ON THE CLOSURE OF THE SUM OF CLOSED SUBSPACES

Irwin E. Schochetman
Mathematics and Statistics
Oakland University
Rochester, Michigan 48309
schochet@oakland.edu

Robert L. Smith
Industrial and Operations Engineering
The University of Michigan
Ann Arbor, Michigan 48109
rlsmith@umich.edu

Sze-Kai Tsui
Mathematics and Statistics
Oakland University
Rochester, Michigan 48309
tsui@oakland.edu

December 6, 1999

Abstract. We give necessary and sufficient conditions for the sum of closed subspaces of a Hilbert space to be closed. Specifically, we show that the sum will be closed if and only if the angle between the subspaces is not zero, or if and only if the projection of either space into the orthogonal complement of the other is closed. We also give sufficient conditions for the sum to be closed in terms of the relevant orthogonal projections. As a consequence, we obtain sufficient conditions for the existence of an optimal solution to an abstract quadratic programming problem in terms of the kernels of the cost and constraint operators.

1991 *Mathematics Subject Classification.* Primary 46C05, Secondary 90C20.

Key words and phrases. closure of sum of closed subspaces, angle between closed subspaces, quadratic program solution existence.

The second author was supported in part by the National Science Foundation under Grant DMI–9713723

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction

In this research/expository article, we suppose H is an arbitrary Hilbert space (real or complex) with K and N closed subspaces of H. We consider the question of when the (ordinary) sum $K+N$ is closed in H. It is obviously true if either $K \subseteq N$ or $N \subseteq K$. It is also true if either K or N is finite dimensional. If K or N is of codimension 1 (i.e., one of the subspaces is a hyperplane through the origin), then $K+N$ is clearly closed. However, it is not closed in general (Example 2.2).

In what follows, we will give necessary and sufficient conditions for $K+N$ to be closed. Although we have found no such conditions in the published literature, conditions were evidently provided without proof in unpublished course notes of Carl Pearcy [3]. In this paper, we state and prove a collection of similar necessary and sufficient conditions in Theorem 2.1 Our first equivalent condition will involve orthogonal complements and projections. Let $L = K^{\perp}$ denote the orthogonal complement of K in H and $M = N^{\perp}$ the orthogonal complement of N in H. Also, let $E_L : H \to L$ the corresponding orthogonal projection onto L and $E_M : H \to M$ the corresponding orthogonal projection onto M. In section 2, we shall see that $E_L(N)$ is closed if and only if $E_M(K)$ is closed, and more importantly for our purposes, that $K+N$ is closed if and only if each of these subspaces is closed (Theorem 2.1(ii)). Thus, the closure of $N+K$ is equivalent to the closure of the orthogonal projection of N into L (resp., K into M). Our next equivalent condition involves the (cosine of the) angle $\theta(K,N)$ between K and N. Its definition is given in section 2, where we shall also see that $K+N$ is closed if and only if $\theta(K,N) < 1$, i.e., the angle between K and N is not equal to 0 (Theorem 2.1(iii)). Finally in section 2, we give some sufficient conditions for Theorem 2.1 to hold in terms of the orthogonal projections $E_K : H \to K$ and $E_N : H \to N$ (Theorem 2.2).

In section 3, we give an application of our main results. This involves an abstract positive semi-definite quadratic programming problem given by minimizing the quadratic objective function $\langle x, Qx\rangle$ subject to the linear equality constraint $Ax = b$, for $x \in H$, where Q and A are bounded linear operators, and Q is also self-adjoint and positive semi-definite. For such a problem, we let K denote the kernel of Q, N the kernel of A and F the feasible region, where $F = N + x$, for any $x \in F$. Then it is known that this problem will admit an optimal solution if the (positive-definite) restriction of Q to $L = K^{\perp}$ is *strictly* positive definite (i.e., coercive) and the projection $E_L(F)$ of F into L is closed [5]. The restriction $Q|L$ is well-known to be strictly positive definite if and only if its (positive) spectrum is bounded away from 0. That leaves the question of when the projection $E_L(F)$ is closed. Our main results give equivalent conditions for this to happen. In particular, it happens precisely when $K+N$ is closed, or if $\theta(K,N) < 1$.

2. Main Results

We begin this section by defining the quantity $\theta(K,N)$, which is essentially the angle between the subspaces K and N. Let

$$S(K,N) = \{(x,y) : x \in K \cap (K\cap N)^{\perp}, \quad y \in N \cap (K \cap N)^{\perp}, \quad \|x\| = \|y\| = 1\}.$$

Note that $S(K,N) = S(N,K)$ and $S(K,N) = \emptyset$ if and only if either $N \subseteq K$ or $K \subseteq N$. Let

$$\theta(K,N) = \begin{cases} \sup\{|\langle x,y\rangle| : (x,y) \in S(K,N)\}, & \text{for } S(K,N) \neq \emptyset, \\ -\infty, & \text{for } S(K,N) = \emptyset, \end{cases}$$

so that $0 \le \theta(K,N) = \theta(N,K) \le 1$. In particular, if $N \cap K = \{0\}$, then $(K\cap N)^{\perp} = H$, $N = N \cap (K\cap N)^{\perp}$, $K = K \cap (K \cap N)^{\perp}$ and

$$\theta(K,N) = \sup\{|\langle x,y\rangle| : x \in N, \quad y \in K, \quad \|x\| = \|y\| = 1\}.$$

If $L = K^{\perp}$ and $M = N^{\perp}$ are hyperplanes through the origin, then $\theta(K,N)$ is the (cosine of the) conventional angle between the one-dimensional subspaces K and N.

Theorem 2.1. *The following are equivalent:*
(i) $E_L(N)$ *is closed in* L.
(ii) $K+N$ *is closed in* H.
(iii) $\theta(K,N)<1$.
(iv) $E_M(K)$ *is closed in* M.

Proof. Given the expository nature of this paper, we find it instructive to show all the implications involved.
(i) $\iff$ *(ii)* : Suppose $E_L(N)$ is closed. Let $x^k=\nu^k+\eta^k$, where $\nu^k\in N$ and $\eta^k\in K, \quad \forall k=1,2,\dots,$ and $x^k\to x\in H$. To show that $x\in N+K$. Since $L=K^\perp$, we have that

$$E_L(x^k)\to E_L(x)\in L,$$

where

$$E_L(x^k)=E_L(\nu^k+\eta^k)=E_L(\nu^k)+E_L(\eta^k)=E_L(\nu^k)\in E_L(N), \quad \forall k=1,2,\dots .$$

By hypothesis, $E_L(x)\in E_L(N)$. Thus, there exists $y\in N$ such that $E_L(x)=E_L(y)$, i.e., $E_L(x-y)=0$, so that $\eta=x-y\in K=L^\perp$. Therefore, $x=y+\eta$, i.e., $x\in N+K$.

Now suppose $N+K$ is closed in H. Let $\{\xi^k\}$ be a sequence in $E_L(N)$ which converges to $\xi\in L$. To show that $\xi\in E_L(N)$. For each k, there exists $y^k\in N$ such that $E_L(y^k)=\xi^k$. Let $\eta^k=\xi^k-y^k, \quad \forall k$. Then

$$E_L(\eta^k)=E_L(\xi^k)-E_L(y^k)=E_L(E_L(y^k))-E_L(y^k)=E_L(y^k)-E_L(y^k)=0,$$

so that $\eta^k\in K, \quad \forall k=1,2,\dots$. Since $\xi^k=y^k+\eta^k\in N+K, \quad \forall K$, and $\xi^k\to\xi$, it follows, by hypothesis, that $\xi\in N+K$. Hence, there exist $\nu\in N$ and $\eta\in K$ for which $\xi=\nu+\eta$. Therefore,

$$E_L(E_L(y^k))=E_L(\xi^k)\to E_L(\xi)=E_L(\nu)+E_L(\eta)=E_L(\nu),$$

and

$$E_L(E_L(y^k))=E_L(y^k)=\xi^k\to\xi,$$

so that $\xi=E_L(\nu)$, for $\nu\in N$, i.e., $\xi\in E_L(N)$.

(ii) $\iff$ *(iii)* : First observe that if $N\subseteq K$ or $K\subseteq N$, then $N+K$ is closed in H and $\theta(N,K)=-\infty$. Thus, we may assume that $N\not\subseteq K$ and $K\not\subseteq N$.

Next we show that, without loss of generality, we may also assume that $N\cap K=\{0\}$. If not, then we may write

$$N=N_1\oplus(N\cap K), \quad K=K_1\oplus(N\cap K),$$

where N_1 (resp. K_1) is the orthogonal complement of $N\cap K$ in N (resp. K). Then

$$\begin{aligned}N+K&=(N_1\oplus(N\cap K))+(K_1\oplus(N\cap K))\\&=(N_1+K_1)\oplus((N\cap K)+(N\cap K))\\&=(N_1+K_1)\oplus(N\cap K).\end{aligned}$$

Thus, $N+K$ is closed if and only if N_1+K_1 is closed, where $N_1\cap K_1=\{0\}$.

Suppose $N+K$ is closed with $N\cap K=\{0\}$. Consider the canonical Hilbert space mapping

$$f:N\oplus K\to N+K,$$

given by

$$f(\nu,\eta) = \nu + \eta, \quad \forall \nu \in N, \quad \forall \eta \in K.$$

This mapping is clearly linear and onto. It is also one-to-one since $N \cap K = \{0\}$. We next show that it is also bounded.

Let $\nu \in N$ and $\eta \in K$. then

$$\begin{aligned}
\|f(\nu,\eta)\|^2 &= \|\nu + \eta\|^2 \\
&= \langle \nu + \eta, \nu + \eta \rangle \\
&= \langle \nu, \nu \rangle + \langle \nu, \eta \rangle + \langle \eta, \nu \rangle + \langle \eta, \eta \rangle \\
&= \langle \nu, \nu \rangle + 2\langle \nu, \eta \rangle + \langle \eta, \eta \rangle \\
&= \|\nu\|^2 + 2\langle \nu, \eta \rangle + \|\eta\|^2 \\
&\leq \|\nu\|^2 + 2|\langle \nu, \eta \rangle| + \|\eta\|^2 \\
&\leq \|\nu\|^2 + 2\|\nu\|\|\eta\| + \|\eta\|^2.
\end{aligned}$$

Moreover,

$$2\|\nu\|\|\eta\| \leq \|\nu\|^2 + \|\eta\|^2.$$

Hence,

$$\|f(\nu,\eta)\|^2 \leq 2(\|\nu\|^2 + \|\eta\|^2) = 2\|(\nu,\eta)\|^2,$$

so that $\|f\| \leq \sqrt{2}$, i.e., f is bounded.

By the Open Mapping Theorem [2, p.57], f has a bounded linear inverse, i.e., there exists $c > 0$ such that

$$c\|(\nu,\eta)\| \leq \|f(\nu,\eta)\| = \|\nu + \eta\| \leq \sqrt{2}\|(\nu,\eta)\|.$$

Consequently,

$$c^2(\|\nu\|^2 + \|\eta\|^2) = c^2\|\nu,\eta)\|^2 \leq \|\nu + \eta\|^2 = \|\nu\|^2 + 2\langle \nu, \eta \rangle + \|\eta\|^2.$$

Hence,

$$c^2(\|\nu\|^2 + \|\eta\|^2) \leq \|\nu\|^2 + 2\langle \nu, \eta \rangle + \|\eta\|^2,$$

i.e.,

$$(c^2 - 1)(\|\nu\|^2 + \|\eta\|^2) \leq 2\langle \nu, \eta \rangle,$$

for all such ν, η.

Now assume in addition that $\|\nu\| = \|\eta\| = 1$. Then

$$2(c^2 - 1) = (c^2 - 1)(\|\nu\|^2 + \|\eta\|^2) \leq 2\langle \nu, \eta \rangle,$$

i.e.,

$$1 - c^2 \geqslant \langle -\nu, \eta \rangle,$$

where $-\nu$ is an arbitrary element of N of norm 1, since ν is such. If $\langle \nu, \eta \rangle \leq 0$, then $\langle -\nu, \eta \rangle = |\langle \nu, \eta \rangle|$, so that $1 - c^2 \geqslant \langle \nu, \eta \rangle$, and $0 < c \leq 1$. If $\langle \nu, \eta \rangle \geqslant 0$, then $\langle \nu, \eta \rangle = |\langle \nu, \eta \rangle|$, so that $1 - c^2 \geqslant -|\langle \nu, \eta \rangle|$. Letting $\nu' = -\nu$, we get

$$1 - c^2 \geqslant -\langle \nu', \eta \rangle = \langle \nu, \eta \rangle = |\langle \nu, \eta \rangle|.$$

Hence, in either case,

$$|\langle \nu, \eta \rangle| \leq 1 - c^2 < 1,$$

for all $\nu \in N, \quad \eta \in K$ with $\|\nu\| = \|\eta\| = 1$. Thus, $\theta(N, K) < 1$.

Now suppose $0 \leq \theta = \theta(N, K) < 1$. Observe that N, K are weakly closed in H, as well as closed. Also, from the definition of θ we have that

$$|\langle \nu, \eta \rangle| \leq \theta \|\nu\| \|\eta\|,$$

so that

$$-2\theta\|\nu\|\|\eta\| \leq 2\langle \nu, \eta \rangle \leq 2\theta\|\nu\|\|\eta\|, \quad \forall \nu \in N, \quad \forall \eta \in K.$$

Suppose $\{\nu^k\}$ is a sequence in N and $\{\eta^k\}$ is a sequence in K such that $x^k = \nu^k + \eta^k \to x$ in H. To show that $x \in N + K$. First we show that

$$\sup_k \|\nu^k\| < \infty$$

and

$$\sup_k \|\eta^k\| < \infty.$$

We have:

$$\begin{aligned}
\|x^k\|^2 &= \|\nu^k + \eta^k\|^2 \\
&= \|\nu^k\| + \|\eta^k\|^2 + 2\langle \nu^k, \eta^k \rangle \\
&\geqslant \|\nu^k\|^2 + \|\eta^k\|^2 - 2\theta\|\nu^k\|\|\eta^k\| \\
&= (\theta\|\nu^k\| - \|\eta^k\|)^2 + (1 - \theta^2)\|\nu^k\|^2,
\end{aligned}$$

i.e.,

$$\|\nu^k + \eta^k\|^2 \geqslant (\theta\|\nu^k\| - \|\eta^k\|)^2 + (1 - \theta^2)\|\nu^k\|^2.$$

Interchanging ν^k and η^k, we also obtain

$$\|\nu^k + \eta^k\|^2 \geqslant (\theta\|\eta^k\| - \|\nu^k\|)^2 + (1 - \theta^2)\|\eta^k\|^2,$$

where $1 - \theta^2 > 0$ by hypothesis. We see from these inequalities that if either $\{\nu^k\}$ or $\{\eta^k\}$ is unbounded, we obtain a contradiction, since the convergent sequence $\{\nu^k + \eta^k\}$ must be bounded.

By the Banach-Alaoglu Theorem [2], passing to subsequences if necessary, we may assume that there exists $\nu \in N$ and $\eta \in K$ such that $\nu^k \rightharpoonup \nu$ and $\eta^k \rightharpoonup \eta$ (weak convergence), as $k \to \infty$. Thus, $\nu^k + \eta^k \rightharpoonup \nu + \eta$ and $\nu^k + \eta^k \to x$, as $k \to \infty$. Necessarily, $x = \nu + \eta \in N + K$, since the weak topology on H separates points.

$(iii) \iff (iv)$: Interchange K and N in (i), (ii) and (iii). □

Example 2.2. For each $j = 1, 2, \ldots$, let $\psi_j = \arcsin(1/j)$ so that $\cos\psi_j = \sqrt{j^2 - 1}/j$. Define

$$H = \{[y_i \ u_i]_{i=1}^{\infty} : y_i, \ u_i \in \mathbb{R}, \ \sum_{i=1}^{\infty}(y_i^2 + u_i^2) < \infty\},$$

$$K = \{[y_i \ 0]_{i=1}^{\infty} : y_i \in \mathbb{R}, \ \sum_{i=1}^{\infty} y_i^2 < \infty\},$$

and

$$N = \{[y_i\ u_i]_{i=1}^{\infty} \in H : y_i = u_i \cot\psi_i, \quad \forall i\}.$$

Clearly, K and N are closed subspaces of the real Hilbert space H with $K \cap N = \{0\}$ and

$$L = K^{\perp} = \{[0\ u_i]_{i=1}^{\infty} : u_i \in \mathbb{R}, \quad \sum_{i=1}^{\infty} u_i^2 < \infty\}.$$

Now, for each $j = 1, 2, \ldots$, define $x^j = (x_i^j)_{i=1}^{\infty}$ by

$$x_i^j = \begin{cases} [0\ 0], & \text{for } i \neq j, \\ [1\ 0], & \text{for } i = j, \end{cases}$$

so that $x^j \in K$ and $\|x^j\| = 1$. Similarly, define $\nu^j = (\nu_i^j)_{i=1}^{\infty}$ by

$$\nu_i^j = \begin{cases} [0\ 0], & \text{for } i \neq j, \\ [\cos\psi_j\ \ \sin\psi_j], & \text{for } i = j, \end{cases}$$

so that $\nu^j \in N$ and $\|\nu^j\| = 1$. Note that

$$\nu_j^j = [\cos\psi_j\ \ \sin\psi_j] = \sin\psi_j [\cot\psi_j\ \ 1], \quad \forall j.$$

We then have

$$\begin{aligned} |\langle x^j, \nu^j \rangle| &= |\sum_{i=1}^{\infty} \langle x_i^j, \nu_i^j \rangle| \\ &= |\langle [1\ 0], [\cos\psi_j\ \ \sin\psi_j] \rangle| \\ &= \cos\psi_j. \quad \forall j = 1, 2, \ldots . \end{aligned}$$

Consequently, we see that

$$\theta(N, K) \geqslant \sup_j \cos\psi_j = 1,$$

i.e., $\theta(N, K) = 1$, so that $E_L(N)$ is not closed by Theorem 2.1.

We next show that $E_L(N)$ is not closed. Suppose it is. Observe that $E_L(N)$ is the set of $([0\ u_j])_{j=1}^{\infty} \in L$ for which there exists $y_j \in \mathbb{R}$ such that $y_j = u_j \cot\psi_j$, $\forall j$, and (y_j) is square summable. For each $j = 1, 2, \ldots$, let $u_j = 1/j$, so that $u = ([0\ u_j]) \in L$. Set $y_j = u_j \cot\psi_j$, and define

$$x_i^j = \begin{cases} [y_i\ u_i], & \text{for } i \leq j \\ [0\ 0], & \text{for } i > j, \end{cases}$$

so that $x^j \in N$, and $u^j = E_L(x^j) \in E_L(N)$, where

$$u_i^j = \begin{cases} [0\ u_i], & \text{for } i \leq j \\ [0\ 0], & \text{for } i > j, \end{cases}$$

for all $j = 1, 2, \ldots$. Clearly, $\{u^j\}$ is a Cauchy sequence in $E_L(N)$. Since $E_L(N)$ is closed, then there must exist $u \in E_L(N)$ such that $u^j \to u$, as $j \to \infty$. Therefore, there must exist $y = (y_j)$ such that $y_j = u_j \cot \psi_j, \quad \forall j$, and $\sum y_j^2 < \infty$, i.e., $([y_j \ u_j]) \in H$. But

$$\begin{aligned}
\|([y_j \ u_j])\|^2 &= \sum_{j=1}^{\infty} (y_j^2 + u_j^2) \\
&= \sum_{j=1}^{\infty} (\cot^2 \psi_j + 1) u_j^2 \\
&= \sum_{j=1}^{\infty} \frac{u_j^2}{\sin^2 \psi_j} \\
&= \infty.
\end{aligned}$$

Contradiction. Hence, $E_L(N)$ is not closed in this example.

We next show that $N + K$ is not closed. Let $[y_i, u_i]$ be an arbitrary element of N, so that $\sum_{i=1}^{\infty} u_i^2 \leq \infty$, in particular. Once again let

$$x_i^j = \begin{cases} [y_i \ u_i], & \text{for } i \leq j \\ [0 \ 0], & \text{for } i > j, \end{cases}$$

so that $x^j \in N$, and

$$z^j = \begin{cases} [y_i \ 0], & \text{for } i \leq j \\ [0 \ 0], & \text{for } i > j, \end{cases}$$

so that $-z^j \in K$, $\forall j$. Consequently, $u^j \in N + K, \quad \forall j$, where

$$u^j = \begin{cases} [0 \ u_i], & \text{for } i \leq j \\ [0 \ 0], & \text{for } i > j. \end{cases}$$

It follows that the sequence $\{u^j\}$ is Cauchy since $\sum_{j=1}^{\infty} u_i^2 < \infty$. By hypothesis, there exists $u \in H$ such that $u^j \to u$ as $j \to \infty$. Necessarily, $u_i^j \to u_i, \quad \forall i$. Hence,

$$u = ([0, u_1], [0, u_2], \ldots).$$

If $u \in K + N$, then there exists $z = ([z_1, 0], [z_2, 0], \ldots) \in K$ and $x \in N$ such that $z + x = u$. Thus,

$$x = u - z = -([z_1, u_1], [z_2, u_2], \ldots).$$

Since this belongs to N, we must have

$$z_i = u_i \cot \psi_i, \quad \forall i.$$

Consequently,

$$\|x\|^2 = \sum_{j=1}^{\infty} (\cot \psi_i^2 + 1) u_i = \infty,$$

which is a contradiction. Therefore, $K + N$ is not closed.

Theorem 2.3. *The following are sufficient for $K + N$ to be closed:*
(i) $E_N E_K = 0$ *or* $E_K E_N = 0$.
(ii) $E_N E_K - E_K E_N = 0$.
(iiia) $\sup\{\|E_N \eta\| : \eta \in K \cap (N \cap K)^\perp, \quad \|\eta\| = 1\} < 1$.
(iiib) $\sup\{\|E_K \nu\| : \nu \in N \cap (N \cap K)^\perp, \quad \|\nu\| = 1\} < 1$.

Proof. (i) If $E_N E_K = 0$, then

$$E_K E_N = (E_N E_K)^* = E_K^* E_N^* = 0,$$

so that $L + M = L \oplus M$, and is therefore closed in H. Similarly for $E_K E_N = 0$.

(ii) We have

$$E_N E_K = E_K E_N = E_{N \cap K},$$

(where, in general, E_X denotes the orthogonal projection of H onto the closed subspace X) and

$$N + K = (N \cap K) \oplus (N \cap (N \cap K)^\perp + K \cap (N \cap K)^\perp).$$

We also have

$$E_N(I - E_N E_K) = E_{N \cap (N \cap K)^\perp},$$

where I is the identity operator on H and

$$I - E_N E_K = E_{(N \cap K)^\perp}.$$

Thus, by hypothesis,

$$E_N(I - E_N E_K) = E_N - E_N^2 E_K = E_N - E_N E_K E_N = (I - E_N E_K) E_N.$$

Similarly,

$$E_K(I - E_N E_K) = E_{K \cap (N \cap K)^\perp}.$$

But

$$E_N(I - E_N E_K)(I - E_N E_K) E_K = 2(E_N E_K - E_K E_N) = 0,$$

by hypothesis. (Note that part (i) is valid for any closed subspace N and K and their corresponding projections E_N and E_K.) Hence, by (i) applied to E_N and $E_{(N \cap K)^\perp}$, we obtain that

$$(N \cap (N \cap K)^\perp) + (K \cap (N \cap K^\perp)$$

is closed in H. Consequently, $N + K$ is closed in H.

(iiia) For convenience, let

$$\beta = \sup\{\|E_N \eta\| : \eta \in K \cap (N \cap K)^\perp, \quad \|\eta\| = 1\}.$$

(We may exclude the case where the defining set is empty. This happens only if $K \subseteq N$, in which case $K + N$ is trivially closed.) Then $\beta < 1$ and $\beta^2 < 1$, i.e., $\delta = 1 - \beta^2 > 0$. Fix

$$\eta \in K \cap (N \cap K)^\perp, \quad \nu \in N \cap (N \cap K)^\perp,$$

such that $\|\eta\| = \|\nu\| = 1$. Then

$$|\langle \eta, \nu \rangle| = \|\eta\| \|\nu\| \cos \psi,$$

where ψ is the angle between η and ν, if this angle is at most $\pi/2$, or the supplement of this angle, if it is greater than $\pi/2$. (Alternately, we can replace η by $-\eta$ where necessary.) Thus, $0 \leq \psi \leq \pi/2$. By the Law of Cosines, we have

$$\|\eta + \nu\|^2 = \|\eta\| + \|\nu\|^2 - 2\|\eta\| \|\nu\| \cos \psi = 2 - 2 \cos \psi.$$

On the other hand, since $\nu \in N$, and $E_N \eta$ is the best approximation in N to η, we have that

$$\|\eta - \nu\|^2 \geqslant \|\eta - E_N \eta\|^2 = \|\eta\|^2 - \|E_N \eta\|^2 = 1 - \|E_N \eta\|^2,$$

by the Pythagorean Identity, where $\eta - E_N \eta$ is orthogonal to $E_N \eta$. Thus,

$$1 - \|E_N \eta\|^2 \leq 2 - 2 \cos \psi.$$

However, by definition of β, we have that

$$\|E_N \eta\|^2 \leq \beta^2,$$

so that

$$1 - \|E_N \eta\|^2 \geqslant 1 - \beta^2 = \delta > 0.$$

Consequently,

$$0 < \delta \leq 2 - 2 \cos \psi,$$

i.e.,

$$0 \leq \cos \psi \leq 1 - \delta/2 < 1,$$

so that

$$|\langle \eta, \nu \rangle| \leq 1 - \delta/2 < 1.$$

This completes the proof of (iiia), since η and ν are arbitrary.

(iiib) The proof in part (iiia) depends only on the fact that N and K are closed subspaces of H. Thus, simply interchange these spaces in the proof. $\square$

3. An Application to Quadratic Programming

We consider the general infinite quadratic programming problem $(\mathcal{G})$ given by:

$$\min \langle x, Qx \rangle$$

subject to $(\mathcal{G})$

$$Ax = b,$$

$$x \in H,$$

where H and M are real Hilbert spaces, $b \in M$, the constraint operator $A : H \to M$ is a bounded linear operator and the cost operator $Q : H \to H$ is a non-zero, (self-adjoint) *positive semi-definite*, bounded linear operator. The feasible region

$$F = \{x \in H : Ax = b\}$$

is a closed, affine subset of H (which we assume to be non-empty), and the kernel

$$N = \{x \in H : Ax = 0\}$$

of A in H is a closed subspace of H. Of course, $F = N + x$, for any $x \in F$.

Recall that Q is positive semi-definite if $\langle x, Qx\rangle \geqslant 0,\ \forall x \in H$, and that Q is *positive definite* if $\langle x, Qx\rangle > 0,\ \ \forall x \in H,\ x \neq 0$. We say that the operator Q is *strictly positive definite* if it is coercive, i.e., if there exists $\sigma_Q > 0$ satisfying

$$\sigma_Q \|x\|^2 \leq \langle x, Qx\rangle, \quad \forall x \in H.$$

This condition is known [1, p.73] to be necessary and sufficient for $(\mathcal{G})$ to admit a (unique) optimal solution for *any* (non-empty) closed, convex subset F of H. Thus, even if F is a closed, convex set in H, and Q is only positive definite, problem $(\mathcal{G})$ may not admit an optimal solution. See [4] for an example.

In this section, we establish sufficient conditions for $(\mathcal{G})$ to admit an optimal solution.

Since Q is positive semi-definite, its kernel K is given by

$$K = \{x \in H : \langle x, Qx\rangle = 0\}.$$

Moreover, since Q is self-adjoint, it follows that K and $L = K^{\perp}$ are invariant under Q. Hence, Q also decomposes into $0 \oplus P$, where 0 is the zero operator on K and $P : L \to L$ is the restriction operator $Q|L$. Note that P is a positive definite, bounded linear operator on L. It need not be strictly positive definite.

Also, since $F \subseteq H$, we have that the image of F under E_K is

$$E_K(F) = \{\eta \in K : \eta + \xi \in F,\ for\ some\ \xi \in L\}.$$

It is non-empty and convex in K, since this is the case for F in H. It is also true that F is closed in H; however, $E_K(F)$ need *not* be closed in K.

Analogously, the image of F under E_L is

$$E_L(F) = \{\xi \in L : \eta + \xi \in F,\ for\ some\ \eta \in K\}.$$

As with $E_K(F)$, the set $E_L(F)$ is non-empty and convex, but *not* necessarily closed in L. Moreover, $F \subseteq E_K(F) \oplus E_L(F)$. The same is true of $N,\ E_K(N)$ and $E_L(N)$.

We may now consider the following related problem $(\mathcal{P})$:

$$\min_{\xi \in E_L(F)} \langle \xi, P\xi\rangle \tag{$\mathcal{P}$}$$

where, as we have seen, P is positive definite on L and $E_L(F)$ is a non-empty, convex subset of L, which may not be closed. Moreover,

$$\langle \xi, P\xi\rangle = \langle x, Qx\rangle,$$

for all $\xi \in L,\ \ \eta \in K$ and $x = \eta + \xi$.

Note that solving $(\mathcal{P})$ is equivalent to solving $(\mathcal{G})$ in the following sense. If $\xi^* \in E_L(F)$ is an optimal solution to $(\mathcal{P})$, then there exists $\eta^* \in E_K(F)$ such that $x^* = \eta^* + \xi^* \in F$ and x^* is optimal for $(\mathcal{G})$. Conversely, if $x^* \in F$ is optimal for $(\mathcal{G})$, then $x^* = \eta^* + \xi^*$, for $\eta^* \in E_K(F)$ and $\xi^* \in E_L(F)$, where ξ^* is optimal for $(\mathcal{P})$.

We are interested in when the feasible region $E_L(F)$ for $(\mathcal{P})$ is closed.

Lemma 3.1. *The following are equivalent for F, K and N:*
(i) $E_L(F)$ *or* $E_L(N)$ *is closed in* L*.*
(ii) $E_M(K)$ *is closed in* M*.*
(iii) $N + K$ *is closed in* H*.*
(iv) $F + K$ *is closed in* H*.*
(v) $\theta(N, K) < 1$*.*

Proof. Apply Theorem 2.1 together with the fact that $F = N + x$, for any fixed $x \in F$. □

Theorem 3.2. *If* $Q|L$ *is strictly positive definite and any one of the conditions in Theorem 3.1 holds, then* $(\mathcal{G})$ *admits an optimal solution.*

Proof. Observe that $(\mathcal{P})$ admits an optimal solution if $P = Q|L$ is strictly positive definite and $E_L(F)$ is closed in L [1, p.73]. □

Acknowledgments. *We wish to thank Peter Duren of the Department of Mathematics at the University of Michigan at Ann Arbor and Stephan Richter of the Department of Mathematics at the University of Tennessee for bringing the information of reference [3] to our attention.*

References

1. Dontchev, A. L. and T. Zolezzi, *Well-posed Optimization Problems,* Lecture Notes in Mathematics, No.1543, Springer-Verlag, 1991.
2. Dunford, N. and J. T. Schwartz, *Linear Operators. Part I*, J. Wiley - Interscience, New York, 1964.
3. Duren, P., Private Communication, University of Michigan, Ann Arbor, Michigan, Dec 12, 1997.
4. Schochetman, I. E., R. L. Smith and S.-K. Tsui, Solution existence for time-varying infinite horizon quadratic programming, *Journal of Mathematical Analysis and Applications* **195** (1995), 135-147.
5. Schochetman, I. E., R. L. Smith and S.-K. Tsui, Solution existence for infinite quadratic programming, *Mathematical Programming with Data Perturbations,* Lecture Notes in Pure and Applied Mathematics, No.195, Marcel Dekker, New York, 1997.