Forecast Horizons for Production Planning with Stochastic Demand

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Abstract

Forecast horizons, i.e long enough planning horizons that ensure agreement of first period optimal production decisions of finite and infinite horizon problems regardless of changes in future demand, are shown to exist in the context of production planning under stochastic demand. The monotonicity of first period optimal production decisions with respect to first order stochastic shifts in demand is the key to the results. Finally, a stopping rule that is ensured to detect the minimal forecast horizon is presented.

1 Introduction.

Production planners face substantial uncertainty in the demand for their products that comes from a variety of sources. For instance, demand could be sensitive to varying economic conditions such as GNP or interest rates, alternatively, technical innovation may imply unexpected early obsolescence. Moreover, it is usually the decisions in the first few periods that are of immediate concern for the decision maker and forecasting is more difficult and costlier for problem parameters farther into the future.

Motivated by these issues, the concept of a *forecast horizon* (see for example, Bès and Sethi (1988)) has long been studied in the litterature. The idea is that problem parameters changes far enough off should not affect the optimal decisions of the first few periods.

In this paper, we provide forecast horizon's existence and computation results for production planning problems that satisfy the following monotonicity property; for any fixed finite planning horizon, there exist first period optimal solutions that are monotone with respect to first order stochastic shifts in demand. In his remarkable work, Topkis (1969) and (1978) developed a general theory of monotonicity of optimal solutions that can applied to many production planning models. These results have also been extensively exploited in the field of mathematical economics (see for example, Hopenhayn and Prescott (1992)).

Monotonicity of optimal production plans under stochastic demand has also been studied by Kleindorfer and Kunreuther (1978), Denardo (1982) and Zhang (1997).

Smith and Zhang (1997) have recently exploited the above mentioned monotonicity property to prove existence of forecast horizons in a deterministic setting.

Our emphasis on monotone optimal solutions as opposed to monotone bounds as in Morton (1978) allows us to effectively detect the *minimal* forecast horizon by means of a stopping rule with an embedded selection procedure, using results by Topkis (1978).

2 Problem Formulation.

Consider a single product firm where a decision for production must be made at the beginning of each period t, $t = 0, 1, 2, \ldots$. Customer orders arrive *during* each time interval. We will convene to call the "demand" at time period t, the total number of customer orders received during time interval $(t, t+1)$, and shall denote it by the random variable D_t.

Thus, if at the beginning of time period t, there are I_t units available on stock and a production decision of x_t is made, the inventory-production system follows the equation :

$$I_{t+1} = I_t + x_t - D_t$$

Moreover, if D_t exceeds $I_t + x_t$, the excess demand is assumed to be backlogged. The one stage cost incurred is :

$$c_t(x_t, I_t, D_t) = c_t(x_t) + h_t(\max\{0, I_{t+1}\}) + p_t(\max\{0, -I_{t+1}\})$$

where :

c_t	production cost function
h_t	holding cost function for excess inventory
p_t	backlogging cost function

We assume that demand at time period t, D_t is a non-negative random variable with compact support $[0, \bar{d}]$ and probability distribution $F_t(.)$.

Hence, the expected stage cost incurred is :

$$E[c_t(x, I, D_t)] = c_t(x) + \int_{\underline{d}}^{I+x} h_t(I + x - u)dF_t(u) + \int_{I+x}^{\bar{d}} p_t(I + x - u)dF_t(u)$$

For a planning horizon of $T + 1$ periods, the production planning problem is then :

$$(P_T) \quad \begin{array}{lll} \min & E[\sum_{t=0}^{T} \alpha^t c_t(x_t, I_t, D_t)] & \\ \text{s.t} & I_{t+1} = I_t + x_t - D_t & t = 0, 1, 2, ..., T \\ & M \geq x_t \geq 0 \text{ integer} & \end{array}$$

where M is the maximal production capacity and $\alpha \in (0, 1)$ the discount factor.

2.1 Stochastic Ordering.

We review briefly, the definition given in Lehmann[1955]. A set S in $\mathcal{R}^n$ is said to be *increasing* if $s \in S$ and $s \leq t$ imply $t \in S$. A distribution F is *stochastically larger* than a distribution G if and only if for every increasing set $I \in \mathcal{R}^n$:

$$\int 1_I(u)dF(u) \geq \int 1_I(u)dG(u)$$

where $1_I(u)$ is the indicator function of the set I.

In the case of distributions on the real line, this order is equivalent to order classically known as *stochastic dominance* :$F \succeq G \Longleftrightarrow F(x) \leq G(x)\ \forall x \in \mathcal{R}$. However, for distributions on spaces of higher dimensions, $F \succeq G \Longrightarrow F(x) \leq G(x)$, but the converse does not hold.

Let $\mathcal{A}$ be a poset, a parameterized family of distributions $\{F_a\}_{a \in \mathcal{A}}$ is *stochastically increasing in* a, if $a \succeq a'$ implies $F_a \succeq F_{a'}$. The next theorem by Topkis[1968] is of great importance :

Theorem (Topkis): A distribution family F_a is stochastically increasing in $a \in \mathcal{A}$ if and only if for every increasing real valued integrable function $v(.)$, the integral $\int v(u)dF_a(u)$ is increasing in a.

2.2 Notation for Parametric Analysis.

We assume that for any time period t, random demand D_t may be any of the random variables in the indexed collection $\{D_a\}_{a\in\mathcal{A}}$, where $\mathcal{A}$ is a *finite* poset. To this collection we associate the family of distribution functions $\{F_a\}_{a\in\mathcal{A}}$, which is assumed to be **stochastically** increasing in a. Moreover, we will denote by $\bar{a}$ and $\underline{a}$, the supremum and the infimum of the set $\mathcal{A}$, respectively, i.e :

$$\bar{a} = \sup_{a\in\mathcal{A}}\{a\} \quad \underline{a} = \inf_{a\in\mathcal{A}}\{a\}$$

Moreover, we assume that $\bar{a}$,$\underline{a} \in \mathcal{A}$. We shall refer to $\underline{a}$ as the "zero" index to which we associate the trivial distribution $F(x) = 1$, for $x \in [0, \bar{d}]$. In other words, to the "zero" index we associate zero demand. Similarly, to the index $\bar{a}$, we associate the trivial distribution :

$$F(x) = \begin{matrix} 0 & x < \bar{d} \\ 1 & x = \bar{d} \end{matrix}$$

Let us denote by $\mathcal{D}_T$ the set of T-long sequences of *independent random demands* from period 0 up to period $T-1$. Formally,

$$\mathcal{D}_T = \prod_{t=0}^{T-1}\{D_a\}_{a\in\mathcal{A}}$$

To every sequence of the form $(D_0, D_1, ..., D_{T-1}) \in \mathcal{D}_T$
we associate the production planning problem $(P_T(D_1, ..., D_{T-1}))$ defined as follows :

$$(P_T(D_0, D_1, ..., D_{T-1})) \quad \begin{matrix} \min & E[\sum_{t=0}^{T-1} \alpha^t c_t(x_t, I_t, D_t)] & \\ \text{s.t} & I_{t+1} = I_t + x_t - D_t & t = 0, 1, 2, ..., T-1 \\ & M \geq x_t \geq 0 \text{ integer} & \end{matrix}$$

We endow the cartesian product set $\mathcal{D}_T$ with the product ordering "$\succeq_T$", i.e :

$$(D_0, D_1, ..., D_{T-1}) \succeq_T (D'_0, D'_1, ..., D'_{T-1})$$
$$\Leftrightarrow$$
$$D_t \succeq D'_t \; t = 0, 1, ..., T-1$$

We shall denote by $x_0^*(D_0, D_1, ..., D_{T-1})$ the first period production decision to an optimal solution to problem $(P_T(D_0, D_1, ..., D_{T-1}))$.

2.3 Infinite Horizon Production Planning.

We now introduce the infinite horizon production planning problem as a suitable modeling refinement to problem (P_T), given the difficulties in defining what should be the "appropiate" finite planning horizon.

As above, in order to parametrize the infinite horizon production planning problem we define the infinite product $\mathcal{D} = \prod_{t=0}^{\infty}\{D_a\}_{a\in\mathcal{A}}$.

Hence, for every infinite sequence $(D_0, D_1, ..., D_{T-1}, ...) \in \mathcal{D}$, we associate the problem :

$$(P(D_0, D_1, ..., D_{T-1}, ...)) \quad \begin{matrix} \min & \lim_{T\to\infty} E[\sum_{t=0}^{T} \alpha^t c_t(x_t, I_t, D_t)] & \\ \text{s.t} & I_{t+1} = I_t + x_t - D_t & t = 0, 1, 2, ... \\ & M \geq x_t \geq 0 \text{ integer} & \end{matrix}$$

¿From this definition, it is straightforward to relate infinite and finite horizon problems via the embedding :

$$(D_0, D_1, ..., D_{T-1}) \in \mathcal{D}_T \hookrightarrow (D_0, D_1, ..., D_{T-1}, 0, 0, ...) \in \mathcal{D}.$$

In words, we extend the finite sequence by appending "zero" demand for all time periods after the planning horizon T.

2.4 Standing Assumptions.

Our first assumption, requires that if it exists, the limit of finite horizon optimal production plans is an optimal solution to the infinite horizon production planning problem.

Assumption 1: Let $(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) \in \mathcal{D}$ and $\{x_0^*(T), x_1^*(T), ..., x_{T-1}^*(T)\}_T$ be a collection of optimal solutions to problems $(P_T(D_0, D_1, ..., D_{T-1}, 0, 0, ...))$.

If there exists a limit plan, say $\{x_0^*, x_1^*, ..., x_{T-1}^*, x_T^*, ...\}$, then it is an optimal solution to the infinite horizon problem $(P(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...))$.

This assumption is fairly standard (see for instance Heyman and Sobel (1984)) and holds for example when costs functions are uniformly bounded or decision spaces are compact.

Assumption 2 :(Monotonicity of Optimal Plans) For every T, there exist an optimal production plan to problem

$(P_T(D_0, D_1, ..., D_{T-1}))$ whose first period decision is monotone in $(D_0, D_1, ..., D_{T-1})$ i.e :

$$(D_0, D_1, ..., D_{T-1}) \succeq_T (D'_0, D'_1, ..., D'_{T-1}) \Rightarrow x_0^*(D_0, D_1, ..., D_{T-1}) \geq x_0^*(D'_0, D'_1, ..., D'_{T-1})$$

Monotonicity of optimal production plans is a pervasive feature in production planning models (see for example, Kleindorfer and Kunreuther (1978)).

3 Review of Solution Concepts.

The gains in modeling accuracy afforded by an infinite horizon are severely compromised by the technical difficulties in accurately forecasting problem parameters. This consideration motivates the problem of finding a finite horizon such that the first optimal decision for such horizon coincide with the infinite horizon counterpart. If such a horizon exists (which is called a solution horizon), it not only provides a rationale to set such horizon as the decision makers planning horizon, but interestingly enough motivates a finite algorithm to solve an infinite problem via a rolling horizon procedure.

Definition 1: Planning Horizon T^* is called a **Solution Horizon** for demand forecast $(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) \in \mathcal{D}$ iff for every $T \geq T^*$, we have :

$$x_0^*(D_0, D_1, ..., D_{T-1}) = x_0^*(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...)$$

where $x_0^*(D_0, D_1, ..., D_{T-1})$ and $x_0^*(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...)$ are first period optimal decisions for the T-planning horizon production planning problem and the infinite horizon production planning problem, respectively.

However, the solution horizon concept is practically of little interest, for its computation may potentially require an infinite forecast of data. Thus, the concept of a forecast horizon (see for example, Bès and Sethi(1987)), that is, a long enough planning horizon that entails the insensitivity of first period optimal production decision with respect to changes in demand distribution at the the tail is very attractive to practitioners. In brief, in order to compute the first period optimal production decision, the planner need only forecast demand distributions for a finite number of periods and this decision is insensitive to changes in demand distribution at the tail.

Definition 2: Planning Horizon T^* is called a **Forecast Horizon** for $\mathcal{D}$ if and only if for $(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) \in \mathcal{D}$ and for every $T \geq T^*$, we have:

$$x_0^*(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) = x_0^*(D'_0, D'_1, ..., D'_{T-1}, D'_T, D'_{T+1}, ...)$$

for every $(D'_0, D'_1, ..., D'_{T-1}, D'_T, D'_{T+1}, ...) \in \mathcal{D}$ such that $D_t = D'_t$ for $0 \leq t < T$.

4 Forecast Horizon Existence and Computation.

We begin our analysis by pointing out a key observation :

Remark :

$$x_0^*(D_0, D_1, ..., D_{T-1}) = x_0^*(D_0, D_1, ..., D_{T-1}, 0, 0, ...)$$

In words, solving the production planning problem with finite horizon T is equivalent to solving the infinite horizon problem whereby we append "zero" demand after T.
Let us define the map $\mathbf{x}_0^* : \mathcal{D} \mapsto \{0, 1, 2, ...M\}$ as follows; for $(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) \in \mathcal{D}$:

$$\mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) = \lim_{T \to \infty} x_0^*(D_0, D_1, ..., D_{T-1}, 0, 0, ...)$$

Lemma 1 : The map $\mathbf{x}_0 : \mathcal{D} \mapsto \{0, 1, 2, ...M\}$ is well-defined, continuous and monotone in $\mathcal{D}$.

Proof : By the monotonicity of the optimal plan (Assumption 2) we have that :

$$x_0^*(D_0, D_1, ..., D_{T-1}, D_T) \geq x_0^*(D_0, D_1, ..., D_{T-1}, 0)$$

since this inequality is preserved by the embedding "$\hookrightarrow$", we have :

$$x_0^*(D_0, D_1, ..., D_{T-1}, D_T, 0, 0, ...) \geq x_0^*(D_0, D_1, ..., D_{T-1}, 0, 0, 0, ...)$$

thus by this monotonicity property and the fact that first period production decisions are bounded the limit exists. Moreover, the map $\mathbf{x}_0(.)$ can also be seen as the uniform limit of the functions $\mathbf{x}_0^T()$ defined as :

$$\mathbf{x}_0^T(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) = x_0^*(D_0, D_1, ..., D_{T-1}, 0, 0, ...)$$

which are trivially continuous (by finiteness of $\mathcal{D}_T$) then it follows the map $\mathbf{x}_0(.)$ inherits continuity.
As for monotonicity, let us pick $(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) \in \mathcal{D}$ and $(D'_0, D'_1, ..., D'_{T-1}, D'_T, D'_{T+1}, ...) \in \mathcal{D}$ such that :

$$D_t \succeq D'_t \quad t = 0, 1, 2, ...$$

By definition of $\mathbf{x}_0(.)$ there exists a planning horizon T_a such that for $T \geq T_a$ we have

$$\mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) = x_0^*(D_0, D_1, ..., D_{T-1})$$

Similarly, there exists a finite planning horizon T_b such that for $T \geq T_b$ we have

$$\mathbf{x}_0(D'_0, D'_1, ..., D'_{T-1}, D'_T, D'_{T+1}, ...) = x_0^*(D'_0, D'_1, ..., D'_{T-1})$$

By monotonicity of optimal plans :

$$\begin{aligned} \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) &= x_0^*(D_0, D_1, ..., D_{T-1}) \\ &\geq \\ x_0^*(D'_0, D'_1, ..., D'_{T-1}) &= \mathbf{x}_0(D'_0, D'_1, ..., D'_{T-1}, D'_T, D'_{T+1}, ...) \end{aligned}$$

for every $T \geq \max\{T_a, T_b\}$. ∎

In view of Assumption 1, the first period decision map defined above "inherits" optimality and motivates the next straightforward result :

Corollary : For every $(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) \in \mathcal{D}$, there exists a *solution horizon*.

4.1 Forecast Horizon Existence.

By exploiting the monotonicity of the map $\mathbf{x}_0(.)$, we prove the existence of a forecast horizon.

Theorem : Under assumptions 1 and 2, there exist a forecast horizon for problem $(P(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...))$.

Proof : Let us consider forecasts $(D_0, D_1, ..., D_{T-1}, \bar{d}, \bar{d}, ...)$ and $(D_0, D_1, ..., D_{T-1}, D_T, 0, 0, ...)$: Then by Lemma 1, it follows that :

$$\mathbf{x}_0(D_0, D_1, ..., D_{T-1}, \bar{d}, \bar{d}, ...) \geq \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) \geq \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, 0, 0, ...)$$

By continuity :

$$\lim_{T\to\infty} \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, \bar{d}, \bar{d}, ...) = \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...)$$

and

$$\lim_{T\to\infty} \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, 0, 0, ...) = \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...)$$

Hence, there exists a finite T^*such that for every $T \geq T^*$ we have :

$$\mathbf{x}_0(D_0, D_1, ..., D_{T-1}, \bar{d}, \bar{d}, ...) = \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...) = \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, 0, 0, ...)$$

In other words for any forecast $(D'_0, D'_1, ..., D'_{T-1}, D'_T, D'_{T+1}, ...) \in \mathcal{D}$ such that $D_t = D'_t$ for $0 \leq t < T$:

$$\mathbf{x}_0(D'_0, D'_1, ..., D'_{T-1}, D'_T, D'_{T+1}, ...) = \mathbf{x}_0(D_0, D_1, ..., D_{T-1}, D_T, D_{T+1}, ...)$$

∎

5 Stopping Rule.

The probable multiplicity of monotone optimal production plans leads to the existence of many forecast horizons. The *minimality* of the Forecast Horizon identified in the above existence proof is thus of great interest.

In his remarkable work, Topkis (1969) and (1978) developed a general monotonicity theory of optimal solutions using lattice programming techniques that not surprisingly encompasses the production planning model of Kleindorfer and Kunreuther's (1978). This theory ensures the existence of a *smallest* and a *largest* optimal solutions that are monotone which will be the basis for our selection procedure that we now introduce.

Assuming costs are uniformly bounded as follows :

$$\sup_t c_t(\cdot) \leq \bar{c}(\cdot) \quad \sup_t h_t(\cdot) \leq \bar{h}(\cdot) \quad \sup_t p_t(\cdot) \leq \bar{p}(\cdot)$$

One can construct a *pessimistic* scenario, in which demand, production and inventory holding costs are at their maximal levels, namely :

$$\begin{array}{ll} \min & \limsup_{N\to\infty} \sum_{t=0}^{N-1} \alpha^t \bar{E}[c(x, I, \bar{d})] \\ s.t & I_{t+1} = I_t + x_t - \bar{d} \\ & M \geq x_t \geq 0 \\ & x_t, I_t \text{ integer} \qquad\qquad t = 0, 1, 2, ... \end{array}$$

where :

$$\bar{E}[c_t(x, I, \bar{d})] = \langle \begin{array}{ll} \bar{c}(x) + \bar{h}(I + x - \bar{d}) & \text{if } x + I \geq \bar{d} \\ \bar{c}(x) + \bar{p}(\bar{d} - x - I) & \text{otherwise} \end{array}$$

The above problem is very easy to solve by means of the functional equation :

$$(DP) \quad V(I) = \min_{M \geq x \geq 0} \{\bar{E}[c_t(x, I, \bar{d})] + \alpha V(x + I - \bar{d})\}$$

Let us now consider the next simpler finite dimensional problem :

$$(\bar{P}_T) \quad \begin{array}{lll} \min & E[\sum_{t=0}^{T} \alpha^t c_t(x_t, I_t, D_t) + \alpha^{T+1} \cdot V(I_T)] & \\ s.t & I_{t+1} = I_t + x_t - D_t & \\ & x_t \geq 0 \text{ integer} & t = 0, 1, ..., T-1 \end{array}$$

By Topkis (1969) and (1978) there exist optimal plans to the problem $(\bar{P}_T)$ such that their first period production decisions are monotone in the demand parameters, let us pick $\bar{x}_0^T$ with the property that $\bar{x}_0^T$ is the *smallest* of such decisions.

Similarly, if we solve:

$$(\underline{P}_T) \quad \begin{array}{lll} \min & E[\sum_{t=0}^{T} \alpha^t c_t(x_t, I_t, D_t)] & \\ s.t & I_{t+1} = I_t + x_t - D_t & \\ & M \geq x_t \geq 0 \text{ integer} & t = 0, 1, ..., T-1 \end{array}$$

we know that there exist and optimal plan such that its first period action say, $\underline{x}_0^T$ is monotonically increasing in T, i.e :

$$\underline{x}_0^{T+1} \geq \underline{x}_0^T$$

and is also the *largest* of all such solutions. By the Forecast Horizon Existence Theorem, we know that these sequences must meet, in other words the algorithm we are to describe below must stop after a finite number of steps.

Step 1.	Solve Functional Equation (DP). T=1
Step 2.	Solve $(\bar{P}_T)$ and $(\underline{P}_T)$ for $\bar{x}_0^T$ and $\underline{x}_0^T$
Step 3.	If $\bar{x}_0^T = \underline{x}_0^T$ then Stop.
	Else T=T+1; Go to Step 2.

Proposition 1 Let T^* be the Forecast Horizon detected by the above procedure, T^* is also the *minimal* Forecast Horizon.

Proof: By contradiction, let us assume there exists $T < T^*$ such that T is the *minimal* Forecast Horizon. By hypothesis:

$$\bar{x}_0^T > \underline{x}_0^T$$

But since $\bar{x}_0^T$ is the first period action of the *smallest* optimal solution to problem $(\bar{P}_T)$ and $\underline{x}_0^T$ is the first period action of the *largest* optimal solution to problem $(\underline{P}_T)$, this implies that the above inequality is valid for any chosen pair of optimal solutions to the problems $(\bar{P}_T)$ and $(\underline{P}_T)$, but this contradicts T being a Forecast Horizon. ∎

6 Conclusion.

We have presented existence and computational results for forecast horizons in the context of production planning with stochastic demand. These results depend critically upon the monotonicity of first period optimal production decisions with respect to first order shifts in demand distributions, a pervasive feature of models with convex costs and backlogging. The minimality of the forecast horizon detected through the proposed stopping rule is obtained via an adequate selection procedure.

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