OPTIMALITY CRITERIA FOR DETERMINISTIC
DISCRETE TIME INFINITE HORIZON OPTIMIZATION

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ABSTRACT. We consider the problem of selecting an optimality criterion, when total costs diverge, in deterministic infinite horizon optimization over discrete time. Our formulation allows for both discrete and continuous state and action spaces, as well as time-varying, i.e., non-stationary, data. The task is to choose a criterion that is neither too over-selective, so that *no* policy is optimal, nor too underselective, so that *most* policies are optimal. We contrast and compare the following optimality criteria: strong, overtaking, weakly overtaking, efficient and average. However, our focus is on the optimality criterion of efficiency. (A solution is *efficient* if it is optimal to each of the states through which it passes.) Under mild regularity conditions, we show that efficient solutions always exist and thus, are not overselective. As to underselectivity, we provide weak state reachability conditions which assure that every efficient solution is also average optimal, thus providing a sufficient condition for average optima to exist. Our main result concerns the case where the discounted per-period costs converge to zero, while the discounted total costs diverge to infinity. Under the assumption that we can reach from any feasible state to any feasible sequence of states in bounded time, we show that every efficient solution is also overtaking, thus providing a sufficient condition for overtaking optima to exist.

1. Introduction

The problem of optimally selecting a sequence of decisions over an infinite horizon is complicated by the criterion issue of imposing preferences over the collection of associated cost streams. Even in the case where the infinite stream of cost flows is discounted, the resulting discounted total costs may all be infinite. Failure of an optimality criterion to distinguish among different policies is a problem of *under-selectivity* of the criterion. At the other extreme is a notion of optimality so strong that none of the feasible policies satisfies its conditions, a problem of *over-selectivity*. In a recent paper, Schochetman and Smith (1998) considered the notion of optimality termed *efficiency* (see Ryan et al (1992)) or sometimes finite optimality (Halkin (1974)). A solution is termed efficient if, roughly speaking, it is optimal to each of the states through which

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it passes. Efficient solutions avoid being overselective in that their existence is assured by mild topological conditions. Nor are they particularly underselective in that the requirement that they be optimal to each state constrains prior states to be along optimal paths to those states. In this paper, we compare and contrast the selectivity of efficiency with more traditional notions of optimality: namely, strong, overtaking, weakly overtaking and average optimality. In particular, we develop a state reachability condition which, in the presence of discounting, assures us that efficient solutions are overtaking optimal. Since efficient solutions always exist, the latter condition provides a new sufficient condition for the existence of overtaking optimal solutions. In the discrete control setting of Schochetman and Smith (1998), it was shown that, under a state reachability condition, every efficient solution is average optimal. Here, we weaken this reachability condition and extend this result to the continuous control case.

The discrete-time, deterministic framework within which we work, and the very general nature of the underlying optimization problem, represent significant departures from the traditional context for the comparison of optimality criteria. We consider an extremely general deterministic infinite horizon optimization problem, formulated as a dynamic programming problem. Essentially, the only restriction in this work, apart from being a deterministic model, is the requirement that the set of feasible decision alternatives be compact at each decision epoch. In particular, we do *not* assume that data are stationary. Moreover, we do *not* assume *complete reachability*, i.e., the ability of the system to transition from any state to any state in the very next period. This is not an uncommon assumption in the literature. Also since we have imposed no linear space structure, we do not make any *convexity* assumptions. In general, our model framework includes production planning under nonstationary demand, parallel and serial equipment replacement under technological change, capacity planning under nonlinear demand, and optimal search in a time-varying environment.

In this paper, we compare and contrast the selectivity of efficiency with the more traditional notions of optimality including strong, overtaking, weakly overtaking, and average optimality. Strong optimality is conferred on any strategy that attains minimum total cost. Of course, it can happen (Example 3.13) that all total costs over the infinite horizon diverge, thus necessitating alternate notions of optimality. Overtaking optimality was introduced in the economic literature by Gale (1965) and von Weiszacker (1967), and later adopted by optimal control theorists. Shortly thereafter, the notion of weakly overtaking optimality was introduced by Brock (1970) for economic growth models, followed by Halkin (1974) for optimal control problems. In the latter paper, Halkin also implicitly defined the notion of finite optimality, which we refer to here as efficiency. Finally, average optimality was extensively studied by Veinott (1966). See also Bertsekas (1987, 1995).

We shall see that the efficiency criterion is not over-selective, since the existence of efficient solutions is assured by relatively mild topological conditions. (We give a reasonable sufficient condition for efficient optimal solutions to exist in our discrete-time, non-stationary, continuous state and control framework.) Nor is it particularly under-selective, since such a strategy must be optimal to every state attained along that path. In the discrete action setting of Schochetman and Smith (1998), it was shown that, under a (rather strong) state-reachability condition, every efficient solution is average optimal. Here, we weaken this state-reachability condition and extend this result to the case of continuous states and controls. Consequently, this provides a sufficient condition for average optimal solutions to exist. Moreover, we give a stronger state-reachability condition which, in the presence of discounting, assures us that efficient solutions are overtaking optimal. Since (as we have noted) efficient solutions commonly exist, this state-reachability condition provides a new sufficient condition for the existence of overtaking optimal solutions. Analogously, we show that a "weaker" reachability condition is sufficient for the existence of average optima.

In section 2, we formulate the state-transition and cost structures of our discrete-time, infinite horizon, deterministic, non-stationary, continuous state and control problem. In section 3, we introduce the optimality criteria of interest (with and without discounting), and compare them in the absence of any additional assumptions. In particular, we present a mild condition which is sufficient to guarantee the existence of efficient solutions (Theorem 3.5). It is also known (Halkin (1974)), that weakly overtaking optima are efficient for continuous time and vector states. We give a discrete-time proof of the fact that overtaking optima are average optimal (Theorem 3.10). We also show by counter-examples that, in general:

- the optimal average value may or may not be attained (Examples 3.13, 3.15),

- overtaking optima need not be strong optima (Example 3.14),
- weakly overtaking optima need not be overtaking optima (Example 3.13),
- average optima need not be overtaking optima (Example 3.14),
- efficient optima need not be weakly overtaking optima (Example 3.15),
- efficiency and average optimality are not comparable criteria in general (Examples 3.13, 3.16), and
- weakly overtaking optimality and average optimality are not comparable in general (Examples 3.14, 3.16).

In section 4, we introduce various state reachability conditions which are considerably weaker than complete reachability. In the presence of *average cost reachability*, we show that efficient solutions are average optimal (Theorem 4.2). In the presence of *total cost reachability*, we show that the overtaking solutions are precisely the efficient solutions (Theorem 4.4). Finally, as a consequence of this fact, we obtain an easily verified sufficient condition involving *bounded time reachability* which guarantees the existence of overtaking optimal solutions (Theorem 4.6).

Some of the results contained herein are known for either the continuous-time setting or the discrete-time setting. In some instances, we give simpler, discrete-time proofs of certain of the continuous-time results. In addition to the references already cited, we recommend Brock and Haurie (1976), Zaslavski (1995), Haurie (1976), Leizarowitz (1987, 1996), Lasserre (1988) and Carlson et al (1991). Finally, in section 5.3 of the last reference, the authors give a discrete-time version of their continuous-time model. However, implicit in this model are stationarity and complete reachability. In addition, states are required to belong to $\mathbb{R}^n$. We make no such assumptions here. Moreover, they do not consider the average optimality criterion at all there.

2. Problem Formulation

We formulate a deterministic infinite horizon optimization problem within a discrete-time framework. Otherwise, our problem is quite general. In particular, it is non-stationary, allows for compact state and action spaces, is discounted or not, and assumes no reachability properties (as part of the problem definition). Moreover, by a familiar device, stochastic infinite horizon problems can be modelled by our framework (see below).

Consider a sequence of decisions, where each decision is made at the beginning of each of a series of equal time periods, indexed by $j = 1, 2, \ldots$. The set of all possible decisions available in period j (irrespective of the period's beginning state) is denoted by Y_j. For convenience, we assume that Y_j is a compactum, i.e., a compact, non-empty metric space with metric ρ_j, $\forall j = 1, 2, \ldots$. Without loss of generality, we may assume that $\rho_j(x_j, y_j) \leq 1$, $\quad \forall x_j, y_j \in Y_j$, $\quad \forall j = 1, 2, \ldots$.

We consider a dynamic system governed by the state equation $s_j = f_j(s_{j-1}, y_j)$, $\quad \forall x_j, y_j \in Y_j$, $\quad \forall j = 1, 2, \ldots$, where s_0 is the fixed and given *initial state* of the system (beginning period 1), s_j is the *state* of the system at the end of period j, i.e., beginning period $j+1$, y_j is the *control* (or action) selected in period j with knowledge of the state s_{j-1}, S_j is the compact metric space of *feasible states* ending period j (with $S_0 = \{s_0\}$), so that $s_j \in S_j$, $\quad \forall j = 1, 2, \ldots$, $Y_j(s_{j-1})$ is the given closed, non-empty subset Y_j of *feasible controls* available in period j when the beginning state is $s_{j-1} \in S_{j-1}$, so that $y_j \in Y_j(s_{j-1}) \subseteq Y_j$, and f_j is the given continuous *state transition function* in period j, where $f_j : F_j \to S_j$, with

$$F_j = \{(s_{j-1}, y_j) \in S_{j-1} \times Y_j : y_j \in Y_j(s_{j-1})\}, \quad \forall j = 1, 2, \ldots.$$

(Note that the non-emptiness of $Y_j(s_{j-1})$, for $s_{j-1} \in S_{j-1}$, is equivalent to the assumption that all finite horizon feasible solutions can be feasibly continued from state s_{j-1} in period j.) We assume that the set-valued mapping $s_{j-1} \twoheadrightarrow Y_j(s_{j-1})$ of S_{j-1} into Y_j has the following (closed graph)

Continuity Property: For each j, if $s_{j-1}^n \to s_{j-1}$ in S_{j-1}, and $y_j^n \to y_j$ in Y_j, as $n \to \infty$, where $y_j^n \in Y_j(s_{j-1}^n)$, $\quad \forall n$, then $y_j \in Y_j(s_{j-1})$.

In this event, each F_j is the closed (hence, compact) graph of the set-valued mapping $s_{j-1} \twoheadrightarrow Y_j(s_{j-1})$ in the compact space $S_{j-1} \times Y_j$. We require that $S_j = f_j(F_j)$ $\quad \forall j = 1, 2, \ldots$, so that, in particular, $S_1 = f_1(F_1)$, where $F_1 = \{s_0\} \times Y_1(s_0)$. Thus, each S_j consists of the set of feasible, i.e., attainable, states in period j.

Remarks. Before proceeding, it is worth noting that continuous-time optimization problems can be adapted to our model. For the sake of simplicity, assume that strategies are the same as state trajectories, i.e.,

decisions are system states. Then proceed as in Zaslavski (1995). Moreover, stochastic optimization problems can also be adapted to our model. Once again, for simplicity, assume decisions are finite in number, so that policies correspond to probability mass functions over underlying stochastic states. Then proceed as in Leizarowitz (1987). We leave it to the interested reader to pursue those cases where decisions are not system states and probability distributions are more general.

The product set $Y = \prod_{j=1}^{\infty} Y_j$ of all potential decision sequences or *strategies* is then a compact topological space relative to the product topology, i.e., the topology of componentwise convergence. The product topology on Y is metrizable with metric d given by

$$d(x, y) = \sum_{j=1}^{\infty} \beta^j \rho_j(x_j, y_j), \qquad \forall x, y \in Y,$$

where β is chosen arbitrarily so that $0 < \beta < 1$.

Now let $y \in Y$ and fix a positive integer N. Then y is *feasible through period* N if $y_j \in Y_j(s_{j-1})$, where $s_j = f_j(s_{j-1}, y_j)$, for all $j = 1, 2, \ldots, N$. Denote all such strategies by X_N, which is thus a closed, non-empty subset of Y. Note that if y is feasible through period N and $M < N$, then y is feasible through period M, i.e., $X_N \subseteq X_M$. Moreover, y is a *feasible* strategy if y is feasible through period N, for each $N = 1, 2, \ldots$. We define the *feasible region* X to be the subset of Y consisting of all those y which are feasible through *each* period N, i.e., $x \in X_N, \quad \forall N$, so that $X = \cap_{N=1}^{\infty} X_N$. This set is closed in Y and non-empty, since $Y_j(s_{j-1})$ is non-empty, for all j, and all $s_{j-1} \in S_{j-1}$. In fact, as a consequence of this assumption, if y is feasible through a given period N, then it may be feasibly extended over all remaining periods.

If y is feasible through period N, then we may define

$$s_1(y) = f_1(s_0, y_1), \quad s_2(y) = f_2(s_1(y), y_2), \quad \ldots, \quad s_N(y) = f_N(s_{N-1}(y), y_N),$$

so that $s_N(y) \in S_N$, and $y \in X_N$ if and only if $y_j \in Y_j(s_{j-1}(y)), \quad \forall j = 1, 2, \ldots, N$. We will refer to each such $s_N(y)$ as the *state through which y passes at the end of period N*. Thus, for each N, we obtain a mapping $s_N : X_N \to S_N$, which is *onto* since S_N consists of feasible states. If $y \in Y$, $z \in X_N$ and $y_j = z_j, \quad \forall j = 1, 2, \ldots, N$, then $y \in X_N$ and $s_N(y) = s_N(z)$. Moreover, if $x \in X$, then for *each* period N, $s_N(x)$ is defined, and $s \in S_N$ implies there exists $x \in X$ for which $s_N(x) = s$. Finally, if $x \in X$, then $(s_{j-1}(x), x_j) \in F_j, \quad \forall j = 1, 2, \ldots$.

2.1 Lemma. *For each N, the mapping $s_N : X_N \to S_N$ is continuous.*

Proof. This follows from the continuity of the f_j. $\qquad \square$

For convenience, we introduce the following notation. If N is a positive integer and $x, y \in Y$, then we define

$$(x \mid_N y) = (x_1, x_2, \ldots, x_N, y_{N+1}, y_{N+2}, \ldots).$$

The following is then immediate.

2.2 Lemma. *If N is a positive integer and $x, y \in X$ are such that $s_N(x) = s_N(y)$, then $z = (x \mid_N y)$ is also in X. Moreover, $s_M(z) = s_M(x), \quad \forall M \leq N$, and $s_M(z) = s_M(y), \quad \forall M > N$.*

Turning to the objective function, we allow the cost of a decision made in period j to also depend (indirectly) on the sequence of previous decisions, or more directly, on the state resulting from these decisions. Specifically, we let $c_j(s_{j-1}, y_j)$ be the (undiscounted) non-negative cost of decision y_j in period j, when s_{j-1} is the state beginning period j. We thus obtain cost functions $c_j : F_j \to [0, \infty)$ which we require to be continuous. Thus, each c_j attains its maximum, denoted by $\|c_j\|, \quad \forall j$. We say that the period costs c_j are *exponentially bounded* if there exists $B > 0$ and $\gamma \geq 1$ such that $\|c_j\| \leq B\gamma^j$, i.e.,

$$0 \leq c_j(s_{j-1}, y_j) \leq B\gamma^j, \quad \forall(s_{j-1}, y_j) \in F_j, \quad \forall j = 1, 2, \ldots.$$

Of course, if $\gamma = 1$, then the period costs are actually *uniformly bounded* by B.

Throughout the following, let α be a discount factor, $0 < \alpha \leq 1$. For each strategy $x \in X$ and positive integer N, we define the associated total N-horizon cost $C_N(x|\alpha)$ by

$$C_N(x|\alpha) = \sum_{j=1}^{N} \alpha^{j-1} c_j(s_{j-1}(x), x_j),$$

so that $0 \leq C_N(x|\alpha) < \infty$, and $C_{N+1}(x|\alpha) \geqslant C_N(x|\alpha)$. If $\alpha < 1$, this cost is *discounted*. If $\alpha = 1$, then the cost is *undiscounted*; in this event, we will write $C_N(x)$ for $C_N(x|1)$, so that $C_N(x|\alpha) \leq C_N(x)$, for all N, x and α. Note that each $C_N(\cdot|\alpha)$ is a continuous real-valued function on X.

Our general problem is to find an infinite horizon feasible strategy $x \in X$ which, in some suitable sense, is optimal, i.e., minimal. The fundamental question is: what does "optimal" mean? There is no guarantee that the total cost of *any* strategy over the infinite horizon will be finite, even if it is discounted. In the next section, we compare and contrast five more-or-less familiar optimality criteria, each of which responds to this question.

3. Optimality Criteria

There are many optimality criteria which exist in the literature, the most popular being *strong optimality*. Others include *overtaking optimality*, *weakly overtaking optimality*, *finite optimality*, also known as *efficiency*, and *average-optimality*. In this paper, we contrast and compare these optimality criteria for our discrete-time problem, with and without discounting. We begin with strong optimality.

For each $x \in X$ and discount factor α, define the infinite horizon total cost $C(x|\alpha)$ by

$$C(x|\alpha) = \sum_{j=1}^{\infty} \alpha^{j-1} c_j(s_{j-1}(x), x_j) = \lim_{N \to \infty} C_N(x|\alpha) = \sup_N C_N(x|\alpha).$$

Thus, the function $C(\cdot|\alpha) : X \to [0, \infty]$ is both the pointwise limit and the supremum of the continuous functions $C_N(\cdot|\alpha)$. Hence, $C(\cdot|\alpha)$ is lower semi-continuous on X (Hewitt and Stromberg (1965), p.89), for each α. As above, we will write $C(x)$ for $C(x|1)$. Thus,

$$0 \leq C(x|\alpha) \leq C(x) \leq \infty, \quad \forall 0 < \alpha \leq 1, \quad \forall x \in X.$$

Consequently, for a given $x \in X$, if $C(x) < \infty$, then $C(x|\alpha) < \infty$, for each α. However, for $0 < \alpha < 1$, if $C(x) = \infty$, it's posssible that $C(x|\alpha) < \infty$. This depends on the behavior of $c_j(s_{j-1}(x), x_j)$ versus that of α^{j-1}, with respect to j. Accordingly, for each x in X for which $C_N(x) > 0$ eventually, i.e., $C(x) > 0$, we define

$$k(x) = \limsup_N \frac{\ln(C_N(x))}{N}.$$

Note that if $C(x) < \infty$, then $k(x) = 0$; if $C(x) = \infty$, then $\ln(C_N(x)) \uparrow \infty$.

3.1 Theorem. *Fix $x \in X$ for which $C(x) > 0$. If $0 < k(x) < \infty$, then $C(x|\alpha) < \infty$, for all α such that $0 < \alpha < e^{-k(x)} < 1$.*

Proof. Fix $0 < \alpha < 1$. For $\sigma = -\ln \alpha$, we have

$$C_N(x|\alpha) = \sum_{j=1}^{N} c_j(s_{j-1}(x), x_j)\alpha^{j-1} = \sum_{j=1}^{N} c_j(s_{j-1}(x), x_j)(e^{-\sigma})^{j-1}, \quad \forall N = 1, 2, \ldots.$$

Applying Theorem 2.5 of Widder (1946), with $\lambda_n = n - 1$ and $a_n = c_n(s_{n-1}(x), x_n)$, we obtain that $C(x|\alpha) < \infty$, for all α satisfying

$$-\ln \alpha > \limsup_N \frac{\ln(C_N(x))}{N - 1} > 0.$$

But
$$\limsup_N \frac{\ln(C_N(x))}{N-1} = \limsup_N \frac{\ln(C_N(x))}{N} = k(x),$$
so that $C(x|\alpha) < \infty$, for all α satisfying $-\ln\alpha > k(x) > 0$; equivalently, $\alpha < e^{-k(x)} < 1$. $\square$

Our *total cost* optimization problem is then formulated as follows:
$$C^*(\alpha) = \inf_{x \in X} C(x|\alpha),$$
so that $0 \le C^*(\alpha) \le \infty$, and $C^*(\alpha) \le C^*(1), \quad \forall 0 < \alpha \le 1$. Note that $C^*(\alpha) < \infty$ if and only if there exists at least one $x \in X$ for which $C(x|\alpha) < \infty$. In any event, $C^*(\alpha)$ is always attained. If $C^*(\alpha) = \infty$, then $C(x|\alpha) = \infty, \quad \forall x \in X$. If $C^*(\alpha) < \infty$, since X is compact and $C(\cdot|\alpha)$ is lower semi-continuous, it follows that $C^*(\alpha)$ is attained.

Strong Optimality: Let $x \in X$. Then x is *strongly optimal* (relative to α) if $C(x|\alpha) = C^*(\alpha) < \infty$, i.e., $C(x|\alpha) < \infty$ and $C(x|\alpha) \le C(y|\alpha), \quad \forall y \in X$.

For each $0 < \alpha \le 1$, we will denote the set of such strongly optimal solutions to our problem by $X^s(\alpha)$. Thus,
$$\emptyset \subseteq X^s(\alpha) \subseteq \{x \in X : C(x|\alpha) < \infty\} \subseteq X,$$
in general, with all inclusions possibly proper. If $C^*(\alpha) < \infty$, then $X^s(\alpha) \ne \emptyset$. It's possible that $C^*(\alpha) = \infty$ (see our examples), equivalently $X^s(\alpha) = \emptyset$, by our definition. (For our purposes here, this is the interesting case.) At the other extreme, if the period costs c_j are exponentially bounded by $B\gamma^j$, then, for $\alpha < 1/\gamma$, we have
$$0 \le C^*(\alpha) \le C(x|\alpha) \le \frac{B\gamma}{1-\alpha\gamma}, \quad \forall x \in X,$$
and $C(\cdot|\alpha)$ is the *uniform* limit of the $C_N(\cdot|\alpha)$, i.e., it's continuous on compact X. Hence, it attains its minimum value, so that $X^s(\alpha) \ne \emptyset$, in particular.

3.2 Lemma. *For each $0 < \alpha \le 1$, the set $X^s(\alpha)$ is closed in X.*

Proof. For a fixed α, this set is the inverse image of the point $C^*(\alpha)$ under the lower semi-continuous mapping $C(\cdot|\alpha)$. Hence, it is necessarily closed (Hewitt and Stromberg (1965), 7.21(d)). $\square$

The following well-studied optimality criteria are particularly useful if $C^*(\alpha) = \infty$, in which case there does not exist a strongly optimal strategy. We recall the familiar notions of overtaking and weakly overtaking optimality.

Let $x, y \in X$. As in the continuous-time case, we will say that x *overtakes* y (relative to α) if
$$\liminf_N [C_N(y|\alpha) - C_N(x|\alpha)] \ge 0,$$
and x *weakly overtakes* y (relative to α) if
$$\limsup_N [C_N(y|\alpha) - C_N(x|\alpha)] \ge 0.$$

Overtaking and Weakly Overtaking Optimality: Let $x \in X$. Then x is *overtaking optimal* if x overtakes y, for all $y \in X$. Similarly for *weakly overtaking optimal*. Clearly, overtaking optimality implies weakly overtaking optimality. Overtaking optimality was originally introduced by von Weiszacker (1965), who called it *catching up optimality*, while weakly overtaking optimality, also called *sporadically catching up optimality*, first appears in Halkin (1974).

Denote the set of such optimal strategies in X by $X^o(\alpha)$ (respectively, $X^w(\alpha)$), so that
$$\emptyset \subseteq X^o(\alpha) \subseteq X^w(\alpha) \subseteq X,$$
in general. Of course, the sets $X^o(\alpha)$ and $X^w(\alpha)$ are different in general (Example 3.13), and they need be not closed in Y (Example 3.14). Both overtaking and weakly overtaking optimality have received considerable attention in the economics and optimal control literature, primarily for continuous-time problems.

The following can be found in Halkin (1974) for the continuous time case.

3.3 Theorem. *Suppose $0 < \alpha \leq 1$. Then, in general, strong optimality implies overtaking optimality. Specifically, if $C^*(\alpha) = \infty$, then*

$$\emptyset = X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^w(\alpha).$$

If $C^*(\alpha) < \infty$, i.e., there exists $x \in X$ for which $C(x|\alpha) < \infty$, then strong optimality and weakly overtaking optimality are equivalent, i.e.,

$$\emptyset \neq X^s(\alpha) = X^o(\alpha) = X^w(\alpha),$$

for such α.

Proof. Let $x \in X^s(\alpha)$. Then $C(x|\alpha) = C^*(\alpha) < \infty$, and $C(x|\alpha) \leq C(y|\alpha), \quad \forall y \in X$. Let $y \in X$. Then either $C(y|\alpha) = \infty$ or $C(y|\alpha) < \infty$. In either case,

$$\liminf_N [C_N(y|\alpha) - C_N(x|\alpha)] = \lim_{N\to\infty} C_N(y|\alpha) - \lim_{N\to\infty} C_N(x|\alpha) = C(y|\alpha) - C(x|\alpha) \geqslant 0.$$

Therefore, $x \in X^o(\alpha)$.

Conversely, assume $C^*(\alpha) < \infty$, i.e., there exists $z \in X^s(\alpha)$, so that $C(z|\alpha) = C^*(\alpha)$. Let $x \in X^w(\alpha)$. Then $\liminf_N [C_N(z|\alpha) - C_N(x|\alpha)] \geqslant 0$, by definition and

$$C(z|\alpha) - C(x|\alpha) \quad = \quad \lim_{N\to\infty} C_N(z|\alpha) - \lim_{N\to\infty} C_N(x|\alpha) \quad = \quad \limsup_N [C_N(z|\alpha) - C_N(x|\alpha)] \quad \geqslant \quad 0,$$

so that $C(x|\alpha) \leq C(z|\alpha)$. Necessarily, $C(x|\alpha) = C^*(\alpha) < \infty$. Thus, $x \in X^s(\alpha)$. $\qquad\square$

Next we turn to the much less well-known finite-optimality notion which we call efficiency. The state-space construction introduced above associated a unique state at the end of each time period with every infinite horizon feasible strategy. Strategies that have the property of *optimally* reaching each of the states through which they pass have been called efficient strategies (see Schochetman and Smith (1998, 1992), Ryan, Bean, and Smith (1992), and also Lasserre (1986) for an early introduction of a similar concept. This efficiency of movement through the state space suggests efficient solutions as candidates for optimality.

Efficiency (Finite Optimality): Let $x \in X$. Then x is *efficient* (relative to α) if, for each $y \in X$, and for each N such that $s_N(y) = s_N(x)$, we have $C_N(x|\alpha) \leq C_N(y|\alpha)$. Also known as *finite optimality*, this criterion was originally introduced in a special case by Halkin (1974), who called it *finite horizon clamped end-point optimality*.

Let $X^e(\alpha)$ denote the subset of X consisting of efficient strategies. It was shown in (Schochetman and Smith (1998), Lemma 3.5) that efficient strategies exist in our context, i.e., $\emptyset \subset X^e(\alpha) \subseteq X$, provided each of the spaces Y_j and S_{j-1} is *discrete*. (Although in Schochetman and Smith (1998) we asssumed that the period costs were uniformly bounded, while here we do not, this has no effect on the definition of efficient strategy.)

Before continuing with our comparisons of optimality criteria, we give a sufficient condition for efficient solutions to exist in the case of *non-discrete* Y_j and S_{j-1}. Fix N, and for each $s \in S_N$, let $X_N(s)$ denote the set of N-horizon feasible strategies which attain state s at the end of period N, i.e.,

$$X_N(s) = \{x \in X_N : s_N(x) = s\} = s_N^{-1}(s).$$

Since s_N is continuous, we thus obtain a partition $\{X_N(s) : s \in S_N\}$ of X_N consisting of compact sets, as well as a set-valued mapping $s \twoheadrightarrow X_N(s)$ of S_N into X_N with compact, non-empty values.

Now, for each N and $s \in S_N$, consider the optimization problem

$$\min_{x \in X_N(s)} C_N(x|\alpha).$$

If we let $X_N^*(s|\alpha)$ denote the set of optimal solutions to this problem, then this set is a closed, non-empty subset of X_N. We thus obtain another compact-valued set mapping of S_N into X_N given by $s \twoheadrightarrow X_N^*(s|\alpha)$. If we define

$$\mathcal{X}_N^*(\alpha) = \cup_{s \in S_N} X_N^*(s|\alpha),$$

so that the $\mathcal{X}_N^*(\alpha)$ are non-empty and nested downward, and

$$\mathcal{X}^*(\alpha) = \cap_{N=1}^\infty \mathcal{X}_N^*(\alpha),$$

then it is not difficult to see that the efficient solutions are precisely the elements of $\mathcal{X}^*(\alpha)$, i.e., $X^e(\alpha) = \mathcal{X}^*(\alpha)$.

The following gives a sufficient condition for the existence of efficient solutions - in the continuous action/state case.

3.4 Theorem. *If, for each N, the set-valued mapping $s \twoheadrightarrow X_N(s)$ is continuous in the sense of Berge (1964, p.116), then efficient solutions exist, i.e., $X^e(\alpha) \neq \emptyset$, and $X^e(\alpha)$ is compact, for all $0 < \alpha \leq 1$.*

Proof. It follows from our hypothesis and Berge (1964) that the set-valued mapping $s \twoheadrightarrow X_N^*(s|\alpha)$ is upper semi-continuous in the sense of Berge (1964, p.109). Consequently, the space $\mathcal{X}_N^*(\alpha)$ is compact (Berge (1964, p.110)), for each N. Hence, $\mathcal{X}^*(\alpha)$ is the intersection of a descending sequence of compact, non-empty sets, and is thus, compact and non-empty.   □

The previous generalizes the following existence result for efficient solutions established in Schochetman and Smith (1998, Lemma 3.5) - for the discrete action/state case.

3.5 Corollary. *If the S_N are discrete, then efficient solutions exist in this case.*

Proof. As is the case for single-valued functions, set-valued functions defined on discrete spaces are continous.

The next result compares $X^w(\alpha)$ with $X^e(\alpha)$.

3.6 Theorem. *In general, weakly overtaking optimality implies effficiency, i.e., $X^w(\alpha) \subseteq X^e(\alpha)$, $\forall 0 < \alpha \leq 1$.*

Proof. The proof given for Theorem 4.1 of Halkin (1974) for continuous time may be adapted here for discrete time. We leave the details for the interested reader.

□

3.7 Corollary. *Suppose $0 < \alpha \leq 1$. If $C^*(\alpha) = \infty$, then*

$$\emptyset = X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^w(\alpha) \subseteq X^e(\alpha).$$

If $C^*(\alpha) < \infty$, then

$$\emptyset \neq X^s(\alpha) = X^o(\alpha) = X^w(\alpha) \subseteq X^e(\alpha),$$

for such α.

Finally, we consider the well-studied notion of average optimality. As is customary, we define the infinite horizon *average cost* (per-period) of $x \in X$ to be

$$A(x|\alpha) = \limsup_N A_N(x|\alpha),   \forall 0 < \alpha \leq 1,$$

where, for all $N = 1, 2, \ldots$,

$$A_N(x|\alpha) = \frac{C_N(x|\alpha)}{N},$$

so that $0 \leq A_N(x|\alpha) \leq C_N(x|\alpha)$, and $A_N(x|\alpha) \leq A_N(x|1)$. Then $A(x|\alpha) \leq C(x|\alpha)$, and

$$0 \leq A(x|\alpha) \leq A(x|1) \leq \infty,$$

in general. Note that the function $A(\cdot|\alpha) = \limsup_N A_N(\cdot|\alpha)$, where $A_N(\cdot|\alpha)$ is continuous, for all N. However, $A(\cdot|\alpha)$ need not be lower semi-continuous, as was the case for $C(\cdot|\alpha)$.

Our average cost optimization problem is then:

$$A^*(\alpha) = \inf_{x \in X} A(x|\alpha).$$

Average optimality: Let $x \in X$. Then x is *average optimal* (relative to α) if $A(x|\alpha) = A^*(\alpha) < \infty$, i.e., $A(x|\alpha) < \infty$ and $A(x|\alpha) \leq A(y|\alpha), \quad \forall y \in X$. This optimality criterion has been studied by a number of authors. For example, see Arapostathis et al (1993) and Bertsekas (1995), as well as the references contained therein.

We will denote the set of average optimal solutions to our problem by $X^a(\alpha)$. As was the case for $X^o(\alpha)$ and $X^w(\alpha)$, the set $X^a(\alpha)$ need not be closed in X (Example 3.13). Of course,

$$\{x \in X : A(x|\alpha) = 0\} \subseteq X^a(\alpha) \subseteq \{x \in X : A(x|\alpha) < \infty\},$$

in general. In particular, $X^a(\alpha) = \emptyset$ if $A^*(\alpha) = \infty$, i.e., $A(x|\alpha) = \infty, \quad \forall x \in X$, or if $A^*(\alpha) < \infty$ and is not attained. Moreover, we have the following properties for $A^*(\alpha)$ versus $C^*(\alpha)$.

- In general, $0 \leq A^*(\alpha) \leq C^*(\alpha) \leq \infty$.
- If $A^*(\alpha) = \infty$, then $C^*(\alpha) = \infty$ also, in which case both $X^s(\alpha)$ and $X^a(\alpha)$ are empty.
- It is possible for $X^s(\alpha)$ to be empty while $X^a(\alpha)$ is not, i.e., $C^*(\alpha) = \infty$, while $A^*(\alpha) < \infty$.
- If $C^*(\alpha) < \infty$, i.e., there exists $x \in X$ such that $C(x|\alpha) < \infty$, then $A(x|\alpha) = 0$, so that $A^*(\alpha) = 0$ and is attained by all such x.
- We have $A^*(\alpha) = C^*(\alpha)$ if and only if $A^*(\alpha) = \infty$ or $C^*(\alpha) = 0$.
- If $A^*(\alpha) < \infty$ is not attained, then $C^*(\alpha) = \infty$ necessarily.

3.8 Lemma. *If the c_j are exponentially bounded by $B\gamma^j$, and $\alpha < 1/\gamma$, then $A(x|\alpha) = 0, \quad \forall x \in X$, so that $A^*(\alpha) = 0$ and $X^a(\alpha) = X$ in this case.*

3.9 Theorem. *Suppose $A^*(\alpha) < \infty$. Then overtaking optimality implies average optimality, so that*

$$X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^a(\alpha),$$

for all such α.

Proof. Suppose $x \in X^o(\alpha)$. Let $y \in X$ and $\epsilon > 0$. Then there exists M sufficiently large such that $C_N(x|\alpha) \leq C_N(y|\alpha) + \epsilon, \quad \forall N \geqslant M$. Consequently,

$$\frac{C_N(x|\alpha)}{N} \leq \frac{C_N(y|\alpha)}{N} + \frac{\epsilon}{N},$$

for all such N. Hence,

$$\limsup_N \frac{C_N(x|\alpha)}{N} \leq \limsup_N \frac{C_N(y|\alpha)}{N},$$

i.e., $A(x|\alpha) \leq A(y|\alpha)$, so that $x \in X^a(\alpha)$, since $A(x|\alpha)$ is necessarily finite by hypothesis. $\square$

In general, weakly overtaking solutions are not average optimal, i.e., $X^w(\alpha)$ need not be contained in $X^a(\alpha)$ (Example 3.14).

3.10 Corollary. *If $C^*(\alpha) < \infty$, so that $A^*(\alpha) = 0$, then*

$$\emptyset \neq X^s(\alpha) = X^o(\alpha) = X^w(\alpha) \subseteq X^a(\alpha) = \{x \in X : A(x|\alpha) = 0\}.$$

Proof. Recall that $X^s(\alpha) \neq \emptyset$ in this case. $\square$

We have shown that for α such that $A^*(\alpha) < \infty$, and without any additional assumptions,

$$\emptyset \subseteq X^s(\alpha) \subseteq X^o(\alpha) \subseteq \begin{cases} X^w(\alpha) \subseteq X^e(\alpha), \\ X^a(\alpha), \end{cases}$$

where:

• $X^s(\alpha) \neq \emptyset$ if and only if $C^*(\alpha) < \infty$;
• $C^*(\alpha)$ is always attained;
• $X^a(\alpha) \neq \emptyset$ if and only if $A^*(\alpha) < \infty$ and is attained;
• $A^*(\alpha)$ may or may not be attained, in general.
• it is always the case that $X^e(\alpha) \neq \emptyset$, if the set-valued mappings $s \twoheadrightarrow X_N(s)$ are continuous, for all N (e.g., discrete state spaces).

Moreover, we will see (by Examples 3.12-3.15) that:

• $X^s(\alpha)$ may or may not be equal $X^o(\alpha)$;
• $X^o(\alpha)$ may or may not be equal $X^w(\alpha)$;
• $X^w(\alpha)$ may or may not be equal $X^e(\alpha)$;
• $X^o(\alpha)$ may or may not be equal $X^a(\alpha)$;
• $X^e(\alpha)$ and $X^a(\alpha)$ are not comparable in general; and
• $X^w(\alpha)$ and $X^a(\alpha)$ are also not comparable, in general;

Thus, the previous inclusions are the *best* possible, barring any additional assumptions.

3.11 Remarks. (1) Observe that if there exists $x \in X$ for which $C(x|\alpha) < \infty$ (i.e., $C^*(\alpha) < \infty$), then $X^s(\alpha)$ is not empty, is equal to $X^o(\alpha) = X^w(\alpha)$, and is contained in both $X^e(\alpha)$ and $X^a(\alpha)$ (Corollaries 3.7 and 3.10), i.e.,

$$\emptyset \neq X^s(\alpha) = X^o(\alpha) = X^w(\alpha) \subseteq \begin{cases} X^e(\alpha), \\ X^a(\alpha). \end{cases}$$

In this case, $X^s(\alpha)$ "dominates" all the other optimal sets in the sense that it is non-empty and contained in each of them. Thus, if $C^*(\alpha) < \infty$, then strong optimality is the optimality criterion of choice because such optimal strategies exist and have all the other properties. However, if $C(x|\alpha) = \infty$, $\forall x \in X$ (i.e., $C^*(\alpha) = \infty$), then $X^s(\alpha) = \emptyset$, and the remaining optimality criteria become important, particularly efficiency, since we have a reasonable sufficient condition for such optima to exist in our model (Theorem 3.4). Needless to say, the strong emphasis here is on the case $C^*(\alpha) = \infty$.
(2) Intuitively speaking, strong optimality is short-term biased, in that the earlier the decision, the greater the impact on the total cost. On the other hand, average optimality is long-term biased because average cost is influenced only by cost to go. However, efficiency appears to be neither short-term nor long-term biased. It is reasonable to expect that a suitable infinite horizon optimization criterion should not be short-term biased. The general concept of bias for optimality criteria has been studied formally by Chichilnisky (1993). We will not pursue this issue here.

We next describe four examples. Without loss of generality, it suffices to consider only the case $\alpha = 1$. If $\alpha \neq 1$, then replace each $c_j(s_{j-1}, y_j)$ by $c_j(s_{j-1}, y_j)/\alpha^{j-1}$ to get the same conclusions.

3.12 Example. Let the data be as follows for $j \geqslant 1$:

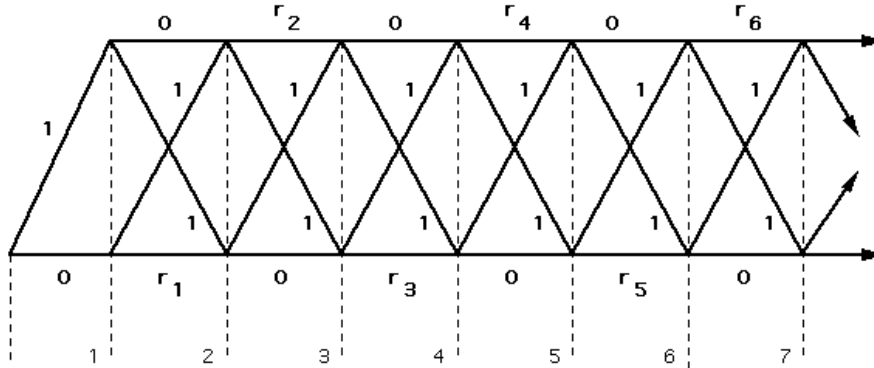$$Y_j = \{0, 1\}, \quad S_j = \{(j, 0), (j, 1)\}, \quad s_0 = (0, 0),$$

$$Y_j(s_{j-1}) = Y_j(j - 1, k) = Y_j = \{0, 1\},$$

$$f_j(s_{j-1}, y_j) = f_j((j - 1, k), y_j) = \begin{cases} (j, k + y_j), & \text{if } k = 0, \\ (j, k - y_j), & \text{if } k = 1, \ j \geqslant 2. \end{cases}$$

To introduce the cost structure, let $r_k = \sum_{j=0}^{k}(1/2)^j$, for $k = 0, 1, \ldots$, so that $r_k \uparrow 2$, as $k \to \infty$. Define

$$c_j(s_{j-1}, y_j) = c_j((j-1, k), y_j) = \begin{cases} 1, & \text{if } y_j \neq 0, \\ 0, & \text{if } y_j = 0, \quad j+k \text{ is odd}, \\ r_j, & \text{if } y_j = 0, \quad j+k \text{ is even}. \end{cases}$$

(See Figure 1.)

Figure 1. State-Space Diagram for Example 3.13

Note that period costs are uniformly bounded. We leave it to the reader to verify that this example has the following properties for $\alpha = 1$, i.e., the undiscounted case:

(i) $C^*(1) = \infty$ and $A^*(1) = 1$, which is attained.

(ii) $\emptyset = X^s(1) = X^o(1) \subset \{\theta\} = X^w(1) = X^e(1) \subset \{x \in X : A(x) = 1\} = X^a(1) = X$, where $\theta = (0, 0, \ldots)$, so that X^a is not contained in X^w, in general.

That is, there is exactly *one* efficient optimal solution, *no* overtaking optimal solution, and *all* feasible solutions are average optimal.

3.13 Example. Let the data be as follows for $j \geqslant 1$:

$$Y_j = \{0, 1\}, \quad S_j = \{(j, 0), (j, 1), \ldots, (j, j)\}, \quad s_0 = (0, 0),$$
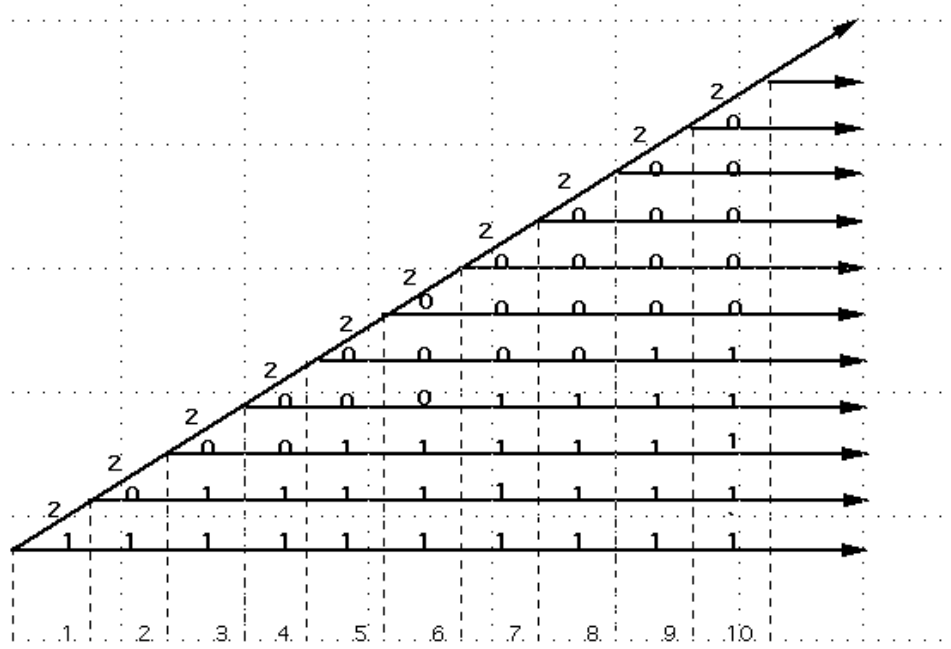
$$Y_j(s_{j-1}) = Y_j(j-1, k) = \begin{cases} \{0\}, & \text{if } 0 \leq k < j - 1, \\ \{0, 1\}, & \text{if } k = j - 1, \end{cases}$$

$$f_j(s_{j-1}, y_j) = f_j((j-1, k), y_j) = \begin{cases} (j, k), & \text{if } 0 \leq k \leq j - 1, \quad y_j = 0, \\ (j, k+1), & \text{if } k = j - 1, \quad y_j = 1. \end{cases}$$

To introduce the cost structure, define

$$c_j(s_{j-1}, y_j) = c_j((j-1, k), y_j) = \begin{cases} 1, & \text{if } 0 \leq k < j - 1, \quad y_j = 0, \\ 0, & \text{if } k = j - 1, \quad y_j = 0, \\ 2, & \text{if } k = j - 1, \quad y_j = 1, \end{cases}$$

for $y_j \in \{0, 1\}$. (See Figure 2.)

Figure 2. State-Space Diagram for Example 3.14

Note that period costs are uniformly bounded. We leave it to the reader to verify that this example has the following properties for the undiscounted case:

(i) $C^*(1) = \infty$ and $A^*(1) = 1$, which is attained.

(ii) $\emptyset = X^s(1) \subset \{x^0\} = X^o(1) = X^w(1) \subset \{x^j : j \geqslant 0\} = X^a(1) \subset X^e(1) = X$, where x^j is equal to 1 in the first j positions and zero thereafter.

That is, there is exactly *one* (weakly) overtaking optimal solution, all but *one* of the feasible strategies are average optimal, and *all* feasible solutions are efficient. Thus, X^w is *properly* contained in X^e, X^e is *not* contained in X^a and X^a is *not* contained in X^w.

3.14 Example. Let the state space structure be as in the previous example, but define the cost structure as follows:

$$c_j(s_{j-1}, y_j) = c_j((j-1, k), y_j) = \begin{cases} \frac{1}{k+1}, & \text{if } 0 \leq k \leq j-1, \ y_j = 0, \\ 1, & \text{if } k = j-1, \ y_j = 1, \end{cases}.$$

(See Figure 3.)
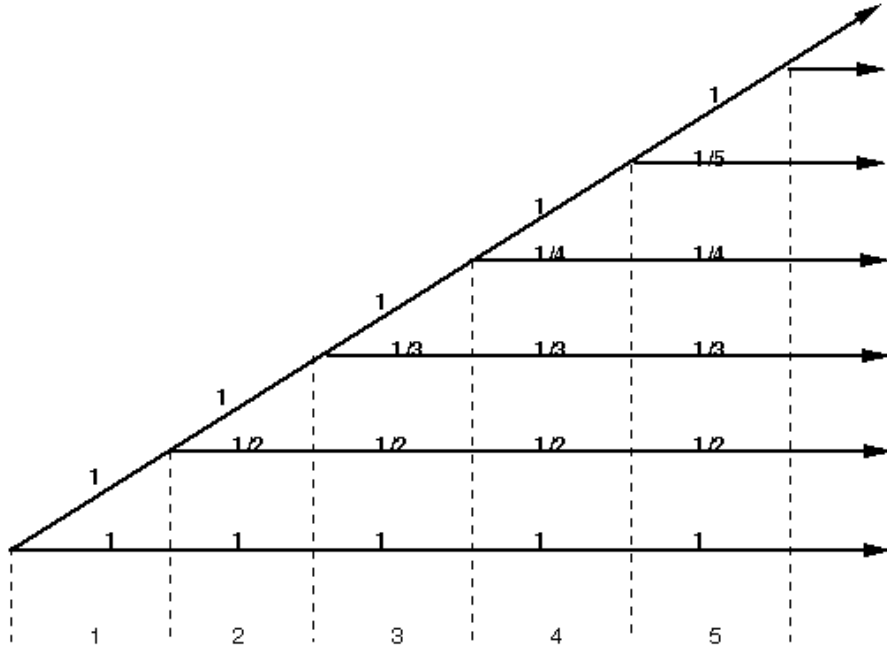
Figure 3. State-Space Diagram for Example 3.15

Note that period costs are uniformly bounded. We leave it to the reader to verify that this example has the following properties for $\alpha = 1$, i.e., the undiscounted case:

(i) $C^*(1) = \infty$ and $A^*(1) = 0$.
(ii) $\emptyset = X^s(1) = X^o(1) = X^w(1) = X^a(1) \subset X^e(1) = X$.

That is, *all* feasible strategies are efficient, and *no* feasible strategy is optimal in any other sense. Thus, X^w is *properly* contained in X^e and X^e is *not* contained in X^a. Moreover, $A^*(1)$ is *not* attained.

3.15 Example. Let the data be as follows for $j \geqslant 1$:

$$Y_j = \begin{cases} \{0,1\}, & \text{if } j = 1, \\ \{0\}, & \text{if } j > 1, \end{cases}$$
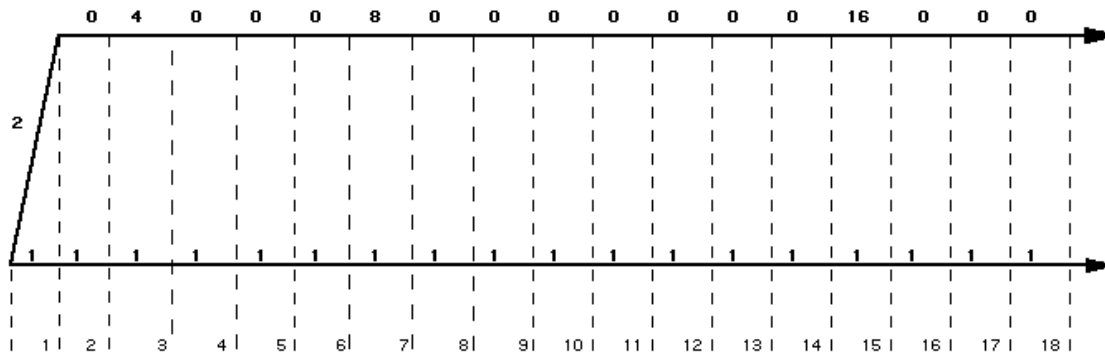
$$S_j = \{(j,0),(j,1)\}, \quad s_0 = (0,0),$$

$$Y_j(s_{j-1}) = Y_j(j-1,k) = \begin{cases} \{0,1\}, & \text{if } j = 1, \\ \{0\}, & \text{if } j \geqslant 2, \end{cases}$$

$$f_j(s_{j-1}, y_j) = f_j((j-1,k), y_j) = \begin{cases} (j, k + y_j), & \text{if } j = 1, \\ (j, k), & \text{if } j \geqslant 2, \end{cases}$$

$$c_j(s_{j-1}, y_j) = c_j((j-1,k), y_j) = \begin{cases} 1, & \text{if } j = 1,\, k = 0,\, y_j = 0, \\ 2, & \text{if } j = 1,\, k = 0,\, y_j = 1, \\ 1, & \text{if } j > 1,\, k = 0,\, y_j = 0, \\ n_j, & \text{if } j > 1,\, k = 1,\, y_j = 0, \end{cases}$$

where $n_j = 0$, if $j + 1$ is not a power of 2, and $n_j = 2^m$, if $j + 1 = 2^m$, for some integer $m \geqslant 1$. (See Figure 4.)

Figure 4. State-Space Diagram for Example 3.16

Note that period costs are not uniformly bounded, but they are exponentially bounded; specifically, $0 \le c_j(s_{j-1}, y_j) \le 2^j$. Clearly, there are just two feasible solutions x^0 and x^1, given by $x^0 = (0, 0, 0, \dots)$ and $x^1 = (1, 0, 0, \dots)$. Moreover, for each $N \geqslant 1$, we have $C_N(x^0) = N$ and

$$C_N(x^1) = \sum_{j=1}^{\lfloor \log_2(N+1) \rfloor} 2^j = 2(2^{\lfloor \log_2(N+1) \rfloor} - 1),$$

so that $C(x^0) = \infty = C(x^1)$, $C^*(1) = \infty$ and $X^s = \emptyset$. For each $N_M = 2^M - 2$, we have

$$C_{N_M}(x^1) = 2(2^{M-1} - 1) = C_{N_M}(x^0), \quad \forall M \geqslant 2,$$

i.e., $C_N(x^0)$ and $C_N(x^1)$ are each equal to N, for all such N. Next suppose that N is strictly between two such indices, i.e., $2^M - 2 < N < 2^{M+1} - 2$, for $M \geqslant 2$. Then

$$C_N(x^1) - C_N(x^0) \geqslant 2^{M+1} - 2 - (2^{M+1} - 3) = 1,$$

for such N. From these facts, it follows that

$$C_N(x^1) - C_N(x^0) \geqslant 0, \quad \forall N,$$

and, in particular, for $N_M = 2^{M+1} - 3$,

$$C_{N_M}(x^1) - C_{N_M}(x^0) \geqslant 1, \quad \forall M \geqslant 1.$$

Consequently,

$$\liminf_N \left(C_N(x^1) - C_N(x^0) \right) \geqslant 0,$$

$$\liminf_N \left(C_N(x^0) - C_N(x^1) \right) \le -1,$$

and

$$\limsup_N \left(C_N(x^0) - C_N(x^1) \right) \geqslant 0,$$

i.e., x^0 overtakes x^1 (so that x^0 weakly overtakes x^1), x^1 does not overtake x^0, and x^1 weakly overtakes x^0. Hence, $X^o(1) = \{x^0\}$ and $X^w = X$. Clearly, $X^e(1) = X$ also since, for each state, only one of the strategies attains that state.

As we have observed, for each $N \geqslant 2$, there exist a unique $M \geqslant 2$ such that $2^M - 2 < N \leq 2^{M+1} - 2$, and $C_N(x^1) = 2^{M+1} - 2$, so that

$$\frac{C_N(x^1)}{2^{M+1} - 2} \leq \frac{C_N(x^1)}{N} \leq \frac{C_N(x^1)}{2^M - 1}.$$

Consequently,

$$A(x^1) \quad = \quad \limsup_N \; C_N(x^1)/N \quad = \quad \lim_{M \to \infty} \frac{2^{M+1} - 2}{2^M - 1} \quad = \quad 2.$$

Thus, $C^*(1) = \infty$, $A^*(1) = 1$ and $X^a(1) = \{x^0\}$, since clearly $A(x^0) = 1$.

We thus obtain the following inclusions:

$$\emptyset = X^s(1) \subset \{x^0\} = X^o(1) = X^a(1) \subset X^w(1) = X^e(1) = X,$$

so that, in particular, X^w and X^e are *not* contained in X^a. There is exactly *one* overtaking (average) optimal solution, *no* strongly optimal solutions, and *all* feasible solutions are weakly overtaking and efficient.

Remark. Example 3.14 shows that there exist problems for which our five optimality criteria are indiscriminate. In such cases, other criteria are called for, of which there are many. See Carlson et al (1991), for example.

4. Reachability Conditions

In this section, we consider certain additional state-reachability conditions for our problem which will prove to be useful for comparing our optimality criteria in the case $C^*(\alpha) = \infty$. These conditions are *controllability* notions. A very strong version of such a notion in the literature is *complete reachability*, which requires that the system be able to transition from any state in any period to any state in the very next period. This was assumed in Zaslavski (1995), and most notably in section 5.3 of Carlson et al (1991). Another strong controllability notion (used in Leizarowitz (1987)) requires that transition from any state at any time to any future state be accomplished by a feasible *stationary* strategy. Our state-reachability conditions are considerably weaker than these.

First, we recall (a slightly weaker version of) the *Bounded Reachability* condition introduced in Schochetman and Smith (1998).

Bounded Time Reachability (BTR): There exists a positive integer R such that for each $1 \leq K < \infty$ and each $x, y \in X$, there exists $K \leq L \leq K + R$ and $z \in X_L$ (depending on K, x, y) for which $s_K(z) = s_K(y)$ and $s_L(z) = s_L(x)$. If such R exists, then our problem is said to satisfy the *Bounded Time Reachability*, i.e., (BTR) property. Roughly speaking, there exists a strategy z which steers the system from state $s_K(y)$ at time K to state $s_L(x)$ at some time L, which is at most R periods from K.

Note that property (BTR) is independent of the cost structure and the discount factor. Consequently, we introduce two other notions of state-reachability which do depend on these data.

Total Cost Reachability (TCR|α): Let $x, y \in X$, $0 < \alpha \leq 1$. For each $\epsilon > 0$, there exists a positive integer M (depending on ϵ), such that for all $N \geqslant M$, there exists $0 \leq K \leq N$ and $z \in X$ (depending on N) such that $s_K(z) = s_K(y)$, $s_N(z) = s_N(x)$ and $C_N(z|\alpha) - C_K(z|\alpha) < \epsilon$. Thus, given $\epsilon > 0$, for sufficiently large N, there exists an earlier period K and a strategy z which steers state $s_K(y)$ at time K to state $s_N(x)$ at time N with cost less than ϵ.

Average Cost Reachability (ACR|α): Let $x, y \in X$. Given $\epsilon > 0$, there exists a positive integer M such that for all $N \geqslant M$, there exists $0 \leq K \leq N$ and $z \in X$ such that $s_K(z) = s_K(y)$, $s_N(z) = s_N(x)$ and $C_N(z|\alpha) - C_K(z|\alpha) < N\epsilon$. Thus, here the "steering" is as in the previous case, but with average cost less than ϵ.

Obviously, these reachability properties do depend on the cost structure and the discount factor. Moreover, the Average Cost Reachability property is weaker than the Total Cost Reachability property, i.e., (TCR|α) $\implies$ (ACR|α), $\forall 0 < \alpha \leq 1$. The converse is false, in general (Example 4.7).

If the c_j are exponentially bounded, say by $Be^{\gamma j}$, and $\alpha < 1/\gamma$, then strong optima exist (section 3), and they are optimal in every other sense. However, if either $\alpha \geqslant 1/\gamma$, or the c_j are not exponentially bounded, then what can we say? The next result sets the stage for our response to this question, which depends on the relationship between α and the c_j.

4.1 Theorem. *Suppose property (BTR) holds.*
(i) If α is such that

$$\lim_{j\to\infty} \frac{\alpha^{j-1}\|c_j\|}{j} = 0,$$

then property (ACR|α) holds. In particular, if

$$\lim_{j\to\infty} \frac{\|c_j\|}{j} = 0,$$

then property (ACR|α) holds for all $0 < \alpha \leq 1$.
(ii) If α is such that

$$\lim_{j\to\infty} \alpha^{j-1}\|c_j\| = 0,$$

then property (TCR|α) holds. In particular, if

$$\lim_{j\to\infty} \|c_j\| = 0,$$

then property (TCR|α) holds for all $0 < \alpha \leq 1$.

Proof. (i) Let $R \geqslant 1$ be as property (BTR). Given $x, y \in X$ and $\epsilon > 0$, let J be sufficiently large such that

$$\frac{\alpha^{j-1}\|c_j\|}{j} < \epsilon/R, \quad \forall j \geqslant J.$$

Let $M = J + R$ and $N \geqslant M$. Set $K = N - R \geqslant J$. By property (BTR), there exists L such that

$$N - R = K \leq L \leq K + R = N,$$

and $w \in X_L$ such that $s_K(w) = s_K(y)$ and $s_L(w) = s_L(x)$. Let $z = (w \mid_L x)$ so that $s_K(z) = s_K(w) = s_K(y)$ and $s_N(z) = s_N(x)$. Also,

$$
\begin{aligned}
C_N(z|\alpha) - C_K(z|\alpha) &= \sum_{j=K+1}^{N} \alpha^{j-1} c_j(s_{j-1}(z), z_j) \\
&\leq \sum_{j=K+1}^{N} \alpha^{j-1}\|c_j\| \\
&\leq \frac{\epsilon}{R} \sum_{j=K+1}^{N} j \\
&\leq N(N - K)\epsilon/R \\
&= \epsilon N.
\end{aligned}
$$

Thus, property (ACR|α) holds. Part (ii) is proved similarly. $\square$

Remark. It's worth noting that, for each $0 < \alpha \leq 1$, it can happen that the hyptheses of Theorem 4.1 hold, together with the property that $C^*(\alpha) = \infty$. For example, it happens when $\alpha = 1$ and $\|c_j\| = B/j$.

In Theorem 4.2 of Schochetman and Smith (1998), we showed that, in the presence of property (BTR), every efficient strategy is average optimal, i.e., $X^e(\alpha) \subseteq X^a(\alpha)$, $\forall 0 < \alpha \leq 1$. We next give a stronger version of this result. Thus, we obtain reasonable sufficient conditions for the existence of average optima - which need not exist in general (Example 3.15). Note that if $A^*(\alpha) = \infty$, then $X^a(\alpha) = \emptyset$, for such α, and, at least in the discrete case, $X^e(\alpha)$ can't possibly be contained therein, since it is non-empty.

4.2 Theorem. *Suppose α is such that property $(ACR|\alpha)$ is satisfied and $A^*(\alpha) < \infty$. Then efficient implies average optimal, i.e., $X^e(\alpha) \subseteq X^a(\alpha)$, so that*

$$X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^w(\alpha) \subseteq X^e(\alpha) \subseteq X^a(\alpha),$$

for such α. If, in addition, the set-valued functions $s \twoheadrightarrow X_N(s)$ are continuous, then there exists efficient optimum which is also average optimal.

Proof. Let $x \in X^e(\alpha)$, and suppose there exists $y \in X$ such that $A(y|\alpha) < A(x|\alpha)$, i.e., $x \notin X^a(\alpha)$. In particular, $A(y|\alpha) < \infty$. Let

$$\epsilon = \begin{cases} \frac{1}{2}(A(x|\alpha) - A(y|\alpha)), & \text{if } A(x|\alpha) < \infty, \\ 1, & \text{if } A(x|\alpha) = \infty, \end{cases}$$

so that $\epsilon > 0$. Also let M be as in property $(ACR|\alpha)$ for x, y and ϵ. Then, for each $N \geqslant M$, there exist $0 \leq K \leq N$ and $z \in X$ such that $s_K(z) = s_K(y)$, $s_N(z) = s_N(x)$ and $C_N(z|\alpha) - C_K(z|\alpha) < N\epsilon$. If $K = N$, then $z = y$, $s_N(z) = s_N(x) = s_N(y)$ and $C_N(y|\alpha) \geqslant C_N(x|\alpha)$, i.e., $A_N(y|\alpha) \geqslant A_N(x|\alpha)$, so that $A_N(y|\alpha) + \epsilon > A_N(x|\alpha)$ in this case.

If $K < N$, define $w = (y \mid_K z)$, so that $w \in X$ and $s_N(w) = s_N(z)$ by Lemma 2.2. Then, necessarily, $C_N(x|\alpha) \leq C_N(w|\alpha)$, since $x \in X^e(\alpha)$ and $w \in X$ with $s_N(w) = s_N(x)$. Moreover, $C_N(y|\alpha) \geqslant C_K(y|\alpha)$, since each $c_j \geqslant 0$. Thus,

$$C_N(y|\alpha) + \sum_{j=K+1}^{N} \alpha^{j-1} c_j(s_{j-1}(z), z_j) \geqslant C_K(y|\alpha) + \sum_{j=K+1}^{N} \alpha^{j-1} c_j(s_{j-1}(z), z_j)$$
$$= C_K(w|\alpha) + [C_N(w|\alpha) - C_K(w|\alpha)]$$
$$= C_N(w|\alpha)$$
$$\geqslant C_N(x|\alpha)$$
$$= \sum_{j=1}^{N} \alpha^{j-1} c_j(s_{j-1}(x), x_j),$$

which implies that

$$\frac{1}{N} C_N(y|\alpha) + \frac{1}{N} \sum_{j=K+1}^{N} \alpha^{j-1} c_j(s_{j-1}(z), z_j) \geqslant \frac{1}{N} \sum_{j=1}^{N} \alpha^{j-1} c_j(s_{j-1}(x), x_j),$$

i.e.,

$$A_N(y|\alpha) + \frac{1}{N}(C_N(z|\alpha) - C_K(z|\alpha)) \geqslant A_N(x|\alpha).$$

Since $C_N(z|\alpha) - C_K(z|\alpha) < N\epsilon$, we have that

$$A_N(y|\alpha) + \epsilon > A_N(y|\alpha) + \frac{1}{N}(C_N(z|\alpha) - C_K(z|\alpha)) \geqslant A_N(x|\alpha),$$

for the case $K < N$. Thus, $A_N(y|\alpha) + \epsilon > A_N(x|\alpha)$, $\quad \forall N \geqslant M$. Consequently,

$$\limsup_N A_N(y|\alpha) + \epsilon = \limsup_N (A_N(y|\alpha) + \epsilon)$$
$$= \limsup_{N \geqslant M} (A_N(y|\alpha) + \epsilon)$$
$$\geqslant \limsup_{N \geqslant M} A_N(x|\alpha)$$
$$= \limsup_N A_N(x|\alpha),$$

so that, $A(y|\alpha) + \epsilon \geqslant A(x|\alpha)$. If $A(x|\alpha) = \infty$, then $A(y|\alpha) = \infty$. Contradiction. Otherwise, substituting for ϵ, we conclude that $A(y|\alpha) \geqslant A(x|\alpha)$. Contradiction. Hence, $A(x|\alpha) \leq A(y|\alpha)$, $\forall y \in X$. This result, together with our assumption that $A^*(\alpha) < \infty$, implies that $A(x|\alpha) < \infty$. To complete the proof, apply Theorem 3.5. $\qquad\square$

We next give a sufficient condition for the efficient strategies and the overtaking optimal strategies to be the same. The following theorem is a central result of this paper, allowing us to conclude the existence of an overtaking optimum (in particular) in the presence of easily verified conditions.

4.3 Theorem. *If α is such that property (TCR|α) is satisfied, then every efficient strategy is overtaking optimal, i.e., $X^e(\alpha) \subseteq X^o(\alpha)$, so that*

$$X^s(\alpha) \subseteq X^o(\alpha) = X^w(\alpha) = X^e(\alpha).$$

If, in addition, $A^*(\alpha) < \infty$, then

$$X^s(\alpha) \subseteq X^o(\alpha) = X^w(\alpha) = X^e(\alpha) \subseteq X^a(\alpha).$$

Proof. By Corollary 3.7, it suffices to show the set inclusion for the first claim. Fix $x \in X^e(\alpha)$ and let $y \in X$. We show that x overtakes y. Let $\epsilon > 0$. By property (TCR|α), there exists a positive integer M such that $\forall N \geqslant M$, there exists $0 \leq K \leq N$ and $z \in X$ such that $s_K(z) = s_K(y)$, $s_N(z) = s_N(x)$ and $C_N(z|\alpha) - C_K(z|\alpha) < \epsilon$. Let $w = (y \mid_K z)$. Then $w \in X$, $s_K(w) = s_K(y)$, $s_N(w) = s_N(z)) = s_N(x)$ and

$$C_N(w|\alpha) - C_K(w|\alpha) = C_N(z|\alpha) - C_K(z|\alpha) < \epsilon.$$

Hence, by the efficiency of x at horizon N, we have

$$\begin{aligned}
C_N(x|\alpha) \leq C_N(w|\alpha) &= C_K(w|\alpha) + C_N(w|\alpha) - C_K(w|\alpha) \\
&< C_K(y|\alpha) + \epsilon \\
&\leq C_N(y|\alpha) + \epsilon.
\end{aligned}$$

Therefore, $x \in X^o(\alpha)$. To complete the proof, apply Theorem 4.2, together with the fact that (TCR|α) $\implies$ (ACR|α). $\qquad\square$

4.4 Example. Let the data be as in Example 3.12. We leave it to the reader to verify that this example has properties (BTR) with $R = 1$, (ACR|α), $\forall 0 < \alpha \leq 1$, and (TCR|$\alpha$), $\forall 0 < \alpha < 1$, i.e., it does *not* have property (TCR|1).

4.5 Examples. Let the data be as in Example 3.13, 3.14 or 3.15. We leave it to the reader to verify that, for each example, and for each $0 < \alpha \leq 1$, all three reachability properties fail.

Before leaving this section, we summarize our main results.

4.6 Theorem. *The following are true for our general optimization problem.*

(i) If the set-valued mappings $s \twoheadrightarrow X_N(s)$ are continuous, then $X^e(\alpha) \neq \emptyset$, i.e., efficient optima exist, for each $0 < \alpha \leq 1$.

(ii) If $C^*(\alpha) = \infty$, then $X^s(\alpha) = \emptyset$.

(iii) If $C^*(\alpha) < \infty$, then it is attained, $A^*(\alpha) < \infty$ and

$$\emptyset \neq X^s(\alpha) \subseteq X^o(\alpha) \subseteq \begin{cases} X^w(\alpha) \subseteq X^e(\alpha), \\ X^a(\alpha), \end{cases}$$

so that there exists a strong optimum which is optimal in every sense.

(iv) If $A^*(\alpha) = \infty$, then $C^*(\alpha) = \infty$, $X^s(\alpha) = X^a(\alpha) = \emptyset$, and

$$X^o(\alpha) \subseteq X^w(\alpha) \subseteq X^e(\alpha).$$

(v) If $A^*(\alpha) < \infty$, then

$$\emptyset \subseteq X^s(\alpha) \subseteq X^o(\alpha) \subseteq \left\{ \begin{array}{l} X^w(\alpha) \subseteq X^e(\alpha), \\ X^a(\alpha). \end{array} \right.$$

(vi) If $A^*(\alpha) < \infty$ and property (ACR|α) holds, then

$$X^s(\alpha) \subseteq X^o(\alpha) \subseteq X^w(\alpha) \subseteq X^e(\alpha) \subseteq X^a(\alpha),$$

and, in particular, efficient optima are average optimal.

(vii) If $A^*(\alpha) < \infty$ and property (TCR|α) holds, then

$$X^s(\alpha) \subseteq X^o(\alpha) = X^w(\alpha) = X^e(\alpha) \subseteq X^a(\alpha),$$

and, in particular, efficient optima are overtaking, weakly overtaking and average optimal.

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