1 (VEINOTT [1964]). *Let $P^*(N) = (P_1^*(N), P_2^*(N), \ldots, P_N^*(N))$ be any optimal solution for a vector $(D_1, D_2, \ldots, D_N)$ of demands. If one of these demands is increased by 1 unit, it is optimal to increase one of these production levels by 1 unit.*

The proof of this lemma can be found in Denardo (1982).

Consider now the demand profile for an $N + 1$-horizon problem where $D_{N+1} = 0$. Since, without loss of optimality, we never leave positive inventory at the end of a horizon, we conclude $I_N^* = 0$ at an optimal solution for $D_{N+1} = 0$. Then by the principle of optimality,

$$P_1^*(N) = P_1^*(N + 1) \tag{9}$$

when $D_{N+1} = 0$. Hence applying the lemma repeatedly as D_{N+1} is increased one unit at a time, we have

$$P_1^*(N) \le P_1^*(N + 1), \text{ for } N = 1, 2, \ldots \tag{10}$$

for any fixed D_{N+1}. Following the same argument, we also have

$$P_n^*(N) \le P_n^*(N + 1),$$
$$\text{for all } 1 \le n \le N,\ N = 1, 2, \ldots. \tag{11}$$

Hence, we have proven the following corollary.

COROLLARY 1. *$P_n^*(N)$ is monotonically increasing in N for any fixed n, $1 \le n \le N$.*

3.2. Optimal Solution and Value Convergence of the N-Horizon Problems

Before we discuss convergence of optimal solutions of the N-horizon problems, we need the following additional notation and assumptions. Let $C(P)$ be the objective function of (Q) for $P \in S$ and $C^* = C(P^*)$. Also let $C(P(N); N)$ be the objective function of $(Q(N))$ for $P(N) \in S(N)$ and $C^*(N) = C(P^*(N); N)$. Furthermore, we adopt the following additional assumptions on (Q):

A1. *There exists a finite cost feasible production schedule to (Q), i.e., $C(P') < \infty$ for some feasible production schedule $P' \in S$.*

A2. *The marginal costs of production are uniformly bounded from above and away from zero, i.e., $0 < \delta_n \le c_n(P_n) - c_n(P_n - 1) \le \gamma_n \le \gamma < \infty$ for all integers $P_n > 0$ and all $n = 1, 2, \ldots$.*

Assumption (A1) is needed for a solution to (P) to exist while (A2) is a regularity condition that bounds optimal production and inventory levels. We now show that $P_n^*(N)$ converges as horizon $N \to \infty$ to an infinite horizon optimal nth decision $P_n^* < \infty$ for all n (Theorem 1). That is, $\lim_{N\to\infty} P_n^*(N) = P_n^*$ under the above conditions. This componentwise convergence of $P^*(N) = (P_1^*(N), P_2^*(N), \ldots, P_N^*(N), 0, 0, \ldots)$ to $P^* = (P_1^*, P_2^*, \ldots)$ as vectors in R^∞ is precisely product convergence in R^∞ (Schochetman and Smith 1992), so we may equivalently write that

$$P^*(N) \to P^* \text{ as } N \to \infty.$$

We establish this convergence by first showing that $P_n^*(N)$ converges to an infinite horizon feasible solution as $N \to \infty$ (Lemmas 2 and 3) and then that value and hence solution convergence holds for all $n = 1, 2, \ldots$ (Lemma 4 and Theorem 1).

LEMMA 2. *There exist finite production bounds $\bar{P}_n$, $n = 1, 2, \ldots$, so that $P_n^*(N) \le \bar{P}_n < \infty$ for all N and $n = 1, 2, \ldots$.*

PROOF. Suppose not, then there exists some n and subsequence N_k^n, $k = 1, 2, \ldots$, such that

$$\lim_{k\to\infty} P_n^*(N_k^n) = \infty. \tag{12}$$

By assumption (A2),

$$\lim_{k\to\infty} c_n(P_n^*(N_k^n)) = \infty \tag{13}$$

and hence

$$\lim_{k\to\infty} C^*(N_k^n) = \infty. \tag{14}$$

However, by (A1), with $P'(N_k^n)$ the first N_k^n decisions in P',

$$C^*(N_k^n) \le C(P'(N_k^n); N_k^n) \le C(P') < \infty. \tag{15}$$

This contradicts equation (14). □

By Corollary 1, at any decision epoch, n, there exists a monotonically increasing sequence of optimal decisions $P_n^*(N)$, $N = 1, 2, \ldots$. By Lemma 2, this sequence of values is bounded from above. Therefore, $P_n^*(N)$ must converge as N goes to infinity, i.e.,

$$\lim_{N\to\infty} P_n^*(N) = \hat{P}_n < \infty \tag{16}$$

exists for all $n = 1, 2, \ldots$.

It remains to show $\hat{P}$ is infinite horizon optimal.

LEMMA 3. $\hat{P} \in S$, *i.e.*, $\hat{P}$ is infinite horizon feasible.

PROOF. Fix $n = 1, 2, \ldots$. Then

$$\sum_{j=1}^{n} \hat{P}_j = \sum_{j=1}^{n} \lim_{N\to\infty} P_j^*(N) = \lim_{N\to\infty} \sum_{j=1}^{n} P_j^*(N)$$

$$\geq \lim_{N\to\infty} \sum_{j=1}^{n} D_j - I_0 = \sum_{j=1}^{n} D_j - I_0. \tag{17}$$

Hence $\hat{P}$ is infinite horizon feasible. □

LEMMA 4. $\lim_{N\to\infty} C^*(N) = C^*$, *i.e., optimal value convergence holds.*

PROOF. Since, without loss of optimality, production is bounded in every period by Lemma 2, this follows from the general optimal value convergence result of Theorem 3.2 in Schochetman and Smith (1989). □

We can now prove our principal result that finite horizon optima monotonically converge upwards as horizon lengthens to an infinite horizon optimal solution.

THEOREM 1. *$\hat{P}$ is infinite horizon optimal, and hence $P^*(N)$ converges monotonically upward to an infinite horizon optimal production schedule, i.e.,*

$$P_n^*(N) \uparrow \hat{P}_n \text{ as } N \to \infty \text{ for all } n = 1, 2, \ldots.$$

PROOF. From (3.16), and nonnegativity of the costs, for any positive integer M,

$$\sum_{n=1}^{M} \alpha^{n-1}[c_n(\hat{P}_n) + h_n(\hat{I}_n)]$$

$$= \lim_{N\to\infty} \left\{ \sum_{n=1}^{M} \alpha^{n-1}[c_n(P_n^*(N)) + h_n(I_n^*(N))] \right\}$$

$$\leq \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \alpha^{n-1}[c_n(P_n^*(N)) + h_n(I_n^*(N))] \right\} = C^*$$

by Lemma 4. Now take the limit as $M \to \infty$ on both sides of the above inequality to get

$$C(\hat{P}) = \lim_{M\to\infty} \left\{ \sum_{n=1}^{M} \alpha^{n-1}[c_n(\hat{P}_n) + h_n(\hat{I}_n)] \right\} \leq C^*. \tag{18}$$

From Lemma 3, $\hat{P} \in S$ and hence $\hat{P}$ is infinite horizon optimal. □

Theorem 1 allows us to easily extend Veinott's monotonicity lemma to the infinite horizon case.

COROLLARY 2. *Suppose Assumptions* (A0) *through* (A2) *hold. Let P^* be any optimal solution for a vector $(D_1, D_2, \cdots)$ of demands. If one of these demands is increased by* 1 *unit, it is optimal to increase one of these production levels by* 1 *unit.*

PROOF. Let $\tilde{P}$ be the optimal infinite horizon production schedule for the demand vector $(D_1, \ldots, D_j + 1, \ldots)$ and let $\tilde{P}_n^*(N)$ be the corresponding optimal production volume in period n under this demand schedule for a planning horizon of N periods. Note that for all $n = 1, 2, \ldots$,

$$\tilde{P}_n = \lim_{N\to\infty} \tilde{P}_n^*(N) \geq \lim_{N\to\infty} P_n^*(N) = \hat{P}_n \tag{19}$$

by Lemma 1 and Theorem 1. Also

$$\sum_{k=1}^{n} \tilde{P}_k = \lim_{N\to\infty} \sum_{k=1}^{n} \tilde{P}_k^*(N) \leq \lim_{N\to\infty} \left(\sum_{k=1}^{n} P_k^*(N) + 1 \right)$$

$$= 1 + \sum_{k=1}^{n} \lim_{N\to\infty} P_k^*(N) = 1 + \sum_{k=1}^{n} \hat{P}_k \tag{20}$$

for all $n = 1, 2, \ldots$ since without loss of optimality ending inventory in period N is zero for all planning horizons $N = 1, 2, \ldots$. (19) together with (20) imply that $\tilde{P}_n = \hat{P}_n$ for all periods n but one in which $\tilde{P}_n = \hat{P}_n + 1$. □

The conclusion of Theorem 1 is called optimal solution convergence while that of Lemma 4 is called optimal value convergence. Optimal value convergence supports a method analogous to successive approximations as applied to homogeneous MDP problems (Denardo 1982). These may be viewed as equivalent to solving successively longer horizon problems as we iterate (the initial guess of value function is seen here as a terminal value at the end of horizon).

Optimal value convergence implies that for N large enough, the corresponding optimal N-horizon plan

$P^*(N)$ achieves a cost arbitrarily close to that achieved by an optimal infinite horizon solution P^*, i.e., $P^*(N)$ and P^* are close in value. But $P^*(N)$ is not an infinite horizon feasible solution. Optimal value convergence is therefore of limited use, approximating infinite horizon optimal cost, but not solutions, while it is the latter we need to implement. Still we may at times be able to extend $P^*(N)$ feasibly over the infinite horizon at small cost to achieve an infinite horizon feasible solution with nearly the same cost as P^*. The solution convergence result of Theorem 1 is however far more powerful, because policies and not just costs are arbitrarily well approximated by sufficiently long finite horizon optimal solutions. In fact, the approximation to early decisions is without error in this case as we note in the next subsection.

3.3. Solution Horizons for Solving the Infinite Horizon Problem

By Theorem 1,

$$\lim_{N\to\infty} P_n^*(N) = P_n^*,\ n = 1, 2, \ldots \tag{21}$$

where $P^* = \hat{P}$ is an infinite horizon optimum. This implies that for any $\epsilon > 0$, there exists a horizon, $N_\epsilon(n)$ such that

$$|P_n^*(N) - P_n^*| < \epsilon, \text{ for all } N \geq N_\epsilon(n). \tag{22}$$

Let $\epsilon = 1$. Then

$$|P_n^*(N) - P_n^*| < 1 \tag{23}$$

so that

$$P_n^*(N) = P_n^* \tag{24}$$

for all $N \geq N_1(n)$. In particular,

$$P_1^*(N) = P_1^*, \text{ for all } N \geq N_1^* \tag{25}$$

where $N_1^* = N_1(1)$ so that N_1^* is a *solution horizon*. That is, there exists a finite horizon N_1^* sufficiently distant that an optimal first period production lot size for any horizon that long or longer yields an infinite horizon optimal first period production lot size.

By forward dynamic programming, let $f_n(i)$ be the present value of the optimal cost from period 1 through period n with ending inventory level i in period n, where $i \geq (I_0 - \Sigma_{j=1}^n D_j)^+$. Then

$$f_n(i) = \min_{0\leq P_n\leq D_n+i} \{f_{n-1}(i + D_n - P_n) + \alpha^{n-1}[c_n(P_n) + h_n(i)]\}$$

where $f_0(i) = 0$ for $i = I_0$ and ∞ otherwise. If we knew the value of the solution horizon N_1^*, we could then solve for $f_{N_1^*}(0)$ to get an infinite horizon optimal first period production level

$$P_1^* = P_1^*(N_1^*).$$

By (24), we can similarly compute the nth period infinite horizon optimal production decision for all $n = 1, 2, \ldots$. We can then recursively find $(P_1^*, P_2^*, \cdots) = P^*$ with zero error. We turn to the computation of solution (and forecast) horizons in §4.

4. Computing Solution and Forecast Horizons

We have shown in the previous section that there exists a solution horizon N_n^* such that

$$P_n^*(N) = P_n^*, \text{ for all } N \geq N_n^*$$

at any decision epoch n. In this section, we seek a method to compute solution horizons for all $n = 1, 2, \ldots$ and a corresponding simple algorithm to compute an optimal infinite horizon solution P_n^* for all n.

Consider $P_1^*(N)$ as N increases. By Corollary 1,

$$P_1^*(N+1) \geq P_1^*(N). \tag{26}$$

Therefore, the optimal first decision either remains the same or increases as N increases. Suppose the latter, that is,

$$P_1^*(N+1) > P_1^*(N). \tag{27}$$

Since, moreover,

$$P_n^*(N+1) \geq P_n^*(N), \text{ for all } 1 \leq n \leq N \tag{28}$$

by Corollary 1, at least one additional unit of inventory is produced in period 1 and held for N periods to satisfy a unit of demand in period $N + 1$. Evidently, by (27) it is then less costly to satisfy a unit of demand in period $N + 1$ by production in period 1 than by production in later periods, and in particular than by production in period $N + 1$. Let σ be a lower bound on the marginal cost of carrying an additional unit of inventory, i.e., by convexity of h, we may set

$$\sigma = \inf_{n \geq 1}\{h_n(1)\} \geq 0.$$

Then by (A0),

$$\sigma(1-\alpha^N)/(1-\alpha) \leq \sum_{n=1}^{N} \alpha^{n-1} h_n(1) \leq \alpha^N \gamma_{N+1} - c_1(1)$$

so that $N \leq N_1^*$ where N_1^* is given by

$$N_1^* = \left\lceil \log_\alpha \left\{ \frac{(1-\alpha)c_1(1) + \sigma}{(1-\alpha)\gamma + \sigma} \right\} \right\rceil \tag{29}$$

where $\lceil X \rceil$ represents the smallest integer strictly greater than X. We conclude then

$$P_1^* = P_1^*(N_1^*) \tag{30}$$

is an infinite horizon first decision depending only on $D_1, D_2, \ldots, D_{N_1^*}$ where N_1^* is given by (29). That is, N_1^* *is a forecast horizon for the first production decision*. Following the same argument, we can compute the forecast horizon for the second production decision, and so on. A tighter bound on a forecast horizon N can be obtained by utilizing specific problem data to compute the greatest number of periods it is economic to carry inventory. That is, a forecast horizon is provided by the largest period of time it can prove profitable to hold a unit of inventory produced in period 1.

Note that N_1^* is independent of all demands. It also only depends on the values of bounds on inventory and marginal production costs. To get a feeling for the magnitude of our forecast horizon, we look at some examples.

In the simple case where production costs are stationary and linear over time,

$$\gamma = \sup_{n \geq 1}\{\sup_{P_n > 0}[c_n(P_n) - c_n(P_n - 1)]\} = c_1(1)$$

and $N_1^* = 1$. In other words, as we would expect, we only need to know the demand in the first period to make the optimal first decision regardless of the inventory costs since no inventory is needed when production cost does not vary over time.

Consider now the case where the production costs are piecewise linear or even nonlinear. In this case if we set

Table 1 **The Forecast Horizon in Days for the First Infinite Horizon Optimal Production Level**

		u				
r	v	1.2	1.4	1.6	1.8	2
0.2	0.2	1	2	3	4	5
0.2	0.1	2	4	6	8	10
0.2	0.05	4	8	12	16	20
0.1	0.2	1	2	3	4	5
0.1	0.1	2	4	6	8	10
0.1	0.05	4	8	12	16	20
0.05	0.2	1	2	3	4	5
0.05	0.1	2	4	6	8	10
0.05	0.05	4	8	12	16	20

$\gamma = uc_1(1)$, $u > 1$ (i.e., the marginal production cost will not exceed $uc_1(1)$) and $\sigma = vc_1(1)$ where v is the inventory charge as the sum of a proportion of production cost, opportunity costs, taxes, insurance costs, the value loss over time (e.g., certain products have to be sold by discount), floor space rental costs, etc., then

$$N_1^* = \left\lceil \log_\alpha \left\{ \frac{1-\alpha+v}{(1-\alpha)u + v} \right\} \right\rceil .$$

For various inventory charges v per day, discount factor $\alpha = 1/(1 + r/365)$ per day where r is the interest rate per year, we computed N_1^* for $u = 1$ to 2. The results are shown in Table 1.

We chose inventory costs unusually high here to illustrate how short these forecast horizons can be. However, even in the case of moderate inventory costs, forecast horizons can be significantly reduced by a more detailed analysis using more precise cost information to provide better bounds on the minimal forecast horizon.[1,2]

[1] We are indebted to Awi Federgruen and an anonymous referee for suggestions that significantly improved the clarity of this paper.

[2] This work was supported in part by the National Science Foundation under grants DDM-9214894, DMI-9501740, and DMI-9713723

References

Aggarwal, A., J. K. Park. 1990. Improved algorithms for economic lot-size problems. Working paper, IBM Thomas J. Watson Research Center, Yorktown Heights, NY.

Bean, J., R. L. Smith. 1984. Conditions for the existence of planning horizons. *Math. Oper. Res.* **9** 391–401.

——, ——. 1993. Conditions for the discovery of solution horizons. *Math. Programming* **59** 215–229.

Bensoussan, A., J. Crouhy, J. M. Proth. 1983. *Mathematical Theory of Production Planning*. Advanced Series in Management. North-Holland, Amsterdam.

Bès, C., S. Sethi. 1988. Concepts of forecast and decision horizons: Applications to dynamic stochastic optimization problems. *Math. Oper. Res.* **13** 295–310.

Blackburn, J, D., H. Kunreuther. 1974. Planning horizons for the dynamic lot size model with backlogging. *Management Sci.* **21** 251–255.

Chand, S. 1982. Lot sizing for products with finite demand horizon and periodic review policy. *European J. Oper. Res.* **11** 145–148.

———, T. E. Morton. 1986. Minimal forecast horizon procedures for dynamic lot size models. *Naval Res. Logist. Quart.* **33** 111–122.

———, S. Sethi, J. M. Proth. 1990. Existence of forecast horizons in undiscounted discrete-time lot size models. *Oper. Res.* **38** 884–892.

———, ———, G. Sorger. 1989. Forecast horizons in the discounted dynamic lot size model. Working paper, University of Toronto, Toronto, Canada.

Denardo, E. 1982. *Dynamic Programming: Models and Applications*. Prentice-Hall, Englewood Cliffs, NJ.

Eppen, G. D., F. J. Gould, B. P. Pashigan. 1969. Extensions of the planning horizon theorem in the dynamic lot size model. *Management Sci.* **15** 268–277.

Federgruen, A., M. Tzur. 1991. A simple forward algorithm to solve general dynamic lot sizing models with *n* periods. *Management Sci.* **37** 909–925.

———, ———. 1992. Fast solution and detection of minimal forecast horizons in dynamic programs with a single indicator of the future: Application to dynamic lot-sizing models. Working paper, Graduate School of Business, Columbia University, New York.

Harris, F. W. 1913. How many parts to make at once. *Factory: The Magazine of Management* **10** 135–136, 152. Reprint 1990, *Oper. Res.* **38** 947–950.

Heyman, D., M. Sobel. 1984. *Stochastic Models in Operations Research*. Vol. II. McGraw-Hill, NY.

Kunreuther, H. C., T. E. Morton. 1973. Planning horizons for production smoothing with deterministic demands. *Management Sci.* **20** 110–125.

———, ———. 1974. General planning horizons for production smoothing with deterministic demands. *Management Sci.* **20** 1037–1046.

Lee, D. R., D. Orr. 1977. Further results on planning horizons in the production smoothing problem. *Management Sci.* **23** 490–498.

Lundin, R. A., T. E. Morton. 1975. Planning horizons for the dynamic lot size model: Zabel vs. protective procedures and computational results. *Oper. Res.* **23** 711–734.

Modigliani, F., F. E. Hohn. 1955. Production planning over time and the nature of the expectation and planning horizon. *Econometrica* **23** 46–66.

Morton, T. 1978a. The nonstationary infinite horizon inventory problem. *Management Sci.* **24** 1474–1482.

———. 1978b. Universal planning horizons for generalized convex production scheduling. *Oper. Res.* **26** 1046–1058.

Schochetman, E. E., R. L. Smith. 1989. Infinite horizon optimization. *Math. Oper. Res.* **14** 559–574.

———, ———. 1992. Finite dimensional approximation in infinite dimensional mathematical programming. *Math. Programming* **54** 307–333.

Thomas, L. J. 1970. Price-production decisions with deterministic demand. *Management Sci.* **16** 747–750.

Thompson, G. L., S. P. Sethi. 1980. Turnpike horizons for production planning. *Management Sci.* **26** 229–241.

Veinott, A. F., Jr. 1964. Production planning with convex costs: A parametric study. *Management Sci.* **10** 441–460.

Wagelmans, A., S. Van Hoesel, A. Kolen. 1989. Economic lot-sizing: An $O(n \log n)$-algorithm that runs in linear time in the Wagner-Whitin case. CORE discussion paper no. 8922. Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

Wagner, H. M., T. M. Whitin. 1958. Dynamic version of the economic lot size model. *Management Sci.* **5** 89–96.

Zabel, E. 1964. Some generalizations of the inventory planning horizon theorem. *Management Sci.* **10** 465–471.

Accepted by Awi Federgruen; received September 3, 1996. This paper has been with the authors $2\frac{1}{2}$ months for 2 revisions.