A Combinatorial Auction to Allocate Traffic

Hao Zhou
(Corresponding Author)
Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, MI, 48109
Phone: (734)355-1074
Email: haozhou@umich.edu

Romesh Saigal
Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, MI, 48109
Phone: (734)763-7544 Fax: (734)764-3451
Email: rsaigal@umich.edu

November 14, 2013

4615 Words + 1 Table + 6 Figures
Total Words: 6365

ABSTRACT

We propose an auction system implemented via V2I devices to toll and allocate traffic. Vehicles bid for paths before entering the network. By solving an optimization problem, the system assigns vehicles to paths and computes the corresponding toll. A mathematical model of this auction is presented and analyzed. We prove that this auction mechanism guarantees truthful reporting and maximizes the social utility. It is then tested on a network with 5100 vehicles. We also discuss the use of the auction as a toll setting mechanism for HOV or HOT lanes.

1. INTRODUCTION

Traffic congestion is a major problem in many parts of the world. For example, in the United States, the cost of increased travel times and fuel consumption alone is estimated to amount to hundreds of dollars per capita per year (1).

Many methods have been proposed for reducing congestion. A commonly studied and implemented method is that of congestion pricing or tolling. A vehicle traveling on a road increases the congestion and thus increases costs on other vehicles, leading to increased social costs. However, in the absence of tolls there is no incentive for individuals to consider the effect of their actions on the system. Tolling, on the other hand, is a price mechanism that shifts the social cost of traveling to individual vehicles, thus makes the traffic system more efficient. The idea of congestion pricing was recognized and advocated by (2), and later promoted by William Vickrey's influential work (3).

As is stated in Vickrey's work, effective congestion pricing requires that tolls be set according to the severity of congestion. This then requires that tolls be a function of the time, location, type of vehicle, etc. Many scholars have proposed a variety of dynamic congestion pricing schemes. For example, Friesz et. al. present a sophisticated method (4) for dynamic congestion pricing. Even though, unfortunately most of these are computationally intensive and difficult to implement, some of them have been successful in the real world. As an example for applications in Singapore, see (5). Overall, (6) provides a good review of these pricing models under various settings like pricing in network, heterogeneity of users, stochastic congestion and so on.

Another proposed methodology for relieving traffic congestion is the use of auctions. Teodorovic et al. (7) propose an auction-based congestion pricing scheme, which lets participating vehicles bid for time-slots for travel in the down-town area of a city. However, the auction is only used for controlling overall flow in an area, and does not allocate traffic at the network level.

Emerging technology such as Vehicle-to-Infrastructure (V2I) communication enables direct information exchange between vehicles and traffic controllers. Milanés et. al. propose an approach that uses Vehicle-to-Infrastructure (V2I) communication (8) to manage traffic. In our paper, we propose a congestion control method based on a combinatorial auction implemented with V2I devices. The auction system determines the toll price according to individual vehicle 'bids', which are collected through a system of V2I devices.

This auction system maintains a set of roads congestion-free throughout the planning period. Thus it can potentially be used as the tolling system of a network of High-Occupancy Toll (HOT) lanes. We want to keep congestion-free links because studies such as (9) have shown that drivers value reliability of journey time no less important than the saving in journey time. Also as is pointed out by (10), if differential charges can yield more reliable journey times, drivers are more willing to accept time-varying tolls.

This paper is organized as follows: a detailed description of the mathematical model and the auction scheme is presented in section 2. In section 3, we test the auction in a small network with 5100 vehicles. In section 5 we analyze the computational complexity of the problem. In Section 4, we discuss issues related to the implementation of the auction in the real world.

2. COMBINATORIAL AUCTIONS

In this paper, a Vehicle-to-Infrastructure (V2I) communication system is used to implement a combinatorial auction designed to efficiently allocate road resources. This V2I system enables a two-way communication between each vehicle and a central controller: vehicles send their "bids" to the central controller, and then the central controller sends back the path assignment and payment information to vehicles. V2I devices can be pre-installed in vehicles, or more conveniently, run as a specifically designed "apps" on smart-phones of drivers.

In a typical auction, multiple buyers are bidding for a single item. The auction we use in here, however, requires multiple buyers (vehicles) bidding for multiple items (roads). This is called a combinatorial auction. Examples applications of combinatorial auctions are the FCC spectrum auctions (11), auctions for airport time slots (12), railroad segments (13), delivery routes(14) and network routing (15).

While bidding a bidder does not necessarily bid his/her true valuation of items. However, with carefully designed mechanism, one can induce all the bidders to bid truthfully. One general type of such mechanism is called VCG mechanism, named after Vickrey (16), Clarke (17) and Groves (18).

In this paper, we implement a type of VCG mechanism for path allocation and toll. To better illustrate this auction mechanism, we start from an "ideal" scenario, where every vehicle is assumed to be equipped with a V2I device and every link is tolled in the network. Then we discuss in the next section how to implement this auction in a more realistic environment, where only part of vehicles and a subset of roads are in the auction. For example, this mechanism can be used as the pricing method for a network with (High Occupancy Vehicle) HOV or HOT (High Occupancy Toll) lanes. Details of this issue will be discussed in the section 4.

2.1. Combinatorial Auction For Entire Network

The auction system is implemented on a set of the links in a network. Vehicle using these links submit their "bids" (the prices s/he is willing to pay) to the central controller. The central controller collects these bids, and solves an optimization problem to assign a path and the corresponding toll to each vehicle. FIGURE 1 illustrates this auction mechanism. On the left of the figure, is the network of links.

The auction system works as follows:

1. Before traveling, each vehicle submits a "bid" to the traffic controller via V2I communication. Each bid consists of the following information:
    - Origin and destination of the travel
    - Estimated time to enter the network
    - Price s/he is willing to pay for each potential path s/he can travel.
2. After the submission deadline, traffic controller collects these bids, and uses this information to solve an optimization problem (details discussed in section 2.4). The controller then sends back the following instructions to each vehicle:
    - Path to take (path assignment)
    - Toll to pay (payment)

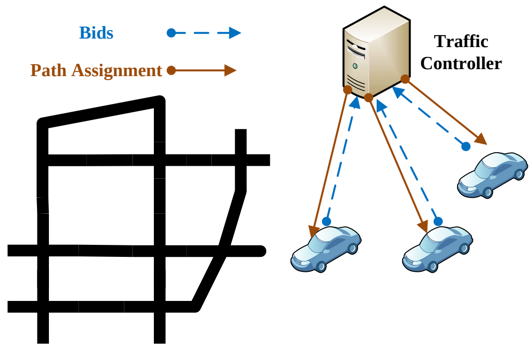

FIGURE 1 Auction Mechanism

3. At start of travel, the vehicle is automatically charged through electronic devices installed in the car. Vehicles must take the assigned path. A penalty fee will be charged for any deviation from the assigned path.

In the rest of this section, we will address the following questions about the auction mechanism.

- What information must the drivers provide to the controller
- How does the controller determine who will be assigned to which path (path assignment)
- What is the toll a driver pays for taking the path. This can be different from what s/he bid. (payment scheme)
- Mechanism to guarantee drivers bid their true valuation
- Is the solution efficient

2.2. Assumptions

We make the following assumptions:

- Infrastructure
    1. Every vehicle is equipped with two-way V2I wireless devices (will be relaxed later)
    2. The bidding process and toll collection is done through wireless communication.
- Traffic Controller

1. No congestion (free flow) for all the links in the auction network
2. Controller has mechanism to prevent drivers from deviating from assigned path (i.e., through penalty)
3. Vehicle must establish communication with controller before entering the network

- Drivers and Vehicles

1. Every vehicle must submit bids for all of the paths it can potentially use
2. All vehicles travel at free flow speed in the network
3. Bids are calculated and submitted by on-board computers
4. Each driver's cost function is independent of other drivers (private value)
5. Once assigned, the driver must accept the path and pay the toll

2.2.1 Remarks on Congestion-free Assumption

Note that for now no congestion is allowed in any link in this auction mechanism. In other words, the toll determined by the auction guarantees traffic flow in every tolled link is within its free-flow capacity. This assumption may seem to be overly restricted in the real world applications, however, as is explained in later sections, this model can be easily extended to a network consisting of two parts: one with tolled links (thus no congestion), and the other with free links (that might be congested). In this settings, the auction mechanism keeps tolled links congestion-free, while still provides toll-free options for drivers unable or unwilling to participate in the auction.

Eliminating congestion from tolled links in this auction model means travel time of every tolled links is always constant. The benefit of this constant travel time are two-fold: first: it reduce the computational complexity of the auction model by allowing linear function for drivers' utility;and second: it provide more predictability of commute time for paying users and meets their expectations of congestion free travel on payment of a toll.

2.3. Mathematical Model

2.3.1 Network

Consider a network that consists of a set of $\mathbf{L} = \{1, 2, \ldots, L\}$ of links. For each link $l \in \mathbf{L}$, the free-flow capacity and travel time are denoted as C_l and T_l, respectively. Note that theoretically congestion can still happen even when flow of a link is below C_l (because the traffic density in certain parts of the link is high). However, if links are short, and time is properly discretized (see section 2.3.3), maintaining link flow below C_l during the planning duration can maintain travel time at T_l.

A path is defined as a sequence of links from one origin to one destination. The set of all possible paths is denoted by $\mathbf{P} = \{1, 2, \ldots, P\}$. We denote a path $p \in \mathbf{P}$ as a sequence of links it contains:

$$p \equiv \{a^p_{(1)}, a^p_{(2)}, \ldots, a^p_{(|p|)}\}$$

where $|p|$ is the number of links contained in p, and $a^p_{(i)}$ is the ith link in path p. And we define the travel time of a path p as T^j[1]:

$$T^j = \sum_{k=1}^{|p|} T_{a^p_{(k)}}$$

2.3.2 Vehicles

There are N vehicles, denoted by set $\mathbf{N} = \{1, 2, \ldots, N\}$, using the traffic network. Vehicle $i \in \mathbf{N}$ will enter the network at time A_i.

The traffic controller's job is to assign a path to each vehicle. The path assignment for vehicle i is a vector $x_i = (x^1_i, x^2_i, \ldots, x^P_i)$, where $x^j_i = 1$ means vehicle i is assigned to path j, and 0 otherwise. Obviously, a valid assignment x_i requires that $\sum_{j=1}^{P} x^j_i = 1$.

Each vehicle has a utility function $U_i(x_i)$ that maps a valid assignment x_i to a real number:

$$U_i(x_i) : \{0, 1\}^P \to \mathbb{R}$$

It can be interpreted as the benefit vehicle i gets when traveling under assignment x_i.

Although this utility function can be of any form, for the purpose of demonstrating our model, we use a linear utility function:

$$U_i(x_i) = \sum_{j \in P} v^j_i x^j_i$$

where v^j_i is the value of traveling in path j, by vehicle i. Note that the actual bid, $\hat{v}^j_i$, made by vehicle i for path j, may be different from the true value v^j_i.

[1] T^j does not change as no congestion is allowed

2.3.3 Time

The entire planning period is discretized into a set of intervals of equal length δ, denoted as $\mathbf{T} = \{1, 2, \ldots, T\}$. δ is set small enough so that the travel time of any link in the network is an integer multiple of δ, but not too small so as to make the problem computationally difficult (issues of computation will be discussed in section 5).

The typical planning period δT can be set to 24 hours, or to the duration of the peak hours when congestion is likely to happen.

2.4. Optimization Problem

Given a path assignment matrix for all drivers, $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, and the bids $\hat{v}_i^j$ for vehicle i and path j, we evaluate the system performance using the sum of the utility of all vehicles (also called social utility function) $\mathbf{U}$:

$$\begin{aligned}\mathbf{U}(\mathbf{x}) &= \sum_{i\in\mathbf{N}} U_i(x_i) \\ &= \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{P}} \hat{v}_i^j x_i^j\end{aligned}$$

Assuming that all links are congestion free, we define the following delay operator τ_l^j for each $l \in \mathbf{L}, j \in \mathbf{P}$:

$$\tau_l^j = \sum_{k=1}^{k:a^j_{(k)}=l} T_{a^j_{(k)}}$$

in other words, τ_l^j is the time to travel to the entrance of link l, given that the vehicle is on path j.

Based on the above social utility function and delay operator τ_l^j, we formulate the following Path Assignment Problem to determine the optimal assignment:

$$\mathbf{U}^* = \text{maximize} \sum_{i \in \mathbf{N}} \sum_{j \in \mathbf{P}} \hat{v}_i^j x_i^j \tag{MAX 1}$$

$$\text{s.t.} \sum_{j \in \mathbf{P}} x_i^j = 1 \qquad \forall i \in \mathbf{N} \tag{1}$$

$$\sum_{j \in \mathbf{P}} \sum_{\substack{i: A_i > t - \tau_l^j - T_l \\ A_i \le t - \tau_l^j}} x_i^j \le C_l \qquad \forall t \in \mathbf{T},\, l \in \mathbf{L} \tag{2}$$

$$x_i^j = 0, 1 \qquad \forall i \in \mathbf{N},\, j \in \mathbf{P} \tag{3}$$

Constraint (1) ensures that each vehicle is assigned to exactly one path. Constraint (2) enforce the number of vehicles in each link l at each time period t does not exceed the total capacity of the link. This is done by summing up all the vehicles that have entered, but not yet exited link l at time t. $t - \tau_l^j - T_l$ is the time a vehicle arrives at the entrance of path j (also the entrance of network), given that it reaches the **entrance** of link l at time t. On the other hand, $t - \tau_l^j$ is the time a vehicle arrives at the entrance of path j, given that it reaches the **exit** of link l at time t. Constraint (3) ensures that the assignment variable x_i^j can only be zero or one.

On solving (MAX 1), we obtain an optimal assignment that maximizes the social utility function. This assignment is then distributed to individual vehicles via V2I wireless communication devices, informing them of the path to take.

2.5. Payment

The optimization problem (MAX 1), generates an optimal solution $\mathbf{x}^*$, determining the path assigned to each vehicle. We now determine the toll price for this assignment. We adopt a scheme similar to traditional VCG mechanism, which determines the toll as an "opportunity cost" it imposes on other vehicles, in other words, the marginal utility price. The procedure of computing toll for vehicle k is as follows:

Define

$$\mathbf{U}_{-k}(\mathbf{x}_{-k}) = \sum_{\substack{i \in \mathbf{N} \\ i \neq k}} \sum_{j \in \mathbf{P}} \hat{v}_i^j \cdot x_i^j$$

where $\mathbf{x}_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N)$. $\mathbf{U}_{-k}$ is the utility when vehicle k has been excluded.

We modify the optimization problem (MAX 1) to exclude vehicle k, and call it (MAX 1-k). Let its optimal value to be $\mathbf{U}^*_{-k}$. Thus :

$$\mathbf{U}^*_{-k} = \text{maximize} \sum_{\substack{i \in \mathbf{N} \\ i \neq k}} \sum_{j \in \mathbf{P}} \hat{v}_i^j x_i^j \tag{MAX 1-k}$$

$$\text{s.t.} \sum_{j \in \mathbf{P}} x_i^j = 1 \quad \forall i \in \mathbf{N}, i \neq k \tag{4}$$

$$\sum_{j \in \mathbf{P}} \sum_{\substack{i: A_i > t - \tau_l^j - \tau_l \\ A_i \leq t - \tau_l^j \\ i \neq k}} x_i^j \leq C_l \quad \forall t \in \mathbf{T},\, l \in \mathbf{L} \tag{5}$$

$$x_i^j = 0, 1 \quad \forall i \in \mathbf{N}, i \neq k,\, j \in \mathbf{P} \tag{6}$$

Thus U^*_{-k} is the optimal social utility when vehicle k is not in the system.

If we denote $\mathbf{x}^*_{-k}$ as the optimal solution from (MAX 1), excluding vehicle k, then the toll π_k for vehicle k is

$$\pi_k = \mathbf{U}^*_{-k} - \mathbf{U}_{-k}(\mathbf{x}^*_{-k}) \tag{7}$$

The first term in equation (7), is the optimal social utility without vehicle k, and the second term is the social utility of the optimal solution x^* of (MAX 1), without the vehicle k. The difference of these two terms is the increase in social utility when vehicle k is not included in the system, justifying it as a toll for vehicle k.

Note that the first term depends only on the bids of vehicles other than k. This is the desirable feature of VCG mechanism, which generates no incentive for vehicles to mis-report their true value (in other words, this mechanism guarantees $v_i^j = \hat{v}_i^j \quad \forall i \in \mathbf{N}, j \in \mathbf{P}$).

Theorem 2.1. *Truthful reporting is an optimal strategy for each vehicle driver in the auction mechanism. Moreover, when each vehicle driver reports truthfully, the outcome of the mechanism is one that maximizes social utility.*

The proof of Theorem 2.1 is in section 7.1.

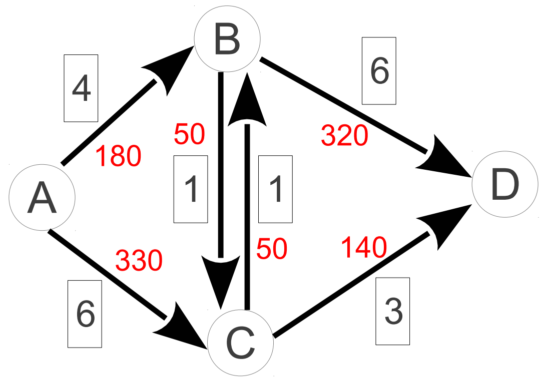

FIGURE 2 Network

3. NUMERICAL EXPERIMENT

3.1. An Example

We test this model on the traffic network shown in FIGURE 2.

There are six links in this network. The free flow travel time T_l of each link l is shown in a box next to it. The free flow capacity C_l is also shown as a red number attached to link l.

We set up the number T_l such that four paths have different free flow travel time: 8 for Path ABCD, 9 for ACD, 10 for ABD, and 13 for ACBD. This makes the interaction of path choice and toll transparent.

To better understand the dynamic of traffic assignment and toll price, we assume that all vehicles are traveling only from A to D.

We assume that the number of vehicles arriving at the entrance follows a Poisson distribution with rate λ. Note that λ can be a function of time.

We assume vehicles' value v_i^j for traveling in path j is a linear function of the free-flow travel time of path j, i.e., $v_i^j = c_i T^j$. Here c_i can be viewed as the vehicle i's "willingness-to-pay" per unit travel time. We generate c_i with a log-normal distribution with mean $\mu = 1$ and standard deviation $\sigma = 0.5$.

To simulate the situation of real traffic, we generate an incoming vehicle flow in the following manner: from time 0 to 20, the number of vehicles arriving gradually increases from 60 to 100 vehicles per minute. Then the rate of arrival stays at 100 vehicles per minute from time 20 to 40 before it gradually decreasing to 60 vehicles per minute at time 60.

The parameters of this test are shown in Table 1

TABLE 1 Parameters of Simulation

Parameter	Meaning
$N = 5100$	Total Number of Vehicles
$\delta = 1$	Minutes per Time Period
$T = 60$	Number of Time Periods
λ	Vehicle arrival rate
$\mu = 1$	Mean of willingness-to-pay
$\sigma = 0.5$	Standard deviation of willingness-to-pay

3.2. Results

We generate input data according to the settings described above, and solve the path assignment (MAX 1) and payment problems (MAX 1-k) using CPLEX 12.0. A 30-CPU computer cluster was used to solve 5100 payment problems in parallel. It took about 5 minutes to solve the path assignment and all payment problems.

We now analyze the traffic flow on each path over time in FIGURE 3.

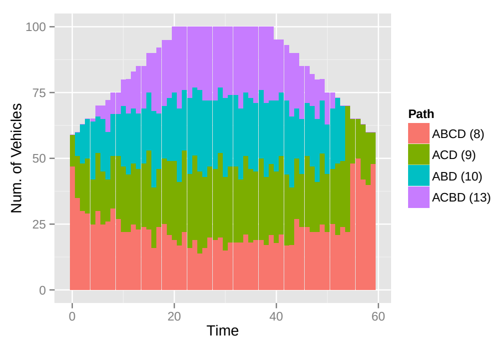

FIGURE 3 Number of Vehicles Using Each Path

As is shown in FIGURE 3, at the beginning when traffic is low, all of the traffic goes through the shortest two paths: ABCD and ACD. As traffic flow increases over time, more and more vehicles are assigned to longer paths.

At the same time, FIGURE 4 shows that the toll price also goes up as the traffic flow increases over time. Also the toll is higher for shorter path, and lower for longer path.

We also analyze the traffic flow of each link during the 60 minutes test period. As is shown in FIGURE 5, while flow in link AC and BD only reach link capacity during the peak time, flow in link AB and CD are very close or at the capacity most of the time during the

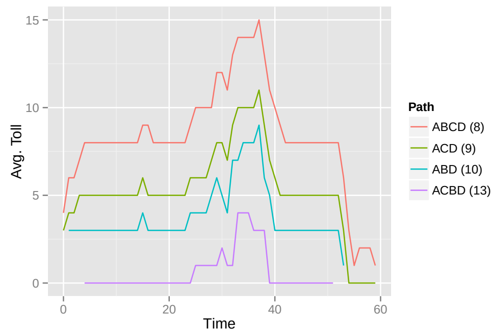

FIGURE 4 Average Toll Price

test.

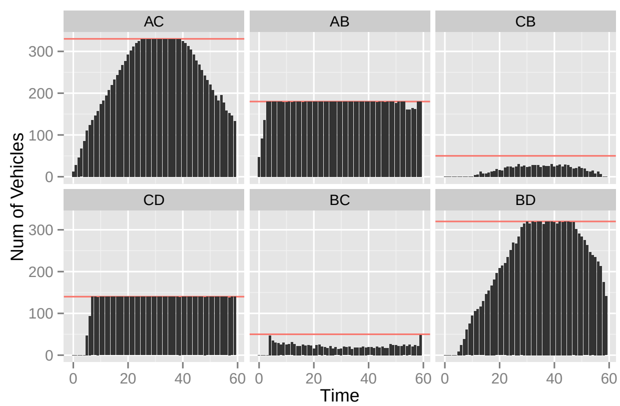

FIGURE 5 Number of Vehicles Using Each Link

To see if the mechanism distributes the toll "fairly", we compare the relationship between payment and bid of each vehicle in FIGURE 6, which consist of four sub-figures, each representing one path. For each path j, we plot all vehicles assigned to this path in the following way: for vehicle k, arrival time A_k is plotted as x coordinate, whereas payment per unit travel time (π_k/T_j) is plotted as y coordinate, and the color of a dot represents the value per unit travel time (c_k), with red being lowest value, and purple being the highest.

Since many vehicles share the same arrival time and payment, to clearly distinguish each vehicle, we add a small random perturbation to each vehicle's x and y coordinates. As is already shown in FIGURE 4, vehicles assigned to shorter paths such as ABCD would pay more than vehicles assigned to longer paths such as ACBD. More importantly, this figure shows that vehicles assigned to shorter paths are driven by mostly "richer" people, i.e., people who has higher value of time c_i: most of the dots in the first sub-figure representing the shortest path ABCD are green and blue, which means these vehicles has value per unit travel time (c_i) greater than 4.

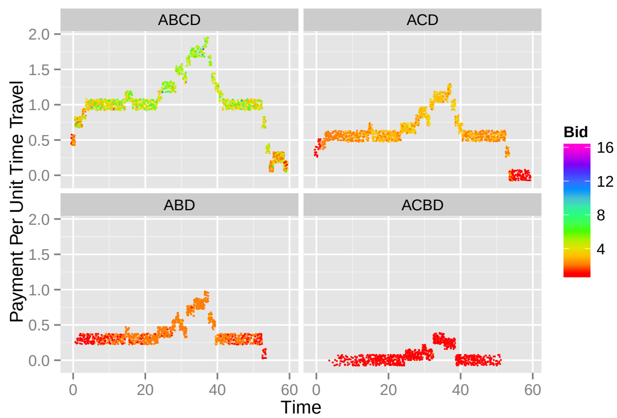

FIGURE 6 Payment and Bid Relationship Over Time

4. IMPLEMENTATION ISSUES

In this model, we require that ALL vehicles be equipped with V2I devices and the required dedicated software for participation in the auction process. However, as we are going to show in this section that such a model can also be implemented as a sub-system embedded in a larger network with only part of vehicles equipped with V2I devices. For example, it can be used as the pricing model for a network of (High Occupancy Vehicle) HOV or HOT (High Occupancy Toll) lanes.

4.1. Alternative Free Paths

In the previous sections, we have assumed that every vehicle is participating in the auction, and each is guaranteed to be assigned to exactly one non-congested path. However, this requirement might be unrealistic in most real-world applications.

One extension of this auction model is to allocate at least one alternative free path between any pair of origin and destination. This alternative path, unlike other paths in the model, is toll-free, but is subject to congestion. We set $\mathbf{P}_f$ as the set of "free" paths. Since these free paths can be congested, we further assume that every vehicle bids zero on this free path, that is, for any vehicle $i \in \mathbf{N}$, $\hat{v}_i^p = 0$ for path $p \in \mathbf{P}_f$.

The rest of the auction mechanism remains the same, except that if a vehicle is assigned to a free path in problem (MAX 1), no payment problem (MAX 1-k) is solved, and the vehicle pays no toll.

4.2. Auction as a Tolling Sub-system for HOV or HOT Lanes

Another issue that arises while implementing V2I devices, or generally, Intelligent Transportation System (ITS), is that early users of the systems gain little or no benefit when the market penetration of that device is low. Although the auction mechanism we present here is implemented as a stand-alone system, where all vehicles are required to be equipped with V2I devices, it can also be used as a tolling sub-system for High Occupancy Vehicle (HOV) or High Occupancy Toll (HOT) lanes.

HOT lanes allow drivers to pay a toll to enter a high occupancy vehicle (HOV) lane when they do not meet the minimum occupancy requirement. Many studies have demonstrated the effectiveness of such system, such as the HOV in I-15 in San Diego by (19) and (20), and SR91 in Orange County, California by (21)

In this case, the set of links $\mathbf{L}$ is defined as the set of HOT lanes. Regular lanes are treated as alternative links as in section 4.1. Only vehicles equipped with bidding devices are allowed to enter the HOT lanes, while vehicles without V2I devices can use regular lanes in the network. Thus, all vehicles can use the roads regardless of whether they are equipped with V2I bidding devices. At the same time, the system creates an incentive for vehicles to participate in the auction since then it generates non-congested travel.

4.3. Rolling Horizon

The current auction is operated off-line, meaning that all vehicles bid and get the assigned paths before starting travel. This limits the usability of the model. However, one can extend this model to a rolling horizon reservation system. In this system, we set up a main auction labeled B_0 which has a "cut-off" time, say, two hours before the start of planning period. Every vehicle that bids before this cut-off time will receive the path assignment and payment information immediately at the cut-off time. Vehicles who miss the cut-off time can still bid upon arrival at the entrance by participating in the following "rolling" auction:

The traffic controller will start a new round of auction B_t at every time period $t \in \mathbf{T}$. Vehicles arriving between time $t-1$ and t who did not bid before the cut-off time can participate in auction B_t. In auction B_t, we solve problems (MAX 1) and (MAX 1-k) by replacing the right-hand-side of constraint (5) by $C_{l,t}$, the "remaining capacity" of link l at time t. $C_{l,t}$ is calculated by subtracting the number of vehicles using link l at time t from the free-flow capacity C_l, using the prior vehicles' assignments from $B_0, B_1, \ldots, B_{t-1}$.

Since $C_{l,t}$ is always less than C_l, vehicles bid in auction B_t are likely to pay higher toll than those who bid in B_0. The illustration of this rolling horizon method is in FIGURE 7

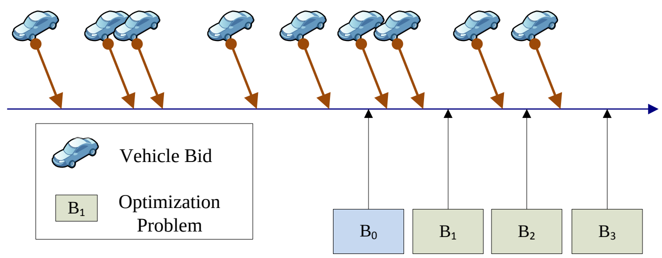

FIGURE 7 Rolling Horizon

5. COMPUTATIONAL ISSUES

Both the path assignment problem (MAX 1) and payment problems (MAX 1-k) are Integer Programming (IP) problems, and thus NP-complete. These are also notoriously hard to solve for large problem size. Although medium-size problems like the example we used in section 3 can be solved relatively fast, it could take considerably longer to solve larger size problems with more vehicles and larger network. In this section, we will analyze the structure of these problems and propose some methods to reduce the complexity of computation.

5.1. Solving Path Assignment Problem

The constraint (3) of path assignment problem (MAX 1) requires that all variables be integer, this makes the problem an IP. A typical way of solving IP is to first solve Linear Programming (LP) relaxation of the IP problem and then use branch-and-bound method to find the optimal integer solution.

5.1.1 Structure of The Path Assignment Problem

Consider the constraints of the LP relaxation of (MAX 1).

$$\sum_{j\in\mathbf{P}} x_i^j = 1 \qquad \forall i \in \mathbf{N} \tag{8}$$

$$\sum_{j\in\mathbf{P}} \sum_{\substack{i:A_i > t-\tau_l^j - T_l \\ A_i \le t-\tau_l^j}} x_i^j + s_{t,l} = C_l \qquad \forall t \in \mathbf{T},\, l \in \mathbf{L} \tag{9}$$

$$x_i^j \ge 0 \qquad \forall i \in \mathbf{N}, i \neq k,\, j \in \mathbf{P} \tag{10}$$

The problem has $N \times P$ variables. There are N constraints in the first group of constraints (8) and $T \times L$ constraints in the second group of constraints (9). In the context of a Simplex method, a basis consists of $N + TL$ basic variables.

Based on the special structure of the basis, we can prove the following theorem:

Theorem 5.1. *The number of non-integer variable in any basic solution of the relaxed IP are bounded by TL.*

The proof of Theorem 2 is in section 7.2.

In the worst case, there will be at most $2TL$ non-integer variables in a solution to the LP relaxation of (MAX 1). In general, the proportion of non-integer solutions is $2TL/NP$. In large network, $T \ll N$ and $L \ll P$, so only a small percentage of variables will be non-integer. In the test case of section 3, at most $(60 \times 4)/(5100 \times 4) = 1.18\%$ of variables will be non-integer.

5.1.2 Reducing Complexity

One method to reduce complexity of (MAX 1) is to use a smaller number of potential paths for each vehicles, and instead of letting vehicles choose from all of the available paths, we limit their choices to, say, at most six paths.

5.2. Solving Payment Problem

Although the initial path assignment problem (MAX 1) may itself be hard to solve, bigger computational challenge is to solve N instances of payment problems (MAX 1-k).

As is shown in (22), in order to maintain truthful reporting property of VCG mechanism, (MAX 1) must be solved to optimality, but the solution of (MAX 1-k) need not be optimum.

So in order to reduce solving time for payment problems, we can set an optimality gap, say, 2%, when solving the payment problems. The branch-and-bound algorithm will stop when the obtained solution is at most 2% away from actual optimal solution.

We have observed that if two vehicles enter the network at the same time and are assigned the same path, they pay the same toll (See Theorem 3). This can be used to reduce running time of our mechanism: instead of solving payment problem for each vehicle, we solve payment problem (MAX 1-k) only once for each time step and each path.

6. CONCLUSIONS AND FUTURE WORK

We have proposed here an auction system implemented via V2I devices to toll and allocate traffic. Participating vehicles "bid" before travel. Traffic controller solves an optimization problem and assign paths and corresponding tolls to these vehicles. A mathematical model of the auction is presented and analyzed. The auctions system is based on VCG mechanism and thus guarantees truthful reporting of bids. The auction scheme is tested on a small network with 5100 vehicles. We also discuss methods for its implementation in the real world, as well as the possibility of implementing it as a tolling sub-system for HOV or HOT lanes.

There are three possible extensions of this work. 1) Changing the auction scheme or developing heuristics that reduces the computational complexity of auction. Here we use the classical VCG mechanism is used to determine the path assignment and toll, but there are other available auction mechanisms that do not involve solving Integer Programming problems. 2) Allow flexible travel time for vehicles. Instead of reporting a fixed travel time, vehicles can report a time window of travel. 3) Introducing stochasticity into the model. Instead of maintaining free-flow for each link in the network, we can allow congestion in certain links. This would requires dynamically forecasting traffic flow in the network, and a more sophisticated model.

7. PROOF OF THEOREMS

7.1. Proof of Truthful Reporting Is a Best Strategy

Theorem 1. Truthful reporting is an optimal strategy for each vehicle driver in the auction mechanism. Moreover, when each vehicle driver reports truthfully, the outcome of the mechanism is one that maximizes social utility.

Proof. This is adapted from (23).

Suppose each driver $i \in \mathbf{N}$ has a intrinsic value v_i^j for traveling in each path $j \in \mathbf{P}$. They report $\hat{v}_i^j$ to the central controller. Now we need to prove that reporting $\hat{v}_i^j = v_i^j, \forall j$ is a best strategy for each driver i.

Consider any fixed profile of reports $\{\hat{v}_i^j\}_{i \neq k}$ for all drivers besides k. Suppose that when driver k reports truthfully, the resulting allocation and payment vectors are denoted by $\mathbf{x}^* = \{x_i^j\}_{i \in \mathbf{N}, j \in \mathbf{P}}$ and $\boldsymbol{\pi}^* = (\pi_1, \pi_2, \ldots, \pi_N)$. But when driver k reports $\hat{v}_k^j$ for each path j, the resulting assignment are denoted as $\hat{\mathbf{x}} = (\hat{x}_1^*, \hat{x}_2^*, \ldots, \hat{x}_N^*)$, whereas the resulting payment is represented by $\hat{\boldsymbol{\pi}} = (\hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_N)$.

When vehicle k reports $\hat{v}_k^j$ for path j, his pay-off is:

$$
\begin{aligned}
& U_k(\hat{x}_k^*) - \hat{\pi}_k \\
=& U_k(\hat{x}_k^*) + \mathbf{U}_{-k}(\hat{\mathbf{x}}_{-k}^*) - \mathbf{U}_{-k}^* \\
\leq& \max_{\mathbf{x} \in S} \{U_k(x_k) + \mathbf{U}_{-k}(\mathbf{x}_{-k})\} - \mathbf{U}_{-k}^* \\
=& U_k(x_k^*) + \mathbf{U}_{-k}(\mathbf{x}_{-k}^*) - \mathbf{U}_{-k}^* \\
=& U_k(x_k^*) - \pi_k^*
\end{aligned}
$$

where S is defined as the set of $\mathbf{x}$ that satisfies constraint (4) to (6). □

7.2. Structure of Path Assignment Problem

Theorem 2. Solving the relaxed IP, the number of non-integer variable in any basic solution are bounded by TL.

Since for each vehicle i, a constraint in group (8) can provide at least one basic variable.

On the other hand, each constraint in group (9) can provide one basic variable.

If none of the links are capacitated, all of the TL slack variables $s_{t,l}$ should be positive, which provide TL basic variables. In this case, for each $i \in \mathbf{N}$, constraint

$$\sum_{j \in \mathbf{P}} x_i^j = 1 \tag{11}$$

only has one basic variable. So the optimal solution would always be integer.

If there are n of the links are capacitated, slack variables corresponding to those links are zero, thus there are n more x_i^j need to be basic variables in the worst case. These additional

x_i^j will have some of the constraints in (8) contain more than one basic variables, thus give non-integer solutions.

7.3. Proof of Fair Price

Theorem 3. If two vehicles k_1 and k_2 that

1. share the same origin and destination,
2. arrive at the entrance of their trip at the same time
3. were assigned to the same path by the traffic controller
4. for vehicle $k = k_1, k_2$, $v_k^1 \leq v_k^2 \leq \cdots \leq v_k^P$ is always true
5. the value of paths by two vehicles satisfies $v_{k_1}^{j_1} - v_{k_1}^{j_2} \leq v_{k_2}^{j_1} - v_{k_2}^{j_2}$ for all $j_1, j_2 \in \mathbf{P}$

then vehicle k_1 would pay no more than k_2

Proof. Consider two vehicles k_1 and k_2 which share the same arrival time $A_{k_1} = A_{k_2}$.

Suppose these two vehicles were assigned the same path j^*, then the payment of these two are:

$$\pi_{k_1} = \mathbf{U}^*_{-k_1} - (\mathbf{U}^* - v_{k_1}^{j^*})$$
$$\pi_{k_2} = \mathbf{U}^*_{-k_2} - (\mathbf{U}^* - v_{k_2}^{j^*})$$

To prove that $\pi_{k_1} - \pi_{k_2} \geq 0$, we will need to show that

$$\mathbf{U}^*_{-k_1} - \mathbf{U}^*_{-k_2} \geq v_{k_2}^{j^*} - v_{k_1}^{j^*}$$

Now if we let j_2 be the path assigned to vehicle k_2 in (MAX 1-k_1), that is, j_2 satisfies $x_{k_2}^{j_2} = 1$ in the optimal solution of (MAX 1-k_1). And similarly, let j_1 satisfy $x_{k_1}^{j_1} = 1$ in the optimal solution of (MAX 1-k_1). We claim that

$$j* \leq j_1 \leq j_2 \tag{12}$$

To prove this, use contradiction. If $j_1 > j_2$, then we can do one of the following

1. assign path j_1 to vehicle k_2 in problem (MAX 1-k_1), that is, let $x_{k_2}^{j_1} = 1$ instead of $x_{k_2}^{j_2} = 1$

2. assign path j_2 to vehicle k_1 in problem (MAX 1-k_2), that is, let $x_{k_1}^{j_2} = 1$ instead of $x_{k_1}^{j_1} = 1$

Define $\mathbf{U}_{-k_1,k_2}$ as the social utility excluding both vehicle k_1 and k_2. Also denote $x_{-k_2})^*_{-k_1}$ as the optimal solution to (MAX 1-k_2), removing vehicle k_1, while $x_{-k_1})^*_{-k_2}$ as the optimal solution to (MAX 1-k_1), removing vehicle k_2.

Note that payment problem (MAX 1-k_1) and (MAX 1-k_2) have identical feasible region (if we treat variables $x_{k_1}^j$ as $x_{k_2}^j$ and vice versa). The only difference between problem (MAX 1-k_1) and (MAX 1-k_2) is the objective coefficient $v_{k_1}^j$ as $v_{k_2}^j$.

For case 1, the change of objective value $\Delta\mathbf{U}_{-k_1}$ is

$$\begin{aligned}\Delta\mathbf{U}_{-k_1} =& \left(\mathbf{U}_{-k_1,k_2}((x_{-k_2})^*_{-k_1}) + v_{k_2}^{j_1}\right) \\ & - \left(\mathbf{U}_{-k_1,k_2}((x_{-k_1})^*_{-k_2}) + v_{k_2}^{j_2}\right) \\ =& \mathbf{U}_{-k_1,k_2}((x_{-k_2})^*_{-k_1}) - \mathbf{U}_{-k_1,k_2}((x_{-k_1})^*_{-k_2}) \\ & + v_{k_2}^{j_1} - v_{k_2}^{j_2}\end{aligned}$$

For case 2, the change of objective value $\Delta\mathbf{U}_{-k_2}$ is

$$\begin{aligned}\Delta\mathbf{U}_{-k_2} =& \left(\mathbf{U}_{-k_1,k_2}((x_{-k_1})^*_{-k_2}) + v_{k_1}^{j_2}\right) \\ & - \left(\mathbf{U}_{-k_1,k_2}((x_{-k_2})^*_{-k_1}) + v_{k_1}^{j_1}\right) \\ =& \mathbf{U}_{-k_1,k_2}((x_{-k_1})^*_{-k_2}) - \mathbf{U}_{-k_1,k_2}((x_{-k_2})^*_{-k_1}) \\ & + v_{k_1}^{j_2} - v_{k_1}^{j_1}\end{aligned}$$

Thus, according to assumption 5

$$\begin{aligned}& \Delta\mathbf{U}_{-k_1} + \Delta\mathbf{U}_{-k_2} \\ =& v_{k_2}^{j_1} - v_{k_2}^{j_2} + v_{k_1}^{j_2} - v_{k_1}^{j_1} \\ >& 0\end{aligned}$$

This means at least one of $\Delta\mathbf{U}_{-k_1}$ or $\Delta\mathbf{U}_{-k_2}$ must be positive. So it is always possible to improve either $\mathbf{U}^*_{-k_1}$ or $\mathbf{U}^*_{-k_2}$, which contradict with the assumption that they are optimal. □

REFERENCES

[1] Schrank, D. and T. Lomax, The 1999 annual mobility report: information for urban America, 1999.

[2] Pigou, A. C., *Wealth and welfare*. Macmillan and Co., limited, 1912.

[3] Vickrey, W. S., Congestion thoery and transport investment. *The American Economic Review*, Vol. 59, No. 2, 1969, pp. 251–260.

[4] Friesz, T. L., C. Kwon, and R. Mookherjee, A computable theory of dynamic congestion pricing. In *Transportation and Traffic Theory 2007. Papers Selected for Presentation at ISTTT17*, 2007.

[5] Toh, R. S. and S.-Y. Phang, Curbing urban traffic congestion in Singapore: A comprehensive review. *Transportation Journal*, 1997.

[6] Lindsey, C. R. and E. T. Verhoef, *Traffic congestion and congestion pricing*. Tinbergen Institute, 2000.

[7] Teodorovic, D., K. Triantis, P. Edara, Y. Zhao, and S. Mladenovic, Auction-based congestion pricing. *Transportation Planning and Technology*, , No. February 2012, 2008, pp. 37–41.

[8] Milanés, V., J. Villagra, and J. Godoy, An intelligent V2I-based traffic management system. *Intelligent Transportation Systems, IEEE Transactions on*, Vol. 13, No. 1, 2012, pp. 49–58.

[9] Lam, T. C. and K. A. Small, The value of time and reliability: measurement from a value pricing experiment. *Transportation Research Part E: Logistics and Transportation Review*, Vol. 37, No. 2, 2001, pp. 231–251.

[10] Bonsall, P., J. Shires, J. Maule, B. Matthews, and J. Beale, Responses to complex pricing signals: Theory, evidence and implications for road pricing. *Transportation Research Part A: Policy and Practice*, Vol. 41, No. 7, 2007, pp. 672–683.

[11] Cramton, P., The FCC spectrum auctions: An early assessment. *Journal of Economics & Management Strategy*, Vol. 6, No. 3, 1997, pp. 431–495.

[12] Rassenti, S. J., V. L. Smith, and R. L. Bulfin, A combinatorial auction mechanism for airport time slot allocation. *The Bell Journal of Economics*, 1982, pp. 402–417.

[13] Brewer, P. J., Decentralized computation procurement and computational robustness in a smart market. *Economic Theory*, Vol. 13, No. 1, 1999, pp. 41–92.

[14] Sheffi, Y., Combinatorial auctions in the procurement of transportation services. *Interfaces*, Vol. 34, No. 4, 2004, pp. 245–252.

[15] Hershberger, J. and S. Suri, Vickrey prices and shortest paths: What is an edge worth? In *Foundations of Computer Science, 2001. Proceedings. 42nd IEEE Symposium on*, IEEE, 2001, pp. 252–259.

[16] Vickrey, W., Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, Vol. 16, No. 1, 1961, pp. 8–37.

[17] Clarke, E. H., Multipart pricing of public goods. *Public Choice*, Vol. 11, No. 1, 1971, pp. 17–33.

[18] Groves, T., Incentives in teams. *Econometrica: Journal of the Econometric Society*, 1973, pp. 617–631.

[19] Brownstone, D., A. Ghosh, T. F. Golob, C. Kazimi, and D. Van Amelsfort, Drivers willingness-to-pay to reduce travel time: evidence from the San Diego I-15 congestion pricing project. *Transportation Research Part A: Policy and Practice*, Vol. 37, No. 4, 2003, pp. 373–387.

[20] Supernak, J., C. Kaschade, and D. Steffey, Dynamic value pricing on I-15 in San Diego: Impact on travel time and its reliability. *Transportation Research Record: Journal of the Transportation Research Board*, Vol. 1839, No. 1, 2003, pp. 45–54.

[21] Sullivan, E. et al., Continuation study to evaluate the impacts of the SR 91 value-priced express lanes: Final report. *Submitted to State of California Department of Transportation Traffic Operations Program HOV Systems Branch, Sacramento CA*, Vol. 94273, 2000.

[22] Nisan, N. and A. Ronen, Computationally feasible VCG mechanisms. *Electronic Commerce: Proceedings of the 2 nd ACM Conference on Electronic Commerce*, Vol. 17, No. 20, 2000, pp. 242–252.

[23] Cramton, P., Y. Shoham, and R. Steinberg, *Combinatorial Auctions*. MIT Press, 2006.