Convergence of Minimum Norm Elements of Projections and Intersections of Nested Affine Spaces in Hilbert Space

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July 22, 2006

Abstract

We consider a Hilbert space, an orthogonal projection onto a closed subspace and a sequence of downwardly directed affine spaces. We give sufficient conditions for the projection of the intersection of the affine spaces into the closed subspace to be equal to the intersection of their projections. Under a closure assumption, one such (necessary and) sufficient condition is that summation and intersection commute between the orthogonal complement of the closed subspace, and the subspaces corresponding to the affine spaces. Another sufficient condition is that the cosines of the angles between the orthogonal complement of the closed subspace, and the subspaces corresponding to the affine spaces, be bounded away from one. Our results are then applied to a general infinite horizon, positive semi-definite, linear quadratic, mathematical programming problem. Specifically, under suitable conditions, we show that optimal solutions exist and, modulo those feasible solutions with zero objective value, they are limits of optimal solutions to finite dimensional truncations of the original problem.

*This author was supported in part by the National Science Foundation under Grant DMI-0322114.

1 Introduction and Problem Formulation

Suppose H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and closed subspace K. Let $K^\perp = H/K$ denote the orthogonal complement of K in H, so that $H = K \oplus K^\perp$, and let $P_{K^\perp} : H \to K^\perp$ be the corresponding orthogonal projection of H onto $K^\perp$. For the sake of notational convenience and simplicity, we will suppress the reference to $K^\perp$and simply write P in place of $P_{K^\perp}$, except in statements of results. If F is an affine space in H of the form $F = N + z$, for N a closed subspace of H and $z \in F$, then $P(F)$ is convex, and it is closed if and only if $P(N)$ is.

We next define the angle $\theta(K, N)$ between the closed subspaces K and N, and its cosine $c(K, N)$ [3]. Let

$$S(K,N) \quad = \quad \{(x,y) : x \in K \cap (K \cap N)^\perp, \quad y \in N \cap (K \cap N)^\perp, \quad \|x\| \le 1, \quad \|y\| \le 1\},$$

so that $S(K, N) = S(N, K)$. Let

$$c(K,N) \quad = \quad \sup\{|\langle x, y\rangle| : (x,y) \in S(K,N)\},$$

so that $0 \le c(K, N) = c(N, K) \le 1$. Consequently, $\theta(K, N)$ is the unique angle in $[0, \pi/2]$ such that $\cos(\theta(K, N)) = c(K, N)$. If $K \subseteq N$, then $c(K, N) = 0$. If $N \cap K = \{0\}$, then

$$(K \cap N)^\perp = H, \qquad N = N \cap (K \cap N)^\perp, \qquad K = K \cap (N \cap K)^\perp$$

and

$$S(K,N) \quad = \{(x,y) : x \in K, \quad y \in N, \quad \|x\| \le 1, \quad \|y\| \le 1\}.$$

If M is a closed subspace of H such that $N \subseteq M$ and $K \cap M = \{0\}$, then $S(K, N) \subseteq S(K, M)$ and $c(K, N) \le c(K, M)$. Moreover, as we shall see in Theorem 1.1 below, if $K + M$ is closed, then $c(K, M) < 1$, so that $c(K, N) < 1$. If $K^\perp$ and $N^\perp$ are hyperplanes through the origin, then $c(K, N)$ is the cosine of the conventional angle between the one-dimensional subspaces K and N. If K or N is finite dimensional, then $c(K, N) < 1$.

In [10], we were interested in finding conditions for $P(F)$ to be norm closed. Here, we are also interested in finding conditions for $P(F)$ to be weakly closed and for $c(K, N)$ to be strictly less than 1. In Theorem 4.1 of [10], we established the equivalence of (i) and (ii) in the following result. (See also Theorem 9.35 of [3].)

Theorem 1.1 *The following are equivalent:*

(i) $P_{K^\perp}(F)$ is closed in H;

(ii) $K + N$ is closed in H;

(iii) $P_{K^\perp}(F)$ is weakly closed in H;

(iv) $K + N$ is weakly closed in H;

(v) $c(K, N) < 1$.

Proof The proofs of the equivalence of (i) through (iv) follows immediately from the fact that, for convex subsets of H, weak closure and norm closure are equivalent [4]. The remaining equivalence follows from Theorem 9.35 of [3]. QED

Here, we further assume that $\{F_j\}_{j=1}^\infty$ is a downwardly nested sequence of affine subspaces, i.e., $F_j = N_j + z_j$, where N_j is a closed subspace of H, $z_j \in F_j$ and $F_{j+1} \subseteq F_j$, for each j. Of course, each $P(F_j)$ is convex, and closed if and only if $K + N_j$ is closed. Let

$$N \;=\; \cap_{j=1}^\infty N_j, \qquad F \;=\; \cap_{j=1}^\infty F_j,$$

and suppose F is non-empty.

• Assume the z_j are norm-bounded, i.e., $\|z_j\| \leq b, \quad \forall j$, for some $b > 0$.

If each $P(F_j)$ is closed, then it contains a unique minimum norm element ξ_j. Moreover, $\cap_{j=1}^{\infty} P(F_j)$ is closed and convex, and thus also contains a unique minimum norm element $\xi^{\dagger}$. It follows from [11] that $\xi_j \to \xi^{\dagger}$, as $j \to \infty$. Furthermore, if $P(F)$ is closed, then it also contains a unique minimum norm element ξ^*. It is unclear if $\xi^{\dagger} = \xi^*$; if true, then $\xi_j \to \xi^*$, as $j \to \infty$, which is what we want. This will be the case if, for example,

$$P(\cap_{j=1}^{\infty} F_j) \;=\; \cap_{j=1}^{\infty} P(F_j), \qquad \text{i.e.,} \qquad P(F) = \cap_{j=1}^{\infty} P(F_j).$$

We wish to find sufficient conditions for this to be the case. Since the forward inclusion is automatically true, the problem reduces to finding conditions for the reverse inclusion to hold. Example 2.6 below shows that the reverse inclusion does *n*ot hold in general. In view of Theorem 1.1, it is also tempting to find sufficient conditions for

$$K + N = K + (\cap_{j=1}^{\infty} N_j) \;=\; \cap_{j=1}^{\infty} (K + N_j),$$

i.e., for intersection and summation to commute. Once again, since the forward inclusion is automatically true, the problem reduces to finding conditions for the reverse inclusion to hold. Example 2.6 below also shows that this reverse inclusion does not hold in general.

In section 2, we present the first of our main results, namely Theorem 2.4. We have that

$$P(F) = \cap_{j=1}^{\infty} P(F_j), \quad \text{if} \quad K + N = \cap_{j=1}^{\infty} (K + N_j),$$

and each $K + N_j$ is weakly closed in H. Conversely, if $P(F) = \cap_{j=1}^{\infty} P(F_j)$, and each $P(F_j)$ is weakly closed, then $K + N = \cap_{j=1}^{\infty} (K + N_j)$. Thus, in this event, we have $\xi_j \to \xi^* = \xi^{\dagger}$, as desired.

Also in section 2, we establish our second main result (Theorem 2.7). We have

$$P(F) = \cap_{j=1}^{\infty} P(F_j), \quad \text{if} \quad \sup_j c(K, N_j) < 1.$$

Thus, in this event also, we have that $\xi_j \to \xi^* = \xi^{\dagger}$, as desired. (It is not clear whether the converse is true.)

We then establish a variant (Theorem 2.14) of the previous result in terms of (postulated) finite dimensional subspaces K_j and M_j of K and N_j, respectively, for use in section 3. Recall that the cosines of the angles between finite dimensional subspaces are automatically less than 1.

At the end of section 2, we apply the previous results in the context of ascending closed subspaces of H - for example, increasing finite dimensional subspaces whose union is dense in the separable Hilbert space H.

In section 3, we give an application of our main results to an infinite dimensional, positive semi-definite, linear-quadratic programming problem (as in [8, 9]). Specifically, under appropriate conditions, we show (Theorem 3.2) that optimal solutions exist. We also characterize them as limits, modulo solutions of zero objective value, of optimal solutions to finite dimensional truncations of the original problem.

2 Main Results

In this section, we first give sufficient conditions for $P(F) = \cap_{j=1}^{\infty} P(F_j)$. Before doing so, we establish some useful preliminary results.

Lemma 2.1 *There exists a subsequence of $\{z_j\}_{j=1}^{\infty}$ which converges weakly to some $z \in F$. Moreover, the set F is affine, and $F = N + z$.*

Proof Since $\{z_j\}_{j=1}^{\infty}$ is bounded, with $z_j \in F_j, \quad \forall j$, by the Hilbert-Banach Theorem, there exists $z \in H$ which we may assume (passing to a subsequence, if necessary) is the weak limit of the z_j. Fix any integer k. Then $z_j \in F_k$, for all $j \geq k$. It follows that $z \in F_k$. Since k is arbitrary, $z \in \cap_{j=1}^{\infty} F_k = F$.

For the second part, if $w \in F$, it follows that $w \in F_j = N_j + z_j$, i.e., $w = m_j + z_j$, for $m_j \in N_j$, $\forall j$. Since $z \in F$, it follows that $z \in F_j = N + z_j$, $\forall j$. Consequently, $z = n_j + z_j$, for $n_j \in N_j$. $\forall j$. Then

$$z = n_j + w - m_j = n_j - m_j + w,$$

so that $z - w \in N_j$, $\forall j$. Hence, $z - w \in N$, i.e., $w \in z - N = N + z$.

Conversely, let $w \in N + z$. Then $x - w \in N_j$, i.e., $x - z_j - m_j = n_j$, so that

$$x = n_j + m_j + z_j \in F_j = N_j + z_j, \qquad \forall j.$$

Thus, $x \in F$, which completes the proof. QED

Lemma 2.2 *The sequence $\{N_j\}_{j=1}^{\infty}$ is also nested downward, i.e., $N_{j+1} \subseteq N_j$, $\forall j$.*

Proof First observe that since $z_{j+1} \in F_{j+1} \subseteq F_j = N_j + z_j$, it follows that $z_j - z_{j+1} \in N_j$, $\forall j$. Now let $n_{j+1} \in N_{j+1}$. Then

$$n_{j+1} + z_{j+1} \in F_{j+1} \subseteq F_j = N_j + z_j,$$

so that $n_{j+1} + z_{j+1} = n_j + z_j$, i.e.,

$$n_{j+1} = n_j + z_j - z_{j+1} \in n_j + N_j = N_j, \qquad \forall j.$$

QED

Remark 2.3 Observe that if $\{F_{j_k}\}_{k=1}^{\infty}$ is any subsequence of $\{F_j\}_{j=1}^{\infty}$, then

$$F = \cap_{k=1}^{\infty} F_{j_k} = \cap_{j=1}^{\infty} F_j,$$

$$P(F) = P(\cap_{k=1}^{\infty} F_{j_k}) = P(\cap_{j=1}^{\infty} F_j),$$

and

$$\cap_{k=1}^{\infty} P(F_{j_k}) = \cap_{j=1}^{\infty} P(F_j).$$

Analogously, if $\{N_{j_k}\}_{k=1}^{\infty}$ is any subsequence of $\{N_j\}_{j=1}^{\infty}$, then, in view of Lemma 2.2,

$$N = \cap_{k=1}^{\infty} N_{j_k} = \cap_{j=1}^{\infty} N_j,$$

$$K + N = K + \cap_{k=1}^{\infty} N_{j_k} = K + \cap_{j=1}^{\infty} N_j,$$

and

$$\cap_{k=1}^{\infty}(K + N_{j_k}) = \cap_{j=1}^{\infty}(K + N_j).$$

Therefore, for our purposes, it suffices to consider subsequences in what follows . In particular, in view of Lemma 2.1, we may restrict attention to a subsequence $\{z_{j_k}\}_{k=1}^{\infty}$ of $\{z_j\}_{j=1}^{\infty}$ which converges weakly to z, that is, $z_{j_k} \rightharpoonup z$, as $k \to \infty$.

We next present our first main result.

Theorem 2.4 *Suppose the results of Theorem 1.1 hold eventually for K and the N_j, i.e., there exists m such that $K + N_j$ is closed for all $j \geq m$. Then $K + N = \cap_{j=1}^{\infty}(K + N_j)$ if and only if $P_{K^\perp}(F) = \cap_{j=1}^{\infty} P_{K^\perp}(F_j)$.*

Proof $\Longrightarrow$: By Remark 2.3, we may assume that $m = 1$. Since $F = \cap_{j=1}^{\infty} F_j$, it suffices to show that $\cap_{j=1}^{\infty} P(F_j) \subseteq P(F)$. Let $x \in \cap_{j=1}^{\infty} P(F_j)$. Then $x \in K^\perp$ and, for each j, $x = P(u_j)$, for unique $u_j \in F_j$, so that $x - u_j = k_j \in K$. Also, for each j, $u_j = n_j + z_j$, for $n_j \in N_j$, so that $x = k_j + u_j = k_j + n_j + z_j$, i.e., $x - z_j = k_j + n_j \in K + N_j$. By hypothesis, each $K + N_j$ is also weakly closed in H. Since the N_j are nested downward, this is also the case for the $K + N_j$. Moreover, for each j, the sequence $\{x - z_i\}_{i\geq j}$ is contained in $K + N_i \subseteq K + N_j$ and weakly converges to $x - z$, which belongs to $K + N_j$, $\forall j$. Thus,

$$x - z \in \cap_{j=1}^{\infty}(K + N_j) = K + N,$$

by hypothesis. Hence, $x - z = k + n$, for $k \in K$ and $n \in N$, so that

$$x = k + (n + z) \in K + F, \quad \text{i.e.,} \quad x = P(x) = P(z + n) \in P(K + F) = P(F),$$

$\Longleftarrow$: It suffices to show that $\cap_{j=1}^{\infty}(K+N_j) \subseteq K+N$. Let $v \in \cap_{j=1}^{\infty}(K+N_j)$. Then, for each j, $v = k_j + n_j$, for some $k_j \in K$ and $n_j \in N_j$. Hence, $v + z_j = k_j + n_j + z_j$, $\forall j$. But

$$k_j + n_j + z_j = v + z_j \rightharpoonup v + z, \qquad \text{as} \quad j \to \infty,$$

where

$$k_j + n_j + z_j \in K + N_j + z_j = K + F_j, \qquad \forall j.$$

Therefore,

$$P(v + z_j) = P(k_j + n_j + z_j) = P(n_j + z_j) \in P(F_j), \qquad \forall j.$$

But $P(v + z_j) \rightharpoonup P(v + z)$, as $j \to \infty$, by Lemma 9.14 of [3]. Since the $P(F_j)$ are weakly closed and descending, we have that $P(v + z) \in P(F_j)$, $\forall j$, i.e., $P(v + z) \in \cap_{j=1}^{\infty}P(F_j) = P(F)$, by hypothesis. Thus, $v+z \in P^{-1}(P(F)) = F+K$, so that there exists $k \in K$ such that $v+z-k \in F$, i.e., $v+z \in F+k = N+z+k$. Hence, $v \in N + k \subseteq N + K$, which completes the proof. QED

The following corollary gives easily verified conditions for the results of Theorem 2.4 to hold. See Corollary 3.4 for another such condition.

Corollary 2.5 *The results of Theorem 2.4 hold under each of the following conditions. Eventually,*

(i) $K \subseteq N_j$.

(ii) there exists m *such that* $N_m \subseteq K$.

(iii) the N_j *are constant.*

Proof By Remark 2.3, we may assume that each condition holds for all j.

(i) By hypothesis, $K \subseteq N$, so that $K + N = N$ and $K + N_j = N_j$, i.e., $K + Nj$ is closed, for all j. Thus, $\cap_{j=1}^{\infty}(K + N_j) = \cap_{j=1}^{\infty}N_j = N = K + N$. Now apply Theorem 2.4.

(ii) In this case, for $m = 1$, $N \subseteq N_1 \subseteq K$, so that $K + N = K$ and $K + N_j = K$, i.e., $K + N_j$ is closed, for all j. Hence, $\cap_{j=1}^{\infty}(K + N_j) = K = K + N$. Apply Theorem 2.4.

(iii) By hypothesis $K + N_j = K + N$, which is closed, $\forall j$. Therefore, $\cap_{j=1}^{\infty}(K + N_j) = K + N$, and the proof is completed by Theorem 2.4. QED

Before continuing, we give an example which shows that $P(\cap_{j=1}^{\infty}F_j) \neq \cap_{j=1}^{\infty}P(F_j)$ and $K + N \neq \cap_{j=1}^{\infty}(K + N_j)$, in general.

Example 2.6 As in Example 2.2 of [10], let $H = \oplus_{j=1}^{\infty}\mathbb{R}^2$,

$$K = \{[x_{i1} \;\; x_{i2}]_{i=1}^{\infty} \in H : x_{i2} = 0, \;\; \forall i\},$$

and

$$N = \{[x_{i1} \;\; x_{i2}]_{i=1}^{\infty} \in H : \; x_{i1} = x_{i2}\sqrt{i^2 - 1}, \;\; \forall i\}.$$

Clearly, K and N are closed subspaces of H with $K \cap N = \{0\}$ and

$$K^{\perp} = \{[x_{i1} \;\; x_{i2}]_{i=1}^{\infty} \in H : \; x_{i1} = 0, \;\; \forall i\}.$$

For each j, let $z_j = 0$ and

$$N_j \;\; = \;\; \{[x_{i1} \;\; x_{i2}]_{i=1}^{\infty} \in H: \; x_{i1} = x_{i2}\sqrt{i^2-1}, \;\; \forall i = 1, \ldots, j\},$$

so that N_j is a closed subspace of H, $F_j = N_j$, $N_{j+1} \subseteq N_j$ and

$$F \;\; = \;\; \cap_{j=1}^{\infty} F_j \;\; = \;\; \cap_{j=1}^{\infty} N_j \;\; = \;\; N.$$

It is not difficult to see that each $P(F_j)$ is closed, so that $\cap_{j=1}^{\infty} P(F_j)$ is closed. However, it was shown in [10] that $P(F)$ is not closed, so they can't possibly be equal. More directly, we exhibit an element of $\cap_{j=1}^{\infty} P(F_j)$ which does not belong to $P(F)$. Let $\xi \in K^{\perp}$ be defined by

$$\xi_i \;\; = \;\; [0 \quad 1/i], \quad \text{for} \;\; i = 1, 2, \ldots.$$

Then $\xi \in P(N_j)$, since $\xi = P(x_j)$, for $x_j \in N_j$ given by

$$x_{ji} = \begin{cases} [\sqrt{i^2-1}/i \quad 1/i], & \text{for} \;\; i = 1, 2, \ldots, j, \\ [0 \quad 1/i], & \text{for} \;\; i = j+1, j+2, \ldots. \end{cases}$$

Thus, $\xi \in \cap_{j=1}^{\infty} P(F_j)$. On the other hand, if $\xi \in P(N)$, then there exists $x \in N$ such that $P(x) = \xi$. Necessarily,

$$x_i \;\; = \;\; [\sqrt{i^2-1}/i \quad 1/i], \qquad \text{for} \;\; i = 1, 2, \ldots.$$

Then

$$\|x\|^2 \;\; = \;\; \sum_{i=1}^{\infty} \left[\frac{i^2-1}{i^2} + \frac{1}{i^2} \right] \;\; = \;\; \infty,$$

i.e., $x \notin H$. Hence, $\xi \notin P(N)$. It also follows that $K + N$ is strictly contained in $\cap_{j=1}^{\infty}(K + N_j)$.

Next we turn to a study of the cosines $c(K, N_j)$ relative to our problem of interest. Recall that the sequence $\{z_j\}_{j=1}^{\infty}$ weakly converges to $z \in F$.

The following is our second main result.

Theorem 2.7 *If the $c(K, N_j)$ are bounded away from 1, i.e., there exists $0 < \alpha < 1$ such that $c(K, N_j) \leq \alpha, \;\; \forall j$, then*

$$P(F) \;\; = \;\; \cap_{j=1}^{\infty} P(F_j).$$

Proof Let $x \in \cap_{j=1}^{\infty} P(F_j)$. Then for each j, there exists $u_j \in F_j$ such that $x = P(u_j)$. Clearly,

$$N_j \;\; = \;\; (K \cap N_j) \oplus [(K \cap N_j)^{\perp} \cap N_j],$$

and

$$F_j \;\; = \;\; (K \cap N_j) \oplus [(K \cap N_j)^{\perp} \cap N_j] + z_j, \qquad \forall j.$$

Thus, $u_j = w_j + y_j + z_j$, where $w_j \in K \cap N_j$ and $y_j \in (K \cap N_j)^{\perp} \cap N_j$, $\forall j$. Note that $\langle w_j, y_j \rangle = 0$, for each j.

Let $v_j = y_j + z_j \in N_j + z_j = F_j$, $\forall j$. Then, since $w_j \in K$, $P(w_j) = 0$, and

$$x \;\; = \;\; P(u_j) \;\; = \;\; P(w_j) + P(y_j + z_j) \;\; = \;\; P(v_j),$$

so that $\|P(v_j)\| = \|P(u_j)\| = \|x\|$, $\forall j$. Moreover,

$$\|P(y_j)\| \;\; = \;\; \|P(v_j - z_j)\| \;\; \leq \;\; \|P(v_j)\| + \|P(z_j)\| \;\; \leq \;\; \|x\| + b, \qquad \forall j,$$

i.e., the sequence $\{P(y_j)\}_{j=1}^{\infty}$ is bounded in $K^{\perp}$.

Now $y_j \in H = K \oplus K^\perp$ implies that there exists $r_j \in K$ for which $y_j = r_j + P(y_j), \quad \forall j$. But

$$K = (K \cap N_j) \oplus [(K \cap N_j)^\perp \cap K].$$

Hence, for each j, there exists $s_j \in K \cap N_j$ and $t_j \in (K \cap N_j)^\perp \cap K$ such that $r_j = s_j + t_j$ and $\langle s_j, y_j \rangle = 0$. Consider the vectors

$$\frac{y_j}{\|y_j\|} \in (K \cap N_j)^\perp \cap N_j \qquad \text{and} \qquad \frac{r_j}{\|t_j\|} \in K.$$

Then

$$\begin{aligned}\left|\left\langle \frac{r_j}{\|t_j\|}, \frac{y_j}{\|y_j\|} \right\rangle\right| &= \left|\left\langle \frac{s_j}{\|t_j\|} + \frac{t_j}{\|t_j\|}, \frac{y_j}{\|y_j\|} \right\rangle\right| \\ &= \left|\left\langle \frac{s_j}{\|t_j\|}, \frac{y_j}{\|y_j\|} \right\rangle + \left\langle \frac{t_j}{\|t_j\|}, \frac{y_j}{\|y_j\|} \right\rangle\right| \\ &= \left|\left\langle \frac{t_j}{\|t_j\|}, \frac{y_j}{\|y_j\|} \right\rangle\right| \\ &\le c(K, N_j) \\ &\le \alpha, \qquad \forall j,\end{aligned}$$

since the last absolute value is of an inner product of a pair of unit vectors from $S(K, N_j)$. Consequently,

$$|\langle r_j, y_j \rangle| \le \|t_j\| \, \|y_j\| \alpha \le \|r_j\| \, \|y_j\| \alpha,$$

i.e.,

$$\left|\left\langle \frac{r_j}{\|r_j\|}, \frac{y_j}{\|y_j\|} \right\rangle\right| \le \alpha, \qquad \forall j.$$

Now let θ_j denote the angle between r_j and y_j in H, where $r_j \in K$, $y_j \in N_j$, $P(y_j) \in K^\perp$ and $y_j = r_j + P(y_j) \in K \oplus K^\perp$. Then

$$|\sin(\theta_j)| = \frac{\|P(y_j)\|}{\|y_j\|},$$

so that

$$\|y_j\| = \frac{\|P(y_j)\|}{|\sin(\theta_j)|},$$

where

$$|\cos(\theta_j)| = \left|\left\langle \frac{r_j}{\|r_j\|}, \frac{y_j}{\|y_j\|} \right\rangle\right| \le \alpha, \qquad \forall j.$$

Thus, $\cos^2(\theta_j) \le \alpha^2$, so that $1 - \cos^2(\theta_j) \ge 1 - \alpha^2$, i.e.,

$$\frac{1}{1 - \cos^2(\theta_j)} \le \frac{1}{1 - \alpha^2} \le \frac{1}{\sqrt{1 - \alpha^2}},$$

so that

$$\|y_j\| \le \frac{\|x\| + b}{|\sin(\theta_j)|} = \frac{\|x\| + b}{\sqrt{1 - \cos^2(\theta_j)}} \le \frac{\|x\| + b}{\sqrt{1 - \alpha^2}}, \qquad \forall j.$$

Therefore, $\{y_j\}_{j=1}^\infty$ is a bounded sequence in H. This is the case also for the sequence $\{v_j\}_{j=1}^\infty$ since

$$\|v_j\| \le \|y_j + z_j\| \le \|y_j\| + \|z_j\| \le \frac{\|x\| + b}{\sqrt{1 - \alpha^2}} + b, \qquad \forall j \ge j_0.$$

Since $\{v_j\}_{j=1}^\infty$ is bounded, with $v_j \in F_j, \ \forall j$, by Lemma 2.1 there exists $v \in F$ which is the weak limit of the v_j. We have $x = P(v_j), \ \forall j$, $\{v_j\}_{j=1}^\infty$ converges weakly to v and P is weakly continuous (Theorem 9.14 of [3]). Hence, x is the weak limit of the $P(v_j)$, i.e., $x = P(v)$. Thus, $P(v) \in P(F)$. QED

Corollary 2.8 *Suppose the* $c(K, N_j)$ *are bounded away from* 1. *Then* $c(K, N) < 1$.

Proof For each j, $P(F_j)$ is closed in H since $c(K, N_j) < 1$ (Theorem 1.1). Therefore,

$$P(F) = \cap_{j=1}^{\infty} P(F_j)$$

is closed in H (Theorem 2.7) and $c(K, N) < 1$ by Theorem 1.1. QED

The following corollary gives a sufficient condition for the $c(K, N_j)$ to be bounded away from 1.

Corollary 2.9 *Suppose that there exists* m *such that* $K \cap N_m = \{0\}$ *and* $K + N_m$ *is closed, then*

$$c(K, N) \ \leq \ c(K, N_j) \ \ \leq \ \ c(K, N_m) \ < \ 1, \quad \forall j \geq m.$$

Proof By hypothesis,

$$\{0\} \ \subseteq \ K + N \ \subseteq \ K + N_j \ \subseteq \ K + N_m \ = \ \{0\}, \quad \forall j \geq m.$$

Hence,

$$S(K, N) \ \subseteq \ S(K, N_j) \ \subseteq \ S(K, N_m), \quad \forall j \geq m,$$

and

$$c(K, N) \ \leq \ c(K, N_j) \ \leq \ c(K, N_m) \ < \ 1, \quad \forall j \geq m,$$

by Theorem 1.1.

Remark 2.10 Parts (i) and (ii) of Corollary 2.5 are special cases of Theorem 2.7, since $c(K, N_j) = 0, \ \forall j$. Moreover, $c(K, N) = 0$, in this case. For part (iii) of Corollary 2.5, if $K + N$ is closed, we have $c(K, N_j) = c(K, N) < 1, \ \forall j$. Obviously, the $c(K, N_j)$ of Example 2.6 are not bounded away from 1.

For the purposes of section 3, it is desirable to have a finite dimensional version of Theorem 2.7. To this end, in view of the definition of $c(K, N)$, we require some results relating intersection and orthogonal complement of closed subspaces of H. Accordingly, let T be a closed subspace of H, so that $H = T \oplus T^{\perp}$. Next, let U and W be closed subspaces of T and V a closed subspace of $T^{\perp}$. Set $K = U \oplus V$ and $N = W \oplus T^{\perp}$. Thus, $T = U \oplus (U^{\perp} \cap T)$ and $T^{\perp} = V \oplus (V^{\perp} \cap T^{\perp})$. In particular, it T is finite dimensional, then so are U and W.

Theorem 2.11 *We have the following:*

(i) $K \cap N = (U \cap W) \oplus V$.

(ii) $(K \cap N)^{\perp} = \left[(U \cap W)^{\perp} \cap T\right] \oplus (V^{\perp} \cap T^{\perp})$.

(iii) $(K \cap N)^{\perp} \cap N = [(U \cap W)^{\perp} \cap W] \oplus (V^{\perp} \cap T^{\perp})$.

(iv) $(K \cap N)^{\perp} \cap K = (U \cap W)^{\perp} \cap U$.

Proof (i) We have

$$K \cap N \ = \ (U \oplus V) \cap (W \oplus T^{\perp}) \ = \ (U \cap W) \oplus (V \cap T^{\perp}) \ = \ (U \cap W) \oplus V.$$

(ii) By (i), we have

$$(K \cap N)^{\perp} \ = \ ((U \cap W) \oplus V)^{\perp} \ = \ \left[(U \cap W)^{\perp} \cap T\right] \oplus (V^{\perp} \cap T^{\perp}).$$

(iii) By (ii), we have

$$(K \cap N)^{\perp} \cap N = \{[(U \cap W)^{\perp} \cap T] \oplus (V^{\perp} \cap T^{\perp})\} \cap (W \oplus T^{\perp})$$
$$= [(U \cap W)^{\perp} \cap W] \oplus (V^{\perp} \cap T^{\perp}).$$

(iv) By (ii), we have

$$\begin{aligned}(K \cap N)^{\perp} \cap K &= \{[(U \cap W)^{\perp} \cap T] \oplus (V^{\perp} \cap T^{\perp})]\} \cap (U \oplus V) \\ &= [(U \cap W)^{\perp} \cap U] \oplus (V^{\perp} \cap V) \\ &= [(U \cap W)^{\perp} \cap U] \oplus \{0\} \\ &= (U \cap W)^{\perp} \cap U.\end{aligned}$$

QED

The following result relates the cosine $c(K, N)$ for K and N with the cosine $c(U, W)$ of their respective subspaces U and W.

Theorem 2.12 *Let K and N be as in Theorem 2.11. Then $c(K, N) = c(U, W)$.*

Proof Let $(x, y) \in S(K, N)$. Then $x \in K \cap (K \cap N)^{\perp}$ with $\|x\| = 1$. By part (iv) of Theorem 2.11, $x \in (U \cap W)^{\perp} \cap U$. On the other hand, $y \in N \cap (K \cap N)^{\perp}$ with $\|y\| = 1$. By part (iii) of Theorem 2.11, $y \in [(U \cap W)^{\perp} \cap U \oplus (V^{\perp} \cap T^{\perp})$ with $\|y\| = 1$. Hence, $y = w + r$, for $w \in (U \cap W)^{\perp}$ and $r \in V^{\perp} \cap T^{\perp}$, and

$$\langle x, y\rangle = \langle x, w + r\rangle = \langle x, w\rangle + \langle x, r\rangle = \langle x, w\rangle,$$

with $\|w\| \leq 1$, since $r \in T^{\perp}$ and $x \in U \subseteq T$. If $w = 0$, then $\langle x, y\rangle = 0$. If $w \neq 0$, then the corresponding unit vector w' belongs to $(U \cap W)^{\perp} \cap W$ and

$$c(U, W) \geq |\langle x, w'\rangle| \geq \frac{1}{\|w\|}|\langle x, w'\rangle| \geq |\langle x, y\rangle|, \qquad \forall (x, y) \in S(K, N).$$

Consequently, $c(K, N) \leq c(U, W)$.

Conversely, let $(u, w) \in S(U, W)$. By Theorem 2.11, it follows that $(u, w) \in S(K, N)$. Hence, $c(K, N) \geq c(U, W)$, and the proof is complete. QED

Remark 2.13 The usefulness of the previous result is illustrated by the following. Suppose T is finite dimensional. In general, K and N are infinite dimensional. Thus, Theorem 2.12 equates $c(K, N)$ with $c(U, W)$, where $c(U, W) < 1$ automatically, since U and W are finite dimensional.

To make use of the previous results, we assume the following in addition.

• Suppose that:

(i) $\{H_j\}_{j=1}^{\infty}$ is a sequence of closed subspaces of H such that $H_j \subseteq H_{j+1}$ and $\cup_{j=1}^{\infty} H_j$ is dense in H;

(ii) $\{K_j\}_{j=1}^{\infty}$ is a sequence of closed subspaces of K such that $K_j \subseteq K_{j+1}$, $K_j \subseteq H_j$ and $\cup_{j=1}^{\infty} K_j$ is dense in K;

(iii) $\{N_j\}_{j=1}^{\infty}$ is a sequence of closed subspaces of H such that $N_{j+1} \subseteq N_j$ and $\cap_{j=1}^{\infty} N_j = N$;

(iv) z is an element of H with the property that $z - z_j \in N_j$, where z_j is the projection of z in H_j, so that $\|z_j\| \leq \|z\|$, $\forall j$;

(v) F_j is the affine subspace $N_j + z_j$ of H, $\forall j$;

(vi) F is the affine subspace $N + z$ of H.

Note that consequently, the F_j are nested downward with $F = \cap_{j=1}^{\infty} F_j$. The following is our finite dimensional version of Theorem 2.7.

Theorem 2.14 *Suppose each N_j is of the form $M_j \oplus (H_j)^\perp$, for M_j a closed subspace of H_j. Suppose also that (i) each K_j is finite dimensional or finite codimensional or (ii) each M_j is finite dimensional or finite codimensional. If the $c(K_j, M_j)$ are bounded away from 1, then $P(F) = \cap_{j=1}^\infty P(F_j)$.*

Proof By (i) or (ii), $K_j + M_j$ is closed in H, $\forall j$ (Corollary 9.37 of [3]). By Theorem 1.1 applied to K_j and M_j, we have that $c(K_j, M_j) < 1$, $\forall j$. But $c(K, N_j) = c(K_j, M_j)$, $\forall j$, by Theorem 2.12. Consequently, by hypothesis, the $c(K, N_j)$ are bounded away from 1 and $P(F) = \cap_{j=1}^\infty P(F_j)$ by Theorem 2.7. QED

Remark 2.15 Although each $0 \leq c(K_j, M_j) < 1$, the upper bound 1 might be an accumulation point of the $c(K_j, M_j)$, $j = 1, 2, \ldots$.

Before leaving this section, it's worth recalling that the hypotheses of the previous results need be satisfied only for subsequences.

3 An Application

In this section, we give the motivation for our main results. Let H and G be (separable) real Hilbert spaces, with $A : H \to G$ a bounded linear operator and $Q : H \to H$ a self-adjoint, positive semi-definite, bounded linear operator. Recall that Q is positive semi-definite if $\langle x, Q(x)\rangle \geq 0$, $\forall x \in H$. Consider the following positive semi-definite, linear quadratic programming problem $(\mathcal{L})$ given by

$$\min \langle x, Q(x)\rangle$$

subject to

$$\begin{aligned} A(x) &= b, \\ x &\in H, \end{aligned}$$

where $b \in G$. Particular applications include the infinite horizon linear quadratic regulator and tracker problems in optimal control theory.

Now let K denote the kernel of Q in H, with orthogonal complement $K^\perp$. Consequently, $H = K \oplus K^\perp$. Note that

$$K = \{\eta \in H : Q(\eta) = 0\} = \{\eta \in H : \langle \eta, Q(\eta)\rangle = 0\}.$$

To see this, since Q is positive semi-definite and self-adjoint, it admits a square root operator $Q^{1/2}$ with the same properties, so that $Q = Q^{1/2}Q^{1/2}$. If $\eta \in H$ is such that $\langle \eta, Q(\eta)\rangle = 0$, then

$$0 = \langle \eta, Q^{1/2}Q^{1/2}(\eta)\rangle = \langle Q^{1/2}(\eta), Q^{1/2}(\eta)\rangle = \|Q^{1/2}(\eta)\|^2,$$

which implies that η is in the kernel of $Q^{1/2}$. However, $Q^{1/2}(\eta) = 0$ implies that $Q(\eta) = Q^{1/2}Q^{1/2}(\eta) = 0$, i.e., the kernel of $Q^{1/2}$ is contained in K. Thus, $\eta \in K$. The reverse inclusion is obvious.

Since Q is self-adjoint, it follows that K and $K^\perp$ are invariant under Q. Hence, the restriction operator $Q|K^\perp = R$ maps $K^\perp$ into itself. Note that R is a positive definite, bounded linear operator on $K^\perp$ which need not be strictly positive definite. It will be if the positive spectrum of Q is bounded away from 0 [9]. Note also that if $x = \eta \oplus \xi$ uniquely, for $x \in H$, $\eta \in K$, $\xi \in K^\perp$, then

$$\langle x, Q(x)\rangle = \langle \eta \oplus \xi, Q(\eta \oplus, \xi)\rangle = \langle \eta, Q(\eta)\rangle + \langle \xi, Q(\xi)\rangle = \langle \xi, Q(\xi)\rangle = \langle \xi, R(\xi)\rangle.$$

Assume that the feasible region F for problem $(\mathcal{L})$, which is the closed affine space

$$F = \{x \in H : A(x) = b\},$$

is non-empty, i.e., b is in the range of A. Then, $F = N + z, \quad \forall z \in F$, where N is the kernel of A in H. Under our additional assumptions, problem $(\mathcal{L})$ has the more compact form

$$\min_{x \in F} \langle x, Q(x) \rangle.$$

Let F^* denote the set of optimal solutions to $(\mathcal{L})$ (possibly empty). (It follows from [1] that F^* is affine.) In the event that $F^* \neq \emptyset$, our objective is to describe the elements of F^*, and approximate them by optimal solutions to finite dimensional truncations of $(\mathcal{L})$ - to the extent possible.

Let $P = P_{K^\perp}$ denote the orthogonal projection of H onto $K^\perp$ as in section 1. Since $F \subseteq H$, we have that the image $P(F)$ of F under P is given by

$$P(F) = \{\xi \in K^\perp : \eta \oplus \xi \in F, \text{ for some } \eta \in K\}.$$

It is non-empty and affine in $K^\perp$, since this is the case for F in H. Although F is closed in H, $P(F)$ need *not* be closed in $K^\perp$. It will be if $K + N$ is closed in H (Theorem 1.1).

Consider the problem $(P(\mathcal{L}))$ given by

$$\min_{\xi \in P(F)} \langle \xi, R(\xi) \rangle,$$

where R is positive definite and $P(F)$ is non-empty and affine. As in [9], solving $(P(\mathcal{L}))$ is equivalent to solving $(\mathcal{L})$ in the following sense. If $\xi \in P(F)$ is optimal for $(P(\mathcal{L}))$, i.e., $\xi \in P(F)^*$, then there exists $\eta \in K$ (not necessarily unique) such that $x = \eta \oplus \xi$ is in F, and is necessarily optimal for $(\mathcal{L})$ since $\langle x, Q(x) \rangle = \langle \xi, R(\xi) \rangle$. Conversely, if $x \in F$ is optimal for $(\mathcal{L})$, then $x = \eta \oplus \xi$ uniquely, for $\eta \in K$, and $\xi \in P(F)$, where ξ is necessarily optimal for $(P(\mathcal{L}))$. Consequently,

$$F^* = P^{-1}(P(F)^*) \cap F.$$

We next turn to the question of optimal solution existence for $(\mathcal{L})$. By the previous discussion, we see that this question is linked to the same question for $(P(\mathcal{L}))$. Note that even if $K + N$, i.e., $P(F)$, is closed in $K^\perp$, $(P(\mathcal{L}))$ need not admit an optimal solution - even though R is positive definite. (See [8] for a counter-example.)

- Assume that $P(F)$ is closed, i.e., $K + N$ is closed, in H. (Recall Theorem 1.1.)

- Assume R is *s*trictly positive definite, i.e., there exists $\gamma > 0$ such that $\gamma \|\xi\|^2 \leq \langle \xi, R(\xi) \rangle, \quad \forall \xi \in K^\perp$.

Hence, as is well-known in this case, $\langle \cdot, R(\cdot) \rangle$ defines a new inner product $\langle \cdot, \cdot \rangle_R$ on $K^\perp$, with associated norm $\|\cdot\|_R$ given by $\|\xi\|_R^2 = \langle \xi, R(\xi) \rangle, \quad \forall \xi \in K^\perp$. Thus, in this case, problem $(P(\mathcal{L}))$ may be reformulated as

$$\min_{\xi \in P(F)} \|\xi\|_R^2.$$

The feasible region $P(F)$ is closed, affine and non-empty. Consequently, an optimal solution to $(P(\mathcal{L}))$ is simply a best approximation in $P(F)$ to the zero element of $K^\perp$ relative to $\|\cdot\|_R$, i.e., a minimum norm element of $P(F)$ relative to $\|\cdot\|_R$. It is well-known that there exists a unique optimal solution ξ^* to $(P(\mathcal{L}))$ in $K^\perp$, so that $P(F)^* = \{\xi^*\}$ and $F^* = P^{-1}(\xi^*) \cap F \neq \emptyset$, in this case.

Next, we approximate ξ^* by optimal solutions to finite dimensional truncations to the original problem - modulo solutions of zero objective value. Let $\{H_j\}_{j=1}^\infty$ be a sequence of closed subspaces of H such that each H_j is invariant under Q, $H_{j+1} \supseteq H_j$ and $\cup_{j=1}^\infty H_j$ is dense in H (H is separable). Let Q_j denote the restriction of Q to H_j and K_j the kernel of Q_j in H_j. For notational convenience in this discussion, let L_j denote the relative complement H_j/K_j of K_j in H_j. Then $H_j = K_j \oplus L_j, \ \forall j$. Let $D_j : H \to H_j$ denote the orthogonal projection onto H_j. Note that $\lim_{j\to\infty} D_j(x) = x, \ \forall x \in H$. Define $A_j = A|H_j$. Similarly, let G_j

be a finite dimensional subspace of G such that $G_{j+1} \supseteq G_j$, $\cup_{j=1}^{\infty} G_j$ is dense in G and $A_j(H_j) \subseteq G_j$, $\forall j$. Let $E_j : G \to G_j$ denote the orthogonal projection onto G_j and $b_j = E_j(b)$. Then $E_j \circ A = A_j \circ D_j$. It is not difficult to see that $\lim_{j\to\infty} b_j = b$ in G.

Define

$$\Phi_j = \{x \in H_j : A(x) = b_j\},$$

which is non-empty affine and closed since H_j is finite dimensional. Note that

$$F = \{x \in H : D_j(x) \in \Phi_j, \quad \forall j\}.$$

Consider the corresponding programming problem (Λ_j) given by

$$\min_{x \in \Phi_j} \langle x, Q(x)\rangle.$$

We may consider the positive definite version $(P(\Lambda_j))$ of (Λ_j) given by

$$\min_{\xi \in S_j(\Phi_j)} \langle \xi, R(\xi)\rangle,$$

where $S_j : H_j \to L_j$ is the orthogonal projection. As above, the space $S_j(\Phi_j)$ is not only affine, it is also *closed* in L_j (finite-dimensional). Since $Q|L_j$ is automatically strictly positive definite, there exists a *unique* optimal solution ξ_j to $(P(\Lambda_j))$ in $S_j(\Phi_j)$, i.e., $S_j(\Phi_j)^* = \{\xi_j\}$. As was the case for $(\mathcal{L})$ and $(P(\mathcal{L}))$, solving (Λ_j) is equivalent to solving $(P(\Lambda_j))$. In fact, since $(P(\Lambda_j))$ has a unique optimal solution, the (non-empty) optimal solution set for (Λ_j) is given by

$$(\Phi_j)^* = \Phi_j \cap S_j^{-1}(S_j(\Phi_j)^*) = \Phi_j \cap S_j^{-1}(\xi_j), \quad \forall j.$$

Next, for each j, consider the following extension $(\mathcal{L}_j)$ of (Λ_j) to a problem in H which approximates $(\mathcal{L})$. Let $(\mathcal{L}_j)$ be the problem given by

$$\min \langle D_j(x), Q(D_j(x))\rangle$$

subject to

$$A(D_j(x)) = b_j,$$
$$x \in H.$$

Note that $(\mathcal{L}_j)$ is essentially finite-dimensional since the objective and constraint functions depend only on H_j, and the feasible region consists of those *square-summable extensions* of the elements of H_j which satisfy the constraint, i.e., the square-summable extensions of the elements of Φ_j. Let

$$F_j = \{x \in H : A(D_j(x)) = b_j\}$$

denote the feasible region for $(\mathcal{L}_j)$, M_j the kernel of A_j in H_j, where $A_j : H_j \to G_j$, and $N_j = M_j \oplus H_j^{\perp}$, so that N_j is the kernel of $A_j \oplus O_j$, where $O_j : H_j^{\perp} \to G_j^{\perp}$ is the zero operator. Then

$$\Phi_j = M_j + z_j, \quad \forall z_j \in \Phi_j,$$

i.e., M_j is the subspace of H_j corresponding to Φ_j, and

$$F_j = \Phi_j \oplus H_j^{\perp} = M_j \oplus H_j^{\perp} + z = N_j + z, \quad \forall z \in F_j,$$

with corresponding subspace of H equal to N_j. Moreover, $N_{j+1} \subseteq N_j$ and $F_{j+1} \subseteq F_j$, for all j. It then follows that $\{F_j\}$ is a sequence of closed affine subspaces of H, $\{N_j\}$ is a sequence of closed subspaces of H, $N = \cap_{j=1}^{\infty} N_j$ and $F = \cap_{j=1}^{\infty} F_j$.

Next, for each j, consider the positive definite version $(P(\mathcal{L}_j))$ of $(\mathcal{L}_j)$, namely

$$\min_{\xi \in P(F_j)} \|\xi\|_R^2 = \min_{\xi \in P(F_j)} \langle \xi, R(\xi)\rangle$$

where, for $K^\perp/L_j$ the relative orthogonal complement of H_j/K_j in $K^\perp$, the set

$$P(F_j) = S_j(\Phi_j) \oplus K^\perp/L_j$$

is closed and affine. Let $T_j : K^\perp \to L_j$ be the orthogonal projection.Then

$$P(F_j)^* = (T_j)^{-1}(\xi_j) \cap P(F_j).$$

Also, $P(F_{j+1}) \subseteq P(F_j)$, and

$$\xi^* \in P(F) \subseteq \cap_{j=1}^\infty P(F_j).$$

As above, solving $(\mathcal{L}_j)$ is equivalent to solving $(P(\mathcal{L}_j))$, i.e.,

$$F_j^* = P^{-1}(\xi_j) \cap F_j.$$

As we shall see, it is unfortunate we cannot conclude that, in general,

$$P(F) = \cap_{j=1}^\infty P(F_j).$$

Observe that, for each j, ξ_j is the unique optimal solution for $(P(\mathcal{L}_j))$, i.e., ξ_j is the unique minimum norm element of $P(F_j)$, since $\|\xi_j\|_R \leq \|\zeta\|_R, \quad \forall \zeta \in P(F_j)$. The set $\cap_{j=1}^\infty P(F_j)$ is closed and affine. Thus, the problem

$$\min_{\xi \in \cap_{j=1}^\infty P(F_j)} \langle \xi, R(\xi) \rangle \quad = \quad \min_{\xi \in \cap_{j=1}^\infty P(F_j)} \|\xi\|_R^2,$$

admits a unique solution $\xi^\dagger$, which is the minimum norm element of $\cap_{j=1}^\infty P(F_j)$ relative to the norm $\| \centerdot \|_R$. It follows from Semple [11] that $\xi_j \to \xi^\dagger$, as $j \to \infty$. We would like it to be the case that $\xi_j \to \xi^*$, as well. Thus, ξ^* and $\xi^\dagger$ are both minimum norm elements relative to $\|\centerdot\|_R$ from $P(F) = P(\cap_{j=1}^\infty F_j)$ and $\cap_{j=1}^\infty P(F_j)$, respectively, where $P(F) \subseteq \cap_{j=1}^\infty P(F_j)$, so that $\|\xi^\dagger\|_R \leq \|\xi^*\|_R$, in general. Since ξ^* and $\xi^\dagger$ are possibly unequal, we next consider the question of when $\xi^* = \xi^\dagger$. Recall the pertinent results in section 2 for sufficient conditions under which $\xi^* = \xi^\dagger$ in general.

Lemma 3.1 *For each j, $K + N_j = (K_j + M_j) \oplus H_j^\perp$. Thus, $K + N_j$ is a closed subspace, i.e., $P_{K^\perp}(F_j) = P(F_j)$ is a closed, affine space. Moreover, $K + N_j$ is weakly closed and $P_{K^\perp}(F_j)$ is weakly closed, $\forall j$.*

Proof We have $K + N_j = (K_j \oplus K/K_j) + (M_j \oplus H_j^\perp)$. We leave it to the interested reader to verify that

$$(K_j \oplus K/K_j) + (M_j \oplus H_j^\perp) \quad = \quad (K_j + M_j) \oplus (K/K_j + H_j^\perp).$$

Hence, $K + N_j = (K_j + M_j) \oplus H_j^\perp$, so that $K + N_j$ is closed, since $K_j + M_j$ is closed, $\forall j$ (both are finite dimensional). Now apply Theorem 1.1. QED

Theorem 3.2 *Suppose (i) $K + \cap_{j=1}^\infty N_j = \cap_{j=1}^\infty (K + N_j)$, or (ii) the $c(K_j, M_j)$ are eventually bounded away from 1. Then $P_{K^\perp}(F) = \cap_{j=1}^\infty P_{K^\perp}(F_j)$, so that $K + N$ is closed, $\xi^* = \xi^\dagger$, $\lim_{j\to\infty} \xi_j = \xi^*$ and*

$$F^* \;=\; P_{K^\perp}^{-1}(\xi^*) \cap F \;=\; P_{K^\perp}^{-1}(\xi^\dagger) \cap F.$$

Proof Apply the results of section 2, particularly Theorems 2.4 and 2.7. QED

Remark 3.3 Of course, it need not be that $P_{K^\perp}(F) = \cap_{j=1}^\infty P_{K^\perp}(F_j)$ in order for $\xi^\dagger = \xi^*$, or for

$$F^* = P_{K^\perp}^{-1}(\xi^*) \cap F = P_{K^\perp}^{-1}(\xi^\dagger) \cap F.$$

The following corollary gives a sufficient condition for the hypotheses of Theorem 3.2 to hold in terms of the problem data and the finite dimensional subspaces.

Corollary 3.4 *If there exists a subsequence of the $c(K_j, M_j)$ consisting of finitely many distinct values, then the hypotheses of Theorem 3.2 hold.*

Under the previous assumptions, there exists w in F^* such that $P(w) = \xi^*$. Also, under the hypotheses of Theorem 3.2, $\xi_j \to \xi^* = \xi^\dagger$. But the optimal solutions F_j^* to $(\mathcal{L}_j)$ satisfy $F_j^* = P^{-1}(\xi_j) \cap F_j$, a non-empty subset of F_j. Thus, for each j, and for each $w^j \in F_j^*$, we have $P(w_j) = \xi_j$. If we could choose the w_j so that they converge to w, then we would be able to approximate an optimal solution to $(\mathcal{L})$ by optimal solutions of the $(\mathcal{L}_j)$, which are "finite dimensional" truncations of $(\mathcal{L}_j)$. In order to do this, the sequence $(w_j)_{j=1}^\infty$ has to be a *convergent selection* from the sets $F_j^* = F_j \cap P^{-1}(\xi_j)$ which converges to an element of $F^* = F \cap P^{-1}(\xi^*)$. (See [6, 7] for construction of such selections.) This will be the subject of future research.

Acknowledgement The authors are deeply indebted to the Reviewer for his/her many helpful suggestions and corrections.

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