ON THE VOLUME OF NON-CENTRAL SECTIONS OF A CUBE

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ABSTRACT. Let Q_n be the cube of side length one centered at the origin in $\mathbb{R}^n$, and let F be an affine $(n-d)$-dimensional subspace of $\mathbb{R}^n$ having distance to the origin less than or equal to $\frac{1}{2}$, where $0 < d < n$. We show that the $(n-d)$-dimensional volume of the section $Q_n \cap F$ is bounded below by a value $c(d)$ depending only on the codimension d but not on the ambient dimension n or a particular subspace F. In the case of hyperplanes, $d = 1$, we show that $c(1) = \frac{1}{17}$ is a possible choice. We also consider a complex analogue of this problem for a hyperplane section of the polydisc.

1. Introduction and main results

Consider a cube of a unit volume in the space $\mathbb{K}^n$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The sections of the cube by linear subspaces are classical objects of study in convex geometry, and precise estimates of their maximal and minimal volume are known. Namely, let $\|\cdot\|_\infty$ and $|\cdot|$ denote the supremum and the euclidean norm on $\mathbb{K}^n$, respectively, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For volume calculations, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and use the volume there. Let

$$Q_n := \{x \in \mathbb{K}^n \mid \|x\|_\infty \leq \alpha\}$$

be the n-dimensional cube (polydisc) of volume 1, i.e. $\alpha = 1/2$ if $\mathbb{K} = \mathbb{R}$ and $\alpha = 1/\sqrt{\pi}$ if $\mathbb{K} = \mathbb{C}$. In the real case, for any linear subspace of $E \subset \mathbb{R}^n$ of dimension $n-d$,

$$1 \leq \mathrm{vol}_{n-d}(Q_n \cap E) \leq 2^{d/2}.$$

The lower estimate is due to Vaaler [Va], and the upper one to Ball [B1]. In the complex case, Oleszkiewicz and Pelczyński [OP] proved that for codimension 1, $1 \leq \mathrm{vol}_{2n-2}(Q_n \cap E) \leq 2$. Less is known about the non-central sections which are the subject of the current paper.

Let us discuss the real case first. Fix a subspace $E \subset \mathbb{R}^n$ and consider sections of the cube by subspaces parallel to E. More precisely, for a vector $v \in E^\perp$, consider a function

$$\Phi(E, v) := \mathrm{vol}_{n-d}\big(Q_n \cap (E + v)\big).$$

Brunn's theorem asserts that Φ is an even function achieving the maximal value at the origin. This, in combination with Ball's theorem, provides an upper bound for the function Φ for all E and v. If $|v| > \frac{1}{2}$, then a non-trivial lower bound for this function is impossible to achieve. Indeed, if E is orthogonal to one of the basic

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vectors e_j, and $v = te_j$ with $t > \frac{1}{2}$, then $Q_n \cap (E + v) = \varnothing$. Our first main result provides a non-trivial lower estimate for the volume of the section for all E and v as long as $|v| \le \frac{1}{2}$. Moreover, this estimate is independent of the ambient dimension n and the space E.

Theorem 1.1. *For any $d \in \mathbb{N}$, there is $\varepsilon(d) > 0$ such that for any $n > d$ and any $(n-d)$-dimensional affine subspace $F \subset \mathbb{R}^n$ whose distance to the origin is smaller than or equal to $1/2$,*

$$\mathrm{vol}_{n-d}(Q_n \cap F) \ge \varepsilon(d).$$

As the discussion above shows, the distance $1/2$ is the maximal possible one.

The value of the bound $\varepsilon(d)$ can be traced from the proof of Theorem 1.1. We believe, however, that this value is quite far from the best possible. A better bound can be obtained for the sections of codimension 1, i.e., whenever $d = n - 1$. We will present this bound in the unified way for both real and complex scalars.

To this end, let us introduce some notation. Given a vector $a \in \mathbb{K}^n$ of length $|a| = 1$ and $t \in \mathbb{K}$, we introduce the *hyperplane section* of the cube

$$S(a,t) := \{x \in \mathbb{K}^n \mid \|x\|_\infty \le \alpha, \ \langle x, a\rangle = \alpha t\} = Q_n \cap H$$

where $H = \{\alpha t \cdot a\} + a^\perp$, and its volume

$$A(a,t) := A_{\mathbb{K}}(a,t) := \left\{ \begin{array}{ll} \mathrm{vol}_{n-1}(S(a,t)) \ , & \mathbb{K} = \mathbb{R} \\ \mathrm{vol}_{2n-2}(S(a,t)) \ , & \mathbb{K} = \mathbb{C} \end{array} \right\} .$$

For $a = (a_j)_{j=1}^n \in \mathbb{K}^n$, let a^* denote the decreasing rearrangement of the sequence $(|a_j|)_{j=1}^n$. Since the volume is invariant under coordinate permutations and sign changes (rotation of coordinate discs in the complex case), we have $A(a,t) = A(a^*, |t|)$. Therefore we will assume in the following that $a = (a_j)_{j=1}^n$, $a_j \ge 0$ and $t \ge 0$.

By Corollary 5 of König, Koldobsky [KK3] we have that

$$A(a,t) \le \sqrt{\frac{2}{1+t^2}} \ , \ \mathbb{K} = \mathbb{R} \qquad \text{and} \qquad A(a,t) \le \frac{2}{1+t^2} \ , \ \mathbb{K} = \mathbb{C},$$

so that $A(a,1) \le 1$ always holds.

As in the general case, if the distance parameter t is strictly bigger than 1, the non-central hyperplane $H = \{\alpha t \cdot a\} + a^\perp$ might not intersect Q_n and $A(a,t)$ might be 0. Assume that $t \in [0,1]$. Our second main result gives explicit bounds for $A(a,t)$ which are independent of the dimension n of the cube and of the direction a.

Theorem 1.2. *Let $a \in \mathbb{K}^n$ with $|a| = 1$. Then*

$$\frac{1}{17} < 0.05974 < A_{\mathbb{R}}(a,1) \le 1 \ ,$$

$$\frac{1}{28} < 0.03699 < A_{\mathbb{C}}(a,1) \le 1 \ .$$

Clearly, the lower bounds are not optimal. However, they cannot be improved by more than a factor of $\simeq 5.2$ in the real case and by a factor of $\simeq 7.3$ in the complex case, see Remark 6.1.

In the rest of the paper, we prove Theorems 1.1 and 1.2. The proof of Theorem 1.1 is contained in Section 2. In the course of it, we represent the function $\Phi(E, v)$ as the density f_X of the projection of a random vector uniformly distributed in Q_n onto the space $E^\perp$. We use both the geometric and the probabilistic definition of this function passing several times from one to another throughout the proof. If the space E is almost orthogonal to a coordinate vector and v is almost parallel to it, we derive the desired estimate by analyzing the characteristic function of f_X and using the log-concavity of this density. The analysis of the characteristic function relies in turn on its representation as the difference of characteristic functions of some other sections of the cube. The opposite case splits into two separate subcases. If the vector v is incompressible, i.e., far from any low-dimensional coordinate subspace, we prove the required bound probabilistically. If this vector is compressible, we rely on the previous analysis to reduce the bound to a similar geometric problem but in dimension depending only on d. The estimate in this case can be obtained directly.

We start preparing the ground for proving Theorem 1.2 in Section 3. In this section, we use the Fourier transform to represent the volume of a hyperplane section as a certain integral over the product of n euclidean spheres S^{k-1} with respect to the Haar measure. Here, $k = 3$ in the real case, and $k = 4$ in the complex case. The estimate of these integrals requires a lower bound for the probability that $|\sum_{j=1}^n a_j U_j| \geq 1$ where $a = (a_1, \dots, a_n) \in S^{n-1}$ and $U_1, \dots, U_n$ are independent random vectors uniformly distributed in S^{k-1}. A similar problem with U_j being scalar random variables has been extensively studied because of its importance in computer science, see e.g., [HK, BTNR, O, BH] and the references therein. However, the methods used there do not seem to be suitable to the vector-valued random variables. In Section 4, we develop a new method based on estimates of the Laplace transform and duality of Orlicz spaces. This method may be of independent interest as it is applicable to a broader class of random vectors. The probability itself is estimated in Section 5. Finally, in Section 6, we apply the toolkit created in three previous sections to complete the proof of Theorem 1.2.

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2. A lower bound for all codimensions

In this section, we prove Theorem 1.1. Let $F \subset \mathbb{R}^n$ be an affine subspace whose distance to the origin is $1/2$. We will represent F as $F = \frac{1}{2}v + E$, where $E \subset \mathbb{R}^n$ is an

$(n-d)$-dimensional linear subspace, and $v \in E^\perp,\ |v| = 1$. Denote by $P : \mathbb{R}^n \to \mathbb{R}^n$ the orthogonal projection onto $E^\perp$.

The strategy of the proof will depend on the position of the space E and the magnitude of the largest coordinate of v. We start from the case when E is almost orthogonal to a coordinate vector and v is almost parallel to this vector.

Lemma 2.1. *For any $d < n$, there exists $\varepsilon_1(d), \delta_1(d)$ such that if $|Pe_1| \geq 1 - \delta_1(d)$ and $v = \frac{Pe_1}{|Pe_1|}$, then*

$$\mathrm{vol}_{n-d}\left(Q_n \cap \left(\frac{1}{2}v + E\right)\right) \geq \varepsilon_1(d).$$

Proof. Assume for a moment that $e_1 \perp E$, and thus $v = e_1$. Then $Q_n \cap \left(\frac{1}{2}v + E\right)$ is a central section of the $(n-1)$-dimensional face of Q_n containing $\frac{1}{2}e_1$. In this case,

$$\mathrm{vol}_{n-d}\left(Q_n \cap \left(\frac{1}{2}v + E\right)\right) \geq 1$$

by Vaaler's theorem [Va]. This means that we can assume that $|Pe_1| < 1$ for the rest of the proof.

A random point $\xi \in Q_n$ can be considered as a random vector of density 1 in the cube. In this probabilistic interpretation, the volume of the section $\mathrm{vol}_{n-d}(Q_n \cap (E + u))$ is the density of the random vector $P\xi$ distributed in $E^\perp$ at the point $u \in E^\perp$. It would be more convenient to consider this random vector distributed in $\mathbb{R}^d$ instead. To this end, notice that the singular value decomposition of P yields the existence of a $d \times n$ matrix R satisfying

$$P = R^\top R, \quad RR^\top = I_d.$$

Therefore, $f_X(u) = \mathrm{vol}_{n-d}(Q_n \cap (E + u))$ can be viewed as the density of the vector $R\xi$ in $\mathbb{R}^d$. We will use the geometric and the probabilistic interpretation interchangeably throughout the proof.

The Fourier transform of the random variable $X = R\xi$ can be written as

$$\phi_X(t) = \int_{\mathbb{R}^d} f_X(x) \exp(-\mathbf{i} \cdot 2\pi \langle x, t\rangle)\, dx = \prod_{j=1}^n \frac{\sin(\pi \langle Re_j, t\rangle)}{\pi \langle Re_j, t\rangle}$$

for $t \in \mathbb{R}^d$, where we used the normalization by 2π for convenience. By the Fourier inversion formula,

$$f_X\left(\frac{1}{2}v\right) = \int_{\mathbb{R}^d} \exp(\mathbf{i}\pi \langle v, t\rangle)\phi_X(t)\, dt = \frac{1}{\pi^d}\int_{\mathbb{R}^d} \cos(\langle v, t\rangle)\phi_X(t/\pi)\, dt.$$

Hence,

$$\begin{aligned}2f_X\left(\frac{1}{2}v\right) &= \frac{2}{\pi^d}\int_{\mathbb{R}^d}\cos(\langle v,t\rangle)\prod_{j=1}^{n}\frac{\sin(\langle Re_j,t\rangle)}{\langle Re_j,t\rangle}\,dt\\ &= \frac{1}{\pi^d}\int_{\mathbb{R}^d}\frac{\sin\left(\left(\frac{1}{|Re_1|}+1\right)\langle Re_1,t\rangle\right)}{\langle Re_1,t\rangle}\prod_{j=2}^{n}\frac{\sin(\langle Re_j,t\rangle)}{\langle Re_j,t\rangle}\,dt\\ &\quad -\frac{1}{\pi^d}\int_{\mathbb{R}^d}\frac{\sin\left(\left(\frac{1}{|Re_1|}-1\right)\langle Re_1,t\rangle\right)}{\langle Re_1,t\rangle}\prod_{j=2}^{n}\frac{\sin(\langle Re_j,t\rangle)}{\langle Re_j,t\rangle}\,dt.\end{aligned}$$

Define $d\times d$ matrices $\Lambda_+,\Lambda_-:\mathbb{R}^d\to\mathbb{R}^d$ by

$$\Lambda_\pm=\left(\left(\frac{1}{|Re_1|}\pm 1\right)^2(Re_1)(Re_1)^\top+\sum_{j=2}^{n}(Re_j)(Re_j)^\top\right)^{-1/2}.$$

As $|Re_1|=|Pe_1|<1$, the matrix Λ_- is well-defined. Then

$$\Lambda_+^{-1}|_{(Re_1)^\perp}=\Lambda_-^{-1}|_{(Re_1)^\perp}=\mathrm{id}|_{(Re_1)^\perp},$$

and so

$$\begin{aligned}\det(\Lambda_+)&=\frac{1}{\det(\Lambda_+^{-1})}=\frac{1}{|\Lambda_+^{-1}Re_1|}=\left((1+|Re_1|)^2+\sum_{j=2}^{n}\langle Re_1,Re_j\rangle^2\right)^{-1/2}\\ &=(2+2|Re_1|)^{-1/2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\det(\Lambda_-)&=\frac{1}{\det(\Lambda_-^{-1})}=\frac{1}{|\Lambda_-^{-1}Re_1|}=\left((1-|Re_1|)^2+\sum_{j=2}^{n}\langle Re_1,Re_j\rangle^2\right)^{-1/2}\\ &=(2-2|Re_1|)^{-1/2}.\end{aligned}$$

Using the change of variables in the integrals above, we can write

$$\begin{aligned}2f_X\left(\frac{1}{2}v\right)=&\frac{1}{\pi^d}\left(\frac{1}{|Re_1|}+1\right)\det(\Lambda_+)\int_{\mathbb{R}^d}\prod_{j=1}^{n}\frac{\sin(\langle\theta_j,t\rangle)}{\langle\theta_j,t\rangle}\,dt\\ &-\frac{1}{\pi^d}\left(\frac{1}{|Re_1|}-1\right)\det(\Lambda_-)\int_{\mathbb{R}^d}\prod_{j=1}^{n}\frac{\sin(\langle\eta_j,t\rangle)}{\langle\eta_j,t\rangle}\,dt,\end{aligned}$$

where

$$\begin{cases}\theta_1=(\frac{1}{|Re_1|}+1)\Lambda_+Re_1,\\ \theta_j=\Lambda_+Re_j,\ \text{for } j>1,\end{cases}\qquad\begin{cases}\eta_1=(\frac{1}{|Re_1|}-1)\Lambda_-Re_1,\\ \eta_j=\Lambda_-Re_j,\ \text{for } j>1,\end{cases}$$

Note that

$$\sum_{j=1}^{n}\theta_j\theta_j^\top=\sum_{j=1}^{n}\eta_j\eta_j^\top=I_d.$$

This allows to view both integrals above as the volumes of certain sections of Q_n by $(n-d)$-dimensional linear subspaces. More precisely,

$$\frac{1}{\pi^d}\int_{\mathbb{R}^d}\prod_{j=1}^{n}\frac{\sin(\langle\theta_j,t\rangle)}{\langle\theta_j,t\rangle}\,dt=\mathrm{vol}_{n-d}(Q_n\cap E_1)$$

and

$$\frac{1}{\pi^d}\int_{\mathbb{R}^d}\prod_{j=1}^{n}\frac{\sin(\langle\eta_j,t\rangle)}{\langle\eta_j,t\rangle}\,dt=\mathrm{vol}_{n-d}(Q_n\cap E_2)$$

for some linear subspaces $E_1, E_2 \subset \mathbb{R}^n$. This can be easily checked using the Fourier inversion formula as above. A theorem of Vaaler [Va] asserts that the volume of any central section of the unit cube is at least 1, and a theorem of Ball [B1] states that it does not exceed $(\sqrt{2})^d$. Therefore,

$$\begin{aligned}2f_X\left(\frac{1}{2}v\right) &\geq \left(\frac{1}{|Re_1|}+1\right)\det(\Lambda_+)-(\sqrt{2})^d\left(\frac{1}{|Re_1|}-1\right)\det(\Lambda_-)\\ &=\frac{1}{\sqrt{2}|Re_1|}\,(1+|Re_1|)^{1/2}-\frac{(\sqrt{2})^{d-1}}{|Re_1|}\,(1-|Re_1|)^{1/2}\geq\varepsilon_1(d),\end{aligned}$$

for some $\varepsilon_1(d) > 0$ whenever $|Pe_1| = |Re_1| \geq 1 - \delta_1(d)$ for an appropriately small $\delta_1(d) > 0$. □

The previous lemma provided a lower bound for the volume of the section if the vector v has the form $\frac{Pe_1}{|Pe_1|}$. We will now extend this bound to the vectors which are close to this one.

Lemma 2.2. *For any $d \in \mathbb{N}$, there exist $\delta_2(d), \varepsilon_2(d)$ such that if $|Pe_1| \geq 1-\delta_1(d)$ and $v = \frac{Pe_1}{|Pe_1|}$, then for any $w \in E^\perp$ with $w \perp v$, $|w| < \delta_2(d)$,*

$$\mathrm{vol}_{n-d}\left(Q_n\cap\left(\frac{1}{2}v+w+E\right)\right)\geq\varepsilon_2(d).$$

Proof. By Lemma 2.1,

$$\mathrm{vol}_{n-d}\left(Q_n\cap\left(\frac{1}{2}v+E\right)\right)\geq\varepsilon_1(d).$$

Also, applying the same lemma to the linear subspace $\tilde{E} := \mathrm{span}(w, E)$, we get

$$vol_{n-d}\left(Q_n\cap\left(\frac{1}{2}v+\tilde{E}\right)\right)\geq\varepsilon_1(d-1).$$

Define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x)=\mathrm{vol}_{n-d}\left(Q_n\cap\left(\frac{1}{2}v+x\frac{w}{|w|}+E\right)\right).$$

Then the previous inequalities read $h(0) \geq \varepsilon_1(d)$, $\int_{\mathbb{R}} h(x)\,dx \geq \varepsilon_1(d-1)$. Assume that $h(|w|) \leq h(0)/2$. Since the function h is even and log-concave, this implies

$h(k|w|) \le 2^{-|k|}h(0)$ for all $k \in \mathbb{Z}$, and hence

$$\varepsilon_1(d-1) \le \int_{\mathbb{R}} h(x)\,dx \le 4|w| \cdot h(0) \le 4|w| \cdot \mathrm{vol}_{n-d}\,(Q_n \cap E) \le 4|w| \cdot (\sqrt{2})^d,$$

where we used Ball's theorem [B1] in the last inequality. This means that the statement of the lemma holds with

$$\delta_2(d) = \frac{\varepsilon_1(d-1)}{4(\sqrt{2})^d} \quad \text{and} \quad \varepsilon_2(d) = \frac{\varepsilon_1(d)}{2}$$

since for $|w| < \delta_2(d)$ we would get a contradiction to our assumption. Thus $h(|w|) > h(0)/2 \ge \varepsilon_2(d)$, so the proof is complete. □

We summarize Lemmas 2.1 and 2.2 in the following corollary.

Corollary 2.3. *For any $d \in \mathbb{N}$, there exist $\delta_3(d), \varepsilon_3(d)$ such that if $v \in E^\perp$, $|v| = 1$ and $\|v\|_\infty \ge 1 - \delta_3(d)$ then*

$$\mathrm{vol}_{n-d}\left(Q_n \cap \left(\frac{1}{2}v + E\right)\right) \ge \varepsilon_3(d).$$

Proof. Without loss of generality, assume that $v_1 = \langle v, e_1 \rangle \ge 1 - \delta$, where $\delta = \delta_3(d)$ will be chosen later. Then

$$|Pe_1| \ge \langle Pe_1, v \rangle = \langle e_1, v \rangle \ge 1 - \delta,$$

and

$$\begin{aligned} \left| v - \frac{Pe_1}{|Pe_1|} \right| &\le |v - Pe_1| + \left| Pe_1 - \frac{Pe_1}{|Pe_1|} \right| \\ &\le \left(|v|^2 - 2\langle v, Pe_1 \rangle + |Pe_1|^2 \right)^{1/2} + \left(1 - |Pe_1|\right) \\ &\le (2 - 2(1-\delta))^{1/2} + \delta. \end{aligned}$$

This means that choosing δ small enough, we can ensure that the conditions of Lemma 2.2 are satisfied. □

Let $X = P\xi$, where ξ is a random vector uniformly distributed in Q_n. The density f_X of the vector X is even and log-concave, so the set $D := \{y \in E^\perp : \ f_X(y) \ge f_X(\frac{1}{2}v)\}$ is convex and symmetric. We need the following simple lemma which would allow us to reduce the estimate of the density of a multi-dimensional projection to a bound on a probability of a half-space.

Lemma 2.4. *Let*

$$D := \left\{ y \in E^\perp : \ f_X(y) \ge f_X\left(\frac{1}{2}v\right) \right\}.$$

Let $S \subset E^\perp$ be a supporting hyperplane to D at v in $E^\perp$, and write $S = \tau u + L$, where L is a linear subspace of $E^\perp$, $u \in E^\perp \cap S^{n-1}$ satisfies $u \perp L$, and $\tau \ge 0$. Then $\tau \le \frac{1}{2}$ and

$$f_X\left(\frac{1}{2}v\right) \ge \max\left(f_X(\tau u), c(d)(\mathbb{P}(\langle \xi, u \rangle \ge \tau))^{1+d/2} \right)$$

for some $c(d) > 0$.

Proof. The inequalities $\tau \leq \frac{1}{2}$ and $f_X(\frac{1}{2}v) \geq f_X(\tau u)$ follows immediately from $\tau u \in S$ and the convexity of D.

To prove the other inequality, denote $\nu = \mathbb{P}(\langle \xi, u\rangle \geq \tau)$ and set

$$K := \left\{ y \in E^\perp : \ \langle y, u\rangle \geq \tau \text{ and } |y| \leq \sqrt{\frac{d}{\nu}} \right\}.$$

Note that $\mathbb{E}\,|X|^2 = \mathbb{E}\,|P\xi|^2 = \sum_{j=1}^n |Pe_j|^2\,\mathbb{E}\,\xi_j^2 = \frac{d}{12}$. Using Markov's inequality, we get

$$\mathbb{P}(X \in K) \geq P(\langle X, u\rangle \geq \tau) - \mathbb{P}\left(|X| \leq \sqrt{\frac{d}{\nu}}\right) \geq \nu - \frac{\mathbb{E}\,|X|^2}{d/\nu} \geq \frac{\nu}{2}.$$

For any $y \in K,\ f_X(y) \leq f_X(\frac{1}{2}v)$ since $K \subset E^\perp \setminus D$. Therefore

$$f_X\left(\frac{1}{2}v\right) \geq \frac{\mathbb{P}(X \in K)}{\mathrm{vol}_d(K)}.$$

As $\mathrm{vol}_d(K) \leq \left(\sqrt{d/\nu}\right)^d \mathrm{vol}_d(B_2^d) \leq C(d)\nu^{-d/2}$, the lemma follows. ☐

To use Lemma 2.4, we have to bound $\mathbb{P}(\langle \xi, u\rangle \geq \tau)$ for a unit vector $u \in S^{n-1}$. This bound is obtained differently depending on whether the vector u is close to a low-dimensional space. We consider the case when it is far from such spaces, i.e., it has enough mass supported on small coordinates. The opposite case will be considered in the proof of Theorem 1.1.

Lemma 2.5. *Let $u \in S^{n-1}$, and let ξ be a random vector uniformly distributed in Q_n. For any $\varepsilon > 0$, there exist $\delta, \eta > 0$ such that if $J_\delta = \{j : \ |u_j| < \delta\}$ and $\sum_{j \in J_\delta} u_j^2 > \varepsilon^2$ then*

$$\mathbb{P}(\langle \xi, u\rangle \geq 1) \geq \eta.$$

Proof. Denote $Y = \sum_{j\in J_\delta} \xi_j u_j,\ Z = \sum_{j \notin J_\delta} \xi_j u_j$, where $\xi_1, \dots, \xi_n$ are i.i.d. random variables uniformly distributed in $[-\frac{1}{2}, \frac{1}{2}]$.

Let g be the standard normal random variable. By the Berry-Esseen theorem,

$$\begin{aligned}\mathbb{P}(\sum_{j\in J_\delta} \xi_j u_j \geq 1) &\geq \mathbb{P}(\sqrt{\sum_{j\in J_\delta} u_j^2} \cdot g \geq 1) - c \max_{j \in J_\delta} \frac{|u_j|}{\sqrt{\sum_{j\in J_\delta} u_j^2}} \\ &\geq \mathbb{P}\left(g \geq \frac{1}{\varepsilon}\right) - c\frac{\delta}{\varepsilon} \geq \tilde{\eta} > 0\end{aligned}$$

if $\delta = \delta(\varepsilon)$ is chosen sufficiently small. Hence,

$$\mathbb{P}(\langle \xi, u\rangle \geq 1) \geq \mathbb{P}\left(Y \geq 1 \text{ and } Z \geq 0\right) \geq \tilde{\eta} \cdot \frac{1}{2} =: \eta,$$

since Y and Z are independent. The lemma is proved. ☐

Having proved these lemmas, we can derive Theorem 1.1.

Proof of Theorem 1.1. Recall that

$$\mathrm{vol}_{n-d}\left(Q_n \cap \left(\frac{1}{2}v + E\right)\right) = f_X\left(\frac{1}{2}v\right),$$

where $X = P\xi$ and ξ is a random vector uniformly distributed in Q_n.

Let u and τ be as in Lemma 2.4. Take

$$\varepsilon = \sqrt{\frac{\varepsilon_3(d)}{2}}$$

and choose the corresponding δ from Lemma 2.5. Define J_δ as in this lemma. If $\sum_{j\in J_\delta} u_j^2 \geq \varepsilon^2$, then by Lemmas 2.4 and 2.5,

$$f_X\left(\frac{1}{2}v\right) \geq c(d)\big(\mathbb{P}(\langle \xi, u\rangle \geq \tau)\big)^{1+d/2} \geq c(d)\eta^{1+d/2}$$

as $\tau \in [0, \frac{1}{2}]$.

Assume now that $\sum_{j\in J_\delta} u_j^2 \leq \varepsilon^2$. If $\|u\|_\infty \geq 1 - \varepsilon_3(d)$, then the statement of the theorem follows from Corollary 2.3 since $f_X(\frac{1}{2}v) \geq f_X(\tau u) \geq f_X(\frac{1}{2}u)$. Thus, we can assume that

$$\|u\|_\infty \leq 1 - \varepsilon_3(d). \tag{2.1}$$

We will use the inequality

$$f_X\left(\frac{1}{2}v\right) \geq c(d)\big(\mathbb{P}(\langle \xi, u\rangle \geq \tau)\big)^{1+d/2} \geq c(d)\left(\mathbb{P}\left(\langle \xi, u\rangle \geq \frac{1}{2}\right)\right)^{1+d/2}$$

again. This shows that to prove the theorem, it is enough to bound $\mathbb{P}\left(\langle \xi, u\rangle \geq \frac{1}{2}\right)$ from below by a quantity depending only on d.

Decompose $\langle \xi, u\rangle = Y + Z$ where $Y = \sum_{j\in J_\delta} \xi_j u_j$, $Z = \sum_{j\notin J_\delta} \xi_j u_j$ as above. Then

$$\begin{aligned}\mathbb{P}\left(\langle \xi, u\rangle \geq \frac{1}{2}\right) &\geq \mathbb{P}\left(Z \geq \frac{1}{2} \text{ and } Y \geq 0\right) = \frac{1}{2}\mathbb{P}\left(Z \geq \frac{1}{2}\right)\\ &= \frac{1}{2}\mathbb{P}\left(\sum_{j\notin J_\delta} \xi_j w_j \geq \theta\right)\end{aligned}$$

where

$$w_j = \frac{u_j}{\sqrt{\sum_{j\notin J_\delta} u_j^2}} \quad \text{and} \quad \theta = \frac{1}{2\sqrt{\sum_{j\notin J_\delta} u_j^2}}.$$

Note that

$$k := |[n] \setminus J_\delta| = |\{j \in [n] : \ |u_j| \geq \delta\}| \leq \delta^{-2},$$

where δ depends only on d. To simplify the notation, assume that $[n]\setminus J_\delta = [k]$. We can recast $\mathbb{P}(\sum_{j\notin J_\delta} \xi_j w_j \geq \theta)$ as

$$\mathbb{P}(\sum_{j\notin J_\delta} \xi_j w_j \geq \theta) = \mathrm{vol}_k(Q_k \cap (w_+^\perp + \theta w)),$$

where $Q_k = [-\frac{1}{2}, \frac{1}{2}]^k$, $w \in S^{k-1}$ is the vector with coordinates w_j, $j \in [k]$, and $w_+^\perp = \{y \in \mathbb{R}^k : \ \langle y, w\rangle \geq 0\}$ is a half-space orthogonal to w. Previously, we reformulated a geometric problem of bounding the volumes of non-central sections of the cube in a probabilistic language. Here, we reduce it back to a similar geometric problem but in dimension k which depends only on d and codimension 1.

By our assumption,

$$\theta \le \frac{1}{2\sqrt{1-\varepsilon^2}},$$

and, in view of (2.1), we have

$$\frac{\sqrt{1-\varepsilon^2}}{2(1-\varepsilon_3(d))} w \in Q_k.$$

Therefore,

$$\begin{aligned}
&\mathrm{vol}_k(Q_k \cap (w_+^\perp + \theta w)) \\
&\ge \mathrm{vol}_k \left(\mathrm{conv} \left(Q_k \cap w^\perp, \frac{\sqrt{1-\varepsilon^2}}{2(1-\varepsilon_3(d))} w \right) \cap (w_+^\perp + \theta w) \right) \\
&= \left(1 - \theta \left(\frac{\sqrt{1-\varepsilon^2}}{2(1-\varepsilon_3(d))} \right)^{-1} \right)^k \cdot \mathrm{vol}_k \left(\mathrm{conv} \left(Q_k \cap w^\perp, \frac{\sqrt{1-\varepsilon^2}}{2(1-\varepsilon_3(d))} w \right) \right) \\
&\ge \left(1 - \frac{1-\varepsilon_3(d)}{1-\varepsilon^2} \right)^k \cdot \frac{1}{k!} \left(\frac{\sqrt{1-\varepsilon^2}}{2(1-\varepsilon_3(d))} \right)^k \mathrm{vol}(Q_k \cap w^\perp) \\
&\ge \left(1 - \frac{1-\varepsilon_3(d)}{1-\varepsilon^2} \right)^k \cdot \frac{1}{k!} \left(\frac{\sqrt{1-\varepsilon^2}}{2(1-\varepsilon_3(d))} \right)^k,
\end{aligned}$$

where the last inequality follows from Vaaler's theorem [Va]. Recalling that $\varepsilon = \sqrt{\varepsilon_3(d)/2}$ and k depends only on d, we see that the quantity above is positive and depends only on d as well. This completes the proof of the theorem. □

3. Volume formulas

To prove Theorem 1.2, we start with the following known volume formulas.

Proposition 3.1. *Let* $a \in \mathbb{R}_+^n,\ |a| = 1,\ t \ge 0$. *Then*

$$A_{\mathbb{R}}(a,t) = \frac{2}{\pi} \int_0^\infty \prod_{j=1}^n \frac{\sin(a_j s)}{a_j s} \cos(ts)\, ds\ , \tag{3.1}$$

$$A_{\mathbb{C}}(a,t) = \frac{1}{2} \int_0^\infty \prod_{j=1}^n j_1(a_j s)\, J_0(ts)\, s\, ds\ ,\ j_1(x) := 2\frac{J_1(x)}{x}\ . \tag{3.2}$$

Here J_0 *and* J_1 *denote the standard Bessel functions.*

Formula (3.1) whose multidimensional version was used in the previous section can be found in Ball's paper [B] on cubic sections, equation (3.2) in Oleszkiewicz, Pelczyński [OP]. The case $t = 0$ of (3.1) goes back to Pólya [P]. A Fourier analytic proof of Proposition 3.1 is outlined in König, Koldobsky [KK1], [KK2].

Due to the oscillating character of the integrands in (3.1) and (3.2), it is difficult to find non-trivial lower bounds for $A(a,t)$ using these equations. Therefore we first prove different formulas for $A(a,t)$.

Proposition 3.2. *Let $(\Omega, \mathbb{P})$ be a probability space and $U_j : \Omega \to S^{k-1} \subset \mathbb{R}^k$, $j = 1, \cdots, n$ be a sequence of independent, random vectors uniformly distributed on the sphere S^{k-1}, where $k = 3$ if $\mathbb{K} = \mathbb{R}$ and $k = 4$ if $\mathbb{K} = \mathbb{C}$. Then for any $a \in \mathbb{R}^n_+$, $|a| = 1$ and $t \geq 0$*

$$(a) \quad A_{\mathbb{R}}(a,t) = \int\limits_{|\sum_{j=1}^n a_j U_j| \geq t} \frac{d\mathbb{P}}{|\sum_{j=1}^n a_j U_j|} \ ,$$

$$(b) \quad A_{\mathbb{C}}(a,t) = \int\limits_{|\sum_{j=1}^n a_j U_j| \geq t} \frac{d\mathbb{P}}{|\sum_{j=1}^n a_j U_j|^2} \ .$$

Proof. (a) Let m denote the normalized Lebesgue surface measure on $S^{k-1} \subset \mathbb{R}^k$ for $k \in \mathbb{N}$, $k \geq 2$. Then for any fixed vector $e \in S^{k-1}$

$$\int_{S^{k-1}} \exp(it < e, u >) \ dm(u) = \frac{\int_0^\pi \cos(t\cos(\phi)) \ \sin(\phi)^{k-2} \ d\phi}{\int_0^\pi \sin(\phi)^{k-2} \ d\phi} = j_{\frac{k}{2}-1}(t) \ ,$$

$j_{\frac{k}{2}-1}(t) = 2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})\frac{J_{\frac{k}{2}-1}(t)}{t^{\frac{k}{2}-1}}$, $t > 0$. Again, $J_{\frac{k}{2}-1}$ denote the standard Bessel functions of index $\frac{k}{2} - 1$. In particular, for $k = 3$ and $k = 4$

$$(3.3) \quad \int_{S^2} \exp(it < e, u >) \ dm(u) = \frac{\sin(t)}{t} \ , \ \int_{S^3} \exp(it < e, u >) \ dm(u) = j_1(t) \ .$$

We may assume that a has at least two non-zero coordinates a_j since otherwise the formulas in (a) and (b) just state $1 = 1$ if $t \leq 1$ and $0 = 0$ if $t > 1$. By (3.3)

$$(3.4) \qquad \prod_{j=1}^n \frac{\sin(a_j s)}{a_j s} = \int_{(S^2)^n} \exp(is < e, \sum_{j=1}^n a_j u_j >) \ dm(u_1) \cdots dm(u_n) \ .$$

This is $O(\frac{1}{s^2})$ as $s \to \infty$, therefore Lebesgue-integrable on $(0, \infty)$. Since (3.4) holds for all $e \in S^2$, we may integrate over e. Using (3.3) again, we find

$$\prod_{j=1}^n \frac{\sin(a_j s)}{a_j s} = \int_{(S^2)^n} \frac{\sin(|\sum_{j=1}^n a_j u_j| s)}{|\sum_{j=1}^n a_j u_j| s} \ dm(u) \ , \ dm(u) := \prod_{j=1}^n dm(u_j) \ .$$

The factor $|\sum_{j=1}^n a_j u_j|$ results from the necessary normalization $\frac{\sum_{j=1}^n a_j u_j}{|\sum_{j=1}^n a_j u_j|} \in S^2$. Hence, using Proposition 3.1,

$$\begin{aligned} A_{\mathbb{R}}(a,t) &= \frac{2}{\pi} \int_0^\infty \left(\int_{(S^2)^n} \frac{\sin(|\sum_{j=1}^n a_j u_j| s)}{|\sum_{j=1}^n a_j u_j| s} \ \cos(ts) \ dm(u) \right) \ ds \\ &= \int_{(S^2)^n} \left(\frac{2}{\pi} \int_0^\infty \frac{\sin(|\sum_{j=1}^n a_j u_j| s)}{|\sum_{j=1}^n a_j u_j| s} \ \cos(ts) \ ds \right) \ dm(u) \\ (3.5) \qquad &= \int\limits_{(S^2)^n, \ |\sum_{j=1}^n a_j u_j| \geq t} \frac{dm(u)}{|\sum_{j=1}^n a_j u_j|} \ , \end{aligned}$$

using that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(As)}{As}\ \cos(ts)\ ds = \left\{ \begin{array}{ccc} 0 & , & 0 < A < t \\ \frac{1}{A} & , & A > t > 0 \end{array} \right\}. \tag{3.6}$$

Note that $m(|\sum_{j=1}^n a_j u_j| = t) = 0$ since a has at least two non-zero coordinates. The integral in (3.6) is only conditionally convergent, which requires justification of interchanging the order of integration in (3.5). This is allowed if

$$\lim_{N\to\infty} \int_{(S^2)^n} \left(\frac{2}{\pi}\int_N^\infty \frac{\sin(|\sum_{j=1}^n a_j u_j| s)}{|\sum_{j=1}^n a_j u_j| s}\ \cos(ts)\ ds \right)\ dm(u) = 0 \tag{3.7}$$

is shown. We have in terms of the Sine integral Si, $Si(x) := \frac{2}{\pi}\int_0^x \frac{\sin(t)}{t}\ dt,\ x \in \mathbb{R}$ that

$$\frac{2}{\pi}\int_N^\infty \frac{\sin(As)}{As}\ \cos(ts)\ ds = \left\{ \begin{array}{ccc} \frac{Si((t-A)N)-Si((t+A)N)}{\pi A} & , & 0 < A < t \\ \frac{\pi - Si((A-t)N)-Si((A+t)N)}{\pi A} & , & A > t > 0 \end{array} \right\}$$

and hence

$$\begin{aligned}
&\int_{(S^2)^n} \left(\frac{2}{\pi}\int_N^\infty \frac{\sin(|\sum_{j=1}^n a_j u_j| s)}{|\sum_{j=1}^n a_j u_j| s}\ \cos(ts)\ ds \right)\ dm(u) \\
&= \int\limits_{(S^2)^n, |\sum_{j=1}^n a_j u_j| < t} \frac{1}{\pi |\sum_{j=1}^n a_j u_j|}\ \left[Si\left((t - |\sum_{j=1}^n a_j u_j|) N \right) \right. \\
&\qquad\qquad \left. - \ Si\left((t + |\sum_{j=1}^n a_j u_j|) N \right) \right]\ dm(u) \\
&+ \int\limits_{(S^2)^n, |\sum_{j=1}^n a_j u_j| > t} \frac{1}{\pi |\sum_{j=1}^n a_j u_j|}\ \left[\pi - Si\left((|\sum_{j=1}^n a_j u_j| - t) N \right) \right. \\
&\qquad\qquad \left. - \ Si\left((t + |\sum_{j=1}^n a_j u_j|) N \right) \right]\ dm(u)
\end{aligned}$$

Since for all $b \in S^2$ and $\beta > 0$

$$\int_{S^2} \frac{dm(u_1)}{|b + \beta u_1|} = \frac{1}{2}\int_{-1}^1 (|b|^2 + \beta^2 + 2\beta |b| v)^{-\frac{1}{2}}\ dv = \left\{ \begin{array}{ccc} \frac{1}{|b|} & , & 0 < \beta < |b| \\ \frac{1}{\beta} & , & \beta > |b| > 0 \end{array} \right\} < \infty$$

and since the Si-function is bounded in modulus by 2, the two integrands involving the Si-function are bounded in modulus by an integrable function independent of N. Since for any $c > 0$ we have that $\lim_{N\to\infty} Si(cN) = \frac{\pi}{2}$, the integrands converge to 0 pointwise. By the Lebesgue theorem, (3.7) follows and (3.5) is proven.

(b) Using (3.3), we find similarly as in (a)

$$\prod_{j=1}^{n} j_1(a_j s) = \int_{(S^3)^n} j_1(|\sum_{j=1}^{n} a_j u_j| s) \ dm(u) \ , \ dm(u) := \prod_{j=1}^{n} dm(u_j),$$

and by Proposition 3.1

$$\begin{aligned} A_{\mathbb{C}}(a,t) &= \frac{1}{2} \int_0^\infty \left(\int_{(S^3)^n} j_1(|\sum_{j=1}^{n} a_j u_j| s) \ J_0(ts) \ dm(u) \right) \ s \ ds \\ &= \int_{(S^3)^n} \left(\frac{1}{2} \int_0^\infty j_1(|\sum_{j=1}^{n} a_j u_j| s) \ J_0(ts) \ s \ ds \right) \ dm(u) \\ &= \int_{(S^3)^n} \left(\int_0^\infty J_1(|\sum_{j=1}^{n} a_j u_j| s) \ J_0(ts) \ ds \right) \ \frac{dm(u)}{|\sum_{j=1}^{n} a_j u_j|} \\ &= \int_{(S^3)^n, \ |\sum_{j=1}^{n} a_j u_j| \geq t} \frac{dm(u)}{|\sum_{j=1}^{n} a_j u_j|^2} \ , \end{aligned} \tag{3.8}$$

since by Gradstein, Ryshik [GR], 6.51.

$$\int_0^\infty J_1(As) \ J_0(ts) \ ds = \left\{ \begin{array}{ccc} 0 & , & 0 < A < t \\ \frac{1}{A} & , & A > t > 0 \end{array} \right\}, \tag{3.9}$$

which is a conditionally convergent integral. To justify exchanging the order of integration in (3.8), we employ the product formula for Bessel functions

$$J_0(u) \ J_0(v) = \frac{1}{\pi} \int_0^\pi J_0(\sqrt{u^2 + v^2 + 2uv\cos(\phi)} \ ) \ d\phi \ , \ u, v \in \mathbb{R} \ ,$$

cf. Watson [W], 11.1. Since $J_0' = -J_1$, differentiating this with respect to u, inserting $u = As$, $v = ts$ and integrating with respect to s yields that for all $N, A, t > 0$

$$\begin{aligned} &\left| \int_0^N J_1(As) J_0(ts) \ ds \right| \\ &= \left| \frac{1}{\pi} \int_0^\pi \left(\int_0^N J_1(\sqrt{A^2 + t^2 + 2At\cos(\phi)} s) ds \right) \frac{A + t\cos(\phi)}{\sqrt{A^2 + t^2 + 2At\cos(\phi)}} d\phi \right| \\ &= \left| \frac{1}{\pi} \int_0^\pi \left(1 - J_0(\sqrt{A^2 + t^2 + 2At\cos(\phi)} N) \right) \frac{A + t\cos(\phi)}{A^2 + t^2 + 2At\cos(\phi)} d\phi \right| \\ &\leq \frac{2}{\pi} \int_0^\pi \frac{|A + t\cos(\phi)|}{A^2 + t^2 + 2At\cos(\phi)} \ d\phi =: I(A,t) \ , \end{aligned}$$

where we also used that $|J_0| \leq 1$ holds. Since

$$\int \frac{A + t\cos(\phi)}{A^2 + t^2 + 2At\cos(\phi)} \ d\phi = \frac{1}{A} \left(\frac{\phi}{2} + \arctan(\frac{A - t}{A + t} \tan(\frac{\phi}{2})) \right) =: \Psi(\phi) \ ,$$

we find for $A > t$ that $I(A,t) = \frac{2}{A}$, $I(A,A) = \frac{1}{A}$ and for $A < t$ that $I(A,t) = \frac{2}{\pi} 2\Psi(\phi_0) \leq \frac{2}{A}$ where $\cos(\phi_0) = -\frac{A}{t}$. Thus $I(A,t) \leq \frac{2}{A}$ which implies using (3.9)

$$\left| \int_N^\infty J_1(As)\ J_0(ts)\ ds \right| \leq \frac{3}{A}\ .$$

Moreover $\lim_{N\to\infty} \int_N^\infty J_1(As)\ J_0(ts)\ ds = 0$ pointwise and $\int_{(S^3)^n} \frac{dm(u)}{|\sum_{j=1}^n a_j u_j|^2} < \infty$, so that we find similarly as in part (a)

$$\lim_{N\to\infty} \int_{(S^3)^n} \left(\int_N^\infty J_1(|\sum_{j=1}^n a_j u_j|s)\ J_0(ts)\ ds \right) dm(u) = 0\ ,$$

and (3.8) follows. We basically replaced the Si-function in part (a) by $\int_0^x J_1(t)\ dt = 1 - J_0(x)$.
Formulas (3.5) and (3.8) yield a concrete realization of the formulas in Proposition 3.2 involving independent, uniformly distributed random vectors on S^{k-1} for $k = 3, 4$. □

4. Exponential estimates and Orlicz spaces duality

To prove Theorem 1.2, we use lower estimates for the probability that certain quadratic forms of random variables on spheres S^{k-1} are non-negative. In this section, we develop a new method of estimating such probabilities. The estimate itself will be obtained in the next section. Our bound relies on the estimate of the norm of the quadratic form in the Orlicz space whose Orlicz function is of an exponential type. The Orlicz function we use is close to the ψ_1 function used in the large deviations theory. The lower bound on probability is obtained in terms of the norm of the indicator function in the dual of this Orlicz space.

We start with a simple lemma showing that a random vector uniformly distributed over the sphere is subgaussian.

Lemma 4.1. *Let U be a random vector uniformly distributed in S^{k-1}. Then the vector U is $(1/\sqrt{k})$-subgaussian, i.e.,*

$$\forall y \in \mathbb{R}^k \quad \mathbb{E}\exp(\langle U, y\rangle) \leq \mathbb{E}\exp\left(\frac{|y|^2}{\sqrt{k}} g\right),$$

where $g \in \mathbb{R}$ denotes the standard normal random variable.

Proof. Due to the rotational invariance, we can assume that $y = \lambda e_1$ for some $\lambda > 0$. Notice that for any $p \in \mathbb{N}$,

$$\mathbb{E}\, \langle U, e_1\rangle^{2p} \leq k^{-p}\, \mathbb{E}\, g^{2p}. \tag{4.1}$$

Indeed, denoting by $g^{(k)}$ the standard Gaussian vector in $\mathbb{R}^k$, we can write

$$\mathbb{E}\, g^{2p} = \mathbb{E} \left\langle g^{(k)}, e_1 \right\rangle^{2p} = \mathbb{E}\, |g^{(k)}|^{2p} \cdot \mathbb{E}\, \langle U, e_1\rangle^{2p}$$

and $\mathbb{E}\,|g^{(k)}|^{2p} \geq (\mathbb{E}\,|g^{(k)}|^2)^p = k^p$ by Jensen's inequality. Decomposing $e^{\lambda x}$ into Taylor series and using (4.1), we derive that

$$\mathbb{E}\exp(\lambda\langle U, e_1\rangle) \leq \mathbb{E}\exp\left(\frac{\lambda}{\sqrt{k}} g\right).$$

The result follows. $\square$

The next lemma provides an estimate of the Laplace transform of the relevant quadratic form.

Lemma 4.2. *Let $U_1, \ldots, U_n$ be i.i.d. random vectors uniformly distributed in S^{k-1}. Let $a = (a_1, \ldots, a_n) \in S^{n-1}$, and define*

$$S := \sum_{1\leq i<j\leq n} a_i a_j \langle U_i, U_j\rangle.$$

Then for any $\lambda \in (-\sqrt{k/2}, \sqrt{k/2})$,

$$\mathbb{E}\exp\left(\lambda\frac{S}{(\mathbb{E}\,S^2)^{1/2}}\right) \leq \left(1 - \frac{2\lambda^2}{k}\right)^{-k/2}.$$

Proof. To simplify the notation, let us estimate $\mathbb{E}\exp(\lambda S)$. Since S is a quadratic form of subgaussian vectors $U_1, \ldots, U_n$, such estimate can be derived from the Hanson-Writght inequality, see [HW, RV]. However, the bound obtained in this way would be too loose for our purposes. Instead, we will use the specific information about this quadratic form to obtain a tighter bound.

Our argument is based on a Laplace transform estimate as in [RV]. Let $g_1^{(k)}, \ldots, g_n^{(k)}$ be independent standard Gaussian vectors in $\mathbb{R}^k$. By Lemma 4.1

$$\mathbb{E}\exp(\langle U_n, y\rangle) \leq \mathbb{E}\exp\left(\frac{1}{\sqrt{k}}\left\langle g_n^{(k)}, y\right\rangle\right).$$

Using this inequality with fixed $U_1, \ldots, U_{n-1}$, we get

$$\begin{aligned}\mathbb{E}\exp(\lambda S) &= \mathbb{E}\exp\left(\lambda\sum_{1\leq i<j\leq n-1} a_i a_j\langle U_i, U_j\rangle + \left\langle\lambda\sum_{1\leq i<j\leq n-1} a_i a_j U_i, U_n\right\rangle\right)\\ &\leq \mathbb{E}\exp\left(\lambda\sum_{1\leq i<j\leq n-1} a_i a_j\langle U_i, U_j\rangle + \left\langle\lambda\sum_{1\leq i<j\leq n-1} a_i a_j U_i, \frac{g_n^{(k)}}{\sqrt{k}}\right\rangle\right),\end{aligned}$$

Repeating the same argument for other U_j, we obtain

$$\begin{aligned}\mathbb{E}\exp(\lambda S) &\leq \mathbb{E}\exp\left(\frac{\lambda}{k}\sum_{1\leq i<j\leq n} a_i a_j\left\langle g_i^{(k)}, g_j^{(k)}\right\rangle\right)\\ &= \left[\mathbb{E}\exp\left(\frac{\lambda}{k}\sum_{1\leq i<j\leq n} a_i a_j g_i g_j\right)\right]^k,\end{aligned}$$

where $g_1, \dots, g_n$ are i.i.d. $N(0,1)$ random variables. To derive the last equality, we notice that $\left\langle g_i^{(k)}, g_j^{(k)} \right\rangle$ is the sum of k i.i.d. random variables distributed like $g_i g_j$. The previous inequality can be rewritten as

$$\mathbb{E}\exp(\lambda S) \le \left[\mathbb{E}\exp\left(\frac{\lambda}{2k}(g^{(n)})^\top B g^{(n)}\right)\right]^k, \tag{4.2}$$

where $g^{(n)} = (g_1, \dots, g_n) \in \mathbb{R}^n$ is the standard Gaussian vector, and B is a symmetric $n \times n$ matrix with the entries $b_{i,j} = a_i a_j$ when $i \neq j$ and 0 otherwise, i.e.,

$$B = aa^\top - \operatorname{diag}(a_1^2, \dots, a_n^2).$$

Denote the eigenvalues of B by $\mu_1 \ge \dots \ge \mu_n$. Then by interlacing $\mu_1 > 0 \ge \mu_2 \ge \dots \ge \mu_n$. Also,

$$\mu_1 \le \|B\|_{\mathrm{HS}} = \left(\sum_{i \neq j} a_i^2 a_j^2\right)^{1/2}, \quad \text{and} \quad \sum_{j=1}^n \mu_j = \operatorname{tr}(B) = 0.$$

By the rotational invariance, we have

$$\mathbb{E}\exp\left(\frac{\lambda}{2k}(g^{(n)})^\top B g^{(n)}\right) = \mathbb{E}\exp\left(\frac{\lambda}{2k}\sum_{j=1}^n \mu_j g_j^2\right) = \prod_{j=1}^n \left(1 - \frac{\lambda \mu_j}{k}\right)^{-1/2}$$

provided that $\frac{\lambda \mu_j}{k} < 1$ for all $j \in [n]$. Since

$$|\mu_j| \le \|B\|_{\mathrm{HS}} = \sqrt{2k}\left(\frac{1}{k}\sum_{1 \le i < j \le n} a_i^2 a_j^2\right)^{1/2} = \sqrt{2k}\left(\mathbb{E}\, S^2\right)^{1/2}, \tag{4.3}$$

this restriction is satisfied if we assume that

$$\frac{\sqrt{2}|\lambda|}{\sqrt{k}}\left(\mathbb{E}\, S^2\right)^{1/2} < 1. \tag{4.4}$$

Assume that this restriction holds. Recall that $0 \ge \mu_2 \ge \dots \ge \mu_n$, and $\sum_{j=2}^n \mu_j = -\mu_1$. Applying the inequality

$$\prod_{j=2}^n (1 + y_j) \ge 1 + \sum_{j=2}^n y_j$$

valid for all $y_2, \dots y_n \in (-1, 1)$ having the same sign, we derive that

$$\prod_{j=1}^n \left(1 - \frac{\lambda\mu_j}{k}\right) \ge \left(1 - \frac{\lambda\mu_1}{k}\right) \cdot \left(1 - \sum_{j=2}^n \frac{\lambda\mu_j}{k}\right) = \left(1 - \frac{\lambda\mu_1}{k}\right) \cdot \left(1 + \frac{\lambda\mu_1}{k}\right)$$
$$= 1 - \left(\frac{\lambda\mu_1}{k}\right)^2.$$

In combination with (4.3), this yields

$$\mathbb{E}\exp\left(\frac{\lambda}{2k}(g^{(n)})^\top B g^{(n)}\right) \le \left(1 - \left(\frac{\lambda\mu_1}{k}\right)^2\right)^{-1/2} \le \left(1 - \frac{\lambda^2 \cdot 2\,\mathbb{E}\, S^2}{k}\right)^{-1/2}.$$

Taking into account (4.2), it shows that if (4.4) holds, then

$$\mathbb{E}\exp(\lambda S) \le \left(1 - \frac{\lambda^2 \cdot 2\,\mathbb{E}\, S^2}{k}\right)^{-k/2}.$$

The result follows if we replace λ by $\lambda/(\mathbb{E}\, S^2)^{1/2}$ in the inequality above. $\square$

We now use the duality of Orlicz norms to estimate the probability that $S > 0$.

Lemma 4.3. *Let Y be a real-valued random variable. Let $\lambda \in (0,1)$, and let $0 < q \le \mathbb{E}\, Y_+$. Assume that $\mathbb{E}\exp(\lambda Y_+) < \infty$. Then*

$$\mathbb{P}(Y > 0) \ge \left[\left(t + \frac{1}{\lambda q}\right)\log\left(1 + \frac{1}{\lambda q t}\right) - \frac{1}{\lambda q}\right]^{-1}$$

for any

$$0 < t \le \frac{1}{\mathbb{E}\exp(\lambda Y_+) - \lambda q - 1}.$$

Proof. Let $t > 0$. Define the functions $L, M : (0,\infty) \to (0,\infty)$ by

$$L(x) = t(e^x - x - 1), \quad M(x) = (t + x)\log\left(1 + \frac{x}{t}\right) - x, \quad x \in (0,\infty).$$

Then L and M are Orlicz functions. Denote by $\|\cdot\|_L$ the norm in the Orlicz space X_L. Then the dual norm is $\|\cdot\|_M$.

If t satisfies the assumption of the lemma, then $\mathbb{E}\, L(\lambda Y_+) \le 1$, and so

$$\|Y_+\|_L \le \frac{1}{\lambda}.$$

Hence, by duality of Orlicz norms,

$$q \le \mathbb{E}(Y_+ \cdot \mathbf{1}_{(0,\infty)}) \le \|Y_+\|_L \cdot \left\|\mathbf{1}_{(0,\infty)}(Y)\right\|_M \le \frac{1}{\lambda} \cdot \left\|\mathbf{1}_{(0,\infty)}(Y)\right\|_M,$$

or $\left\|\frac{1}{\lambda q}\mathbf{1}_{(0,\infty)}(Y)\right\|_M \ge 1$. This inequality reads

$$1 \le \left\|\frac{1}{\lambda q}\mathbf{1}_{(0,\infty)}(Y)\right\|_M = \left[\left(t + \frac{1}{\lambda q}\right)\log\left(1 + \frac{1}{\lambda q t}\right) - \frac{1}{\lambda q}\right] \cdot \mathbb{P}(Y > 0),$$

which proves the lemma. $\square$

In order to apply Lemma 4.3, we need an upper bound for $\mathbb{E}\exp(\lambda Y_+)$. This is our next task.

Lemma 4.4. *Let Y be a real-valued random variable such that $\mathbb{E}\, Y = 0$. Then for any $\lambda > 0$,*

$$\mathbb{E}\exp(\lambda Y_+) \le \mathbb{E}\exp(\lambda Y) + \frac{1}{4}\,\mathbb{E}\exp(-\lambda Y).$$

Proof. Denote $p = \mathbb{P}(Y \ge 0)$. Then

$$\mathbb{E}\exp(\lambda Y_+) = \mathbb{E}\exp(\lambda Y) + (1 - p) - \mathbb{E}[\exp(\lambda Y)\mathbf{1}_{(-\infty,0)}(Y)].$$

We can estimate the last term from below by Cauchy-Schwarz inequality:

$$\begin{aligned}(1-p)^2 &= \big(\mathbb{E}\,\mathbf{1}_{(-\infty,0)}(Y)\big)^2 \\ &\le \mathbb{E}[\exp(\lambda Y)\mathbf{1}_{(-\infty,0)}(Y)]\cdot\mathbb{E}[\exp(-\lambda Y)\mathbf{1}_{(-\infty,0)}(Y)] \\ &\le \mathbb{E}[\exp(\lambda Y)\mathbf{1}_{(-\infty,0)}(Y)]\cdot\mathbb{E}\exp(-\lambda Y).\end{aligned}$$

This implies that

$$\mathbb{E}\exp(\lambda Y_+) \le \mathbb{E}\exp(\lambda Y) + 1 - p - (1-p)^2\big(\mathbb{E}\exp(-\lambda Y)\big)^{-1}.$$

The proof finishes by maximizing this expression over $p \in \mathbb{R}$. □

Remark 4.5. If $\mathbb{E}\exp(-\lambda Y) > 2$, the maximum of the function above is attained outside of the interval $[0,1]$. In this case one can obtain a better bound

$$\mathbb{E}\exp(\lambda Y_+) \le \mathbb{E}\exp(\lambda Y) - \big(\mathbb{E}\exp(-\lambda Y)\big)^{-1} + 1$$

by taking $p = 0$. However, we are not going to use this improvement.

5. Tail estimates

To estimate $A(a,1)$ from below, we need a lower estimate of $\mathbb{P}(|\sum_{j=1}^n a_j U_j| \ge 1)$ and tail estimates for the random vectors $\sum_{j=1}^n a_j U_j$ in Proposition 3.2.

Proposition 5.1. *Let $(U_j)_{j=1}^n$ be a sequence of independent random vectors uniformly distributed on the sphere $S^{k-1} \subset \mathbb{R}^k$ for $k \ge 2$. Let $a \in \mathbb{R}^n_+$, $|a| = 1$. Then*

$$\mathbb{P}(|\sum_{j=1}^n a_j U_j| \ge 1) \ge \frac{2\sqrt{3}-3}{3+\frac{4}{k}} =: \gamma_k\ .$$

For $k = 3, 4$ we have the better numerical estimates

$$\mathbb{P}(|\sum_{j=1}^n a_j U_j| \ge 1) \ge 0.1268 \quad , \quad k = 3\ ,$$

$$\mathbb{P}(|\sum_{j=1}^n a_j U_j| \ge 1) \ge 0.1407 \quad , \quad k = 4\ .$$

Remark 5.2. (a) In the case of the Rademacher variables $(r_j)_{j=1}^n$, Oleszkiewicz [O] showed that

$$\mathbb{P}(|\sum_{j=1}^n a_j r_j| \ge 1) \ge \frac{1}{10}$$

holds. His beautiful scalar proof does not seem to generalize to our case of spherical variables.

(b) The estimate of Proposition 5.1 is not optimal. It is unclear whether the minimum occurs for $a^n = \frac{1}{\sqrt{n}}(1, \cdots, 1)$. In the Rademacher case, $k = 1$, this is not true, as Zhubr showed around 1995 for $n = 9$ (unpublished). For $k \ge 2$ and $n \to \infty$,

a^n yields that no better lower bound than the following is possible: By the central limit theorem

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} U_j \to \mathcal{N}(0, \Sigma) \ , \ \Sigma = \frac{1}{k} \mathrm{Id}_k$$

with density function $f(x) = \left(\frac{k}{2\pi}\right)^{\frac{k}{2}} \exp(-\frac{k}{2}|x|^2)$. Therefore

$$\lim_{n\to\infty} \mathbb{P}(\frac{1}{\sqrt{n}} | \sum_{j=1}^{n} U_j | \geq 1) = |S^{k-1}| \left(\frac{k}{2\pi}\right)^{\frac{k}{2}} \int_1^\infty r^{k-1} \exp(-\frac{k}{2} r^2) \ dr$$

$$= \frac{1}{\Gamma(\frac{k}{2})} \int_{\frac{k}{2}}^\infty s^{\frac{k}{2}-1} \exp(-s) \ ds =: \phi(k)$$

The sequence $(\phi(k))_{k\geq 2}$ is increasing, with

$$\phi(2) = \frac{1}{e} \simeq 0.3679 < \phi(3) \simeq 0.3916 < \phi(4) = \frac{3}{e^2} \simeq 0.4060 \text{ and } \lim_{k\to\infty} \phi(k) = \frac{1}{2} \ .$$

Proof of Proposition 5.1. (a) Let $S := \sum_{1\leq i<j\leq n} a_i a_j < U_i, U_j >$. Since $| \sum_{j=1}^n a_j U_j|^2 = 1 + 2S$,

$$\mathbb{P}(| \sum_{j=1}^{n} a_j U_j | \geq 1) = \mathbb{P}(S \geq 0) \ .$$

Since $\mathbb{E}(U_j) = 0$, $\mathbb{E}(S) = 0$. By Proposition 2.3 of Veraar's paper [V] on lower probability estimates for centered random variables we have the estimate

$$\mathbb{P}(S \geq 0) \geq (2\sqrt{3} - 3) \ \frac{\mathbb{E}(S^2)^2}{\mathbb{E}(S^4)} \ .$$

We claim that $(3 + \frac{4}{k}) \ \mathbb{E}(S^2)^2 \geq \mathbb{E}(S^4)$ so that the statement of Proposition 5.1 for general k (not being 3 or 4)

$$\mathbb{P}(S \geq 0) \geq \gamma_k$$

will follow.

(b) To prove the claim, we calculate $\mathbb{E}(S^2)$ and $\mathbb{E}(S^4)$.

$$\mathbb{E}(S^2) = \sum_{i<j} \sum_{l<m} a_i a_j a_l a_m \ \mathbb{E}(< U_i, U_j >< U_l, U_m >) \ .$$

The expectation terms on the right are non-zero only if $i = l < j = m$. Thus

$$\mathbb{E}(S^2) = \sum_{1\leq i<j\leq n} a_i^2 a_j^2 \ \mathbb{E}(< U_i, U_j >^2) = (\frac{1}{k} \sum_{1\leq i<j\leq n} a_i^2 a_j^2 \ )$$

since $\mathbb{E}(< U_i, U_j >^2) = \int_{S^{k-1}} v_1^2 \ dm(v) = \frac{1}{k} \int_{S^{k-1}} |v|^2 \ dm(v) = \frac{1}{k}$.

For $\mathbb{E}(S^4)$, we have to evaluate $\mathbb{E}(\prod_{l=1}^4 < U_{i_l}, U_{j_l} >)$ with $i_l < j_l$, $l = 1, 2, 3, 4$. By the independence of the variables U_j, this is non-zero only if products of squares, fourth powers or cyclic combinations show up in the index combinations, yielding cases such as

$$\mathbb{E}(< U_1, U_2 >^2 < U_3, U_4 >^2) = \mathbb{E}(< U_1, U_2 >^2) \ \mathbb{E}(< U_3, U_4 >^2) = \frac{1}{k^2} \ ,$$

$$\mathbb{E}(< U_1, U_2 >^2 < U_1, U_3 >^2) = \mathbb{E}(< U_1, U_2 >^2) \ \mathbb{E}(< U_1, U_3 >^2) = \frac{1}{k^2} \ ,$$

$$\mathbb{E}(< U_1, U_2 >^4) = \int_{S^{k-1}} v_1^4 \ dm(v) = \frac{\int_0^\pi \cos(t)^4 \ \sin(t)^{k-2} \ dt}{\int_0^\pi \sin(t)^{k-2} \ dt} = \frac{3}{k(k+2)} \ ,$$

or

$$\mathbb{E}(< U_1, U_2 >< U_2, U_3 >< U_3, U_4 >< U_1, U_4 >) = (\int_{S^{k-1}} v_1^2 \ dm(v))^3 = \frac{1}{k^3} \ .$$

Each product of squares $< U_i, U_j >^2 < U_l, U_m >^2$ with $i < j$, $l < m$ and $(i,j) \neq (l,m)$ occurs $\binom{4}{2} = 6$ times and each cyclic combination $4! = 24$ times in the fourth power expansion of S. Therefore

$$\mathbb{E}(S^4) = \frac{6}{k^2} (\sum_{i<j, l<m, (i,j)\neq(l,m)} a_i^2 a_j^2 a_l^2 a_m^2 \ ) + \frac{3}{k(k+2)} \sum_{i<j} a_i^4 a_j^4$$
$$+ \frac{24}{k^3} (\sum_{i<j<l<m} a_i^2 a_j^2 a_l^2 a_m^2 \ ) \ .$$

Expanding $(\sum_{i<j} a_i^2 a_j^2)(\sum_{l<m} a_l^2 a_m^2)$, besides cases of equalities of indices, increasing index combinations show up 6 times, namely $i < j < l < m, l < m < i < j, i < k < j < l, i < k < l < j, k < i < j < l, k < i < l < j$ which implies $24 \ (\sum_{i<j<l<m} a_i^2 a_j^2 a_l^2 a_m^2) \leq 4 \ (\sum_{i<j} a_i^2 a_j^2)^2$. Hence

$$\mathbb{E}(S^4) \leq (3 + \frac{4}{k}) \left(\frac{1}{k} \sum_{1 \leq i < j \leq n} a_i^2 a_j^2 \right)^2 - (\frac{3}{k^2} - \frac{3}{k(k+2)}) \sum_{1 \leq i < j \leq n} a_i^4 a_j^4$$
$$\leq (3 + \frac{4}{k}) \left(\frac{1}{k} \sum_{1 \leq i < j \leq n} a_i^2 a_j^2 \right)^2 = (3 + \frac{4}{k}) \ \mathbb{E}(S^2)^2 \ .$$

This proves the claim for general k. To prove the better numerical estimates for $k = 3, 4$, we use an Orlicz-space duality instead of the $L_2 - L_2$-duality employed by Veraar.

(c) Using the Lemmas of the previous section, we now prove the better estimates for $\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) = \mathbb{P}(S \geq 0)$ in the cases $k = 3, 4$. Set

$$Y = \frac{S}{(\mathbb{E}\, S^2)^{1/2}}.$$

Combining Lemmas 4.2 and 4.4, we obtain

$$\mathbb{E} \exp(\lambda Y_+) \leq \frac{5}{4} \left(1 - \frac{2\lambda^2}{k} \right)^{-k/2}$$

for any $\lambda \in (0, \sqrt{k/2})$. Also, since $\mathbb{E}\, S = 0$,

$$\mathbb{E}\, Y_+ = \frac{1}{2} \frac{\mathbb{E}\, |S|}{(\mathbb{E}\, S^2)^{1/2}} \geq \frac{1}{2} \left(\frac{(\mathbb{E}\, S^2)^2}{\mathbb{E}\, S^4} \right)^{1/2} \geq \frac{1}{2} \left(\frac{1}{3 + 4/k} \right)^{1/2} = \frac{1}{2} \sqrt{\frac{k}{3k+4}},$$

where we used Hölder's inequality and the moment estimate from part (b). Hence, Lemma 4.3 can be applied with

$$q = \frac{1}{2}\sqrt{\frac{k}{3k+4}} \quad \text{and} \quad t = \frac{1}{\frac{5}{4}\left(1-\frac{2\lambda^2}{k}\right)^{-k/2} - \lambda q - 1}.$$

Substituting these values in the estimate of Lemma 4.3, and using a numerical maximization of the right hand side estimate, we obtain the desired lower bounds 0.1268 with $\lambda \approx 0.7111$ for $k = 3$ and 0.1407 with $\lambda \approx 0.7508$ for $k = 4$. □

Remark 5.3. At the limit $k \to \infty$, our approach yields a bound

$$P(S \geq 0) > 0.205475,$$

which is about $\frac{1}{3}$ better than the bound $\frac{2}{\sqrt{3}} - 1 \ \sim 0.154700$ following from Veraar's inequality. For $k \geq 100$, our lower bound is greater than 0.2.

We will now consider the upper tail of $|\sum_{j=1}^n a_j U_j|$. A bound for the upper tail follows directly from Lemma 4.1 and the Hanson-Wright inequality. Yet, as we strive for good constants, we need a tighter estimate.

Proposition 5.4. *$(U_j)_{j=1}^n$ be a sequence of independent, random vectors uniformly distributed on the sphere S^{k-1} for $k \in \mathbb{N}$, $k \geq 2$. Let $a \in \mathbb{R}_+^n$, $|a| = 1$. Then for any $t > 1$*

$$\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq t) \leq t^k \exp(\frac{k}{2} - \frac{k}{2}t^2) \ .$$

Proof. The Khintchine inequality for the variables (U_j) states for any $p \geq 2$

$$||\sum_{j=1}^n a_j U_j||_{L_p(S^{k-1})} \leq b_{p,k}|a| = b_{p,k} := \sqrt{\frac{2}{k}} \left(\frac{\Gamma(\frac{p+k}{2})}{\Gamma(\frac{k}{2})}\right)^{\frac{1}{p}} \ ,$$

cf. König, Kwapień [KKw], Theorem 3. The constants $b_{p,k}$ are the best possible. We find for $c > 0$

$$\begin{aligned}\int_{S^{k-1}} \exp(c|\sum_{j=1}^n a_j U_j|^2) \ d\mathbb{P} &= \sum_{m=0}^\infty \frac{c^m}{m!} \int_{S^{k-1}} |\sum_{j=1}^n a_j U_j|^{2m} \ d\mathbb{P} \\ &\leq \sum_{m=0}^\infty \frac{c^m}{m!} \ b_{2m,k}^{2m} =: f_k(c) \ .\end{aligned}$$

We evaluate $f_k(c)$ explicitly. For $0 < c < \frac{k}{2}$, we have

$$f_k(c) = \sum_{m=0}^\infty \frac{c^m}{m!} \ \left(\frac{2}{k}\right)^m \ \frac{\Gamma(m+\frac{k}{2})}{\Gamma(\frac{k}{2})} = \sum_{m=0}^\infty \binom{-\frac{k}{2}}{m} \left(-\frac{2c}{m}\right)^m = \left(1 - \frac{2c}{k}\right)^{-\frac{k}{2}} \ .$$

Therefore for any fixed $t > 1$

$$\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq t) \exp(ct^2) \leq \int_{S^{k-1}} \exp(c|\sum_{j=1}^n a_j U_j|^2) \ d\mathbb{P} = f_k(c) \ ,$$

$$\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq t) \leq f_k(c) \exp(-ct^2) =: g_k(c) \ .$$

For a given $t > 1$, g_k is minimal for $\bar{c} = \frac{k}{2}(1 - \frac{1}{t^2})$. This yields

$$\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq t) \leq t^k \exp(\frac{k}{2} - \frac{k}{2} t^2) \ .$$

□

6. A lower bound for hyperplane sections

In this section, we prove Theorem 1.2.

We first consider the real case. By Proposition 3.2

$$A_{\mathbb{R}}(a, 1) = \int_{|\sum_{j=1}^n a_j U_j| \geq 1} \frac{d\mathbb{P}}{|\sum_{j=1}^n a_j U_j|} \ ,$$

where the $(U_j)_{j=1}^n$ are independent random vectors uniformly distributed on the sphere $S^2 \subset \mathbb{R}^3$. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1)^2 &= (\int_{|\sum_{j=1}^n a_j U_j| \geq 1} d\mathbb{P} \ )^2 \\
&\leq A_{\mathbb{R}}(a, 1) \int_{|\sum_{j=1}^n a_j U_j| \geq 1} |\sum_{j=1}^n a_j U_j| \ d\mathbb{P} \\
&= A_{\mathbb{R}}(a, 1) \int_0^\infty \mathbb{P}(|\sum_{j=1}^n a_j U_j| \ \chi_{[|\sum_{j=1}^n a_j U_j| \geq 1]} \geq s) \ ds \\
&= A_{\mathbb{R}}(a, 1) \ \left(\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + \int_1^\infty \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq s) \ ds \right) \ .
\end{aligned}$$

For any $t > 1$,

$$\begin{aligned}
\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1)^2 &\leq A_{\mathbb{R}}(a, 1) \ \left(t \ \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + \int_t^\infty \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq s) \ ds \right) \\
&\leq A_{\mathbb{R}}(a, 1) \ \left(t \ \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + \int_t^\infty s^3 \exp(\frac{3}{2} - \frac{3}{2} s^2) \ ds \right) \\
&= A_{\mathbb{R}}(a, 1) \ \left(t \ \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + \frac{3t^2 + 2}{9} \exp(\frac{3}{2} - \frac{3}{2} t^2) \right) \ ,
\end{aligned}$$

where we applied the tail estimate of Proposition 5.4 for $k = 3$. Therefore

$$A_{\mathbb{R}}(a,1) \geq \frac{\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1)^2}{t\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + \frac{3t^2+2}{9}\exp(\frac{3}{2} - \frac{3}{2}t^2)}$$
$$\geq \frac{\gamma_3^2}{t\gamma_3 + \frac{3t^2+2}{9}\exp(\frac{3}{2} - \frac{3}{2}t^2)} \ ,$$

with $\gamma_3 := 0.1268$, using Proposition 5.1 for $k = 3$. Choosing $t = 1.92$ yields

$$A_{\mathbb{R}}(a,1) \geq 0.05974 > \frac{1}{17} \ ,$$

which is the claim of Theorem 1.2 in the real case.

In the complex case, by Proposition 3.2

$$A_{\mathbb{C}}(a,1) = \int\limits_{|\sum_{j=1}^n a_j U_j| \geq 1} \frac{d\mathbb{P}}{|\sum_{j=1}^n a_j U_j|^2} \ ,$$

where the $(U_j)_{j=1}^n$ now are independent uniformly distributed random vectors on $S^3 \subset \mathbb{R}^4$. We then find, using Propositions 5.4 and 5.1 for $k = 4$

$$\begin{aligned}
&\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1)^2 = \Big(\int\limits_{|\sum_{j=1}^n a_j U_j| \geq 1} d\mathbb{P} \Big)^2 \\
&\leq A_{\mathbb{C}}(a,1) \int\limits_{|\sum_{j=1}^n a_j U_j| \geq 1} |\sum_{j=1}^n a_j U_j|^2 \, d\mathbb{P} \\
&= A_{\mathbb{C}}(a,1) \left(\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + 2 \int_1^\infty s \, \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq s) \, ds \right) \\
&\leq A_{\mathbb{C}}(a,1) \left(t^2 \, \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + 2 \int_1^\infty s \, \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq s) \, ds \right) \\
&\leq A_{\mathbb{C}}(a,1) \left(t^2 \, \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + 2 \int_t^\infty s^5 \exp(2 - 2s^2) \, ds \right) \\
&= A_{\mathbb{C}}(a,1) \left(t^2 \, \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + \frac{2t^4 + 2t^2 + 1}{4} \exp(2 - 2t^2) \right) \ ,
\end{aligned}$$

for any $t > 1$. Hence

$$A_{\mathbb{C}}(a,1) \geq \frac{\mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1)^2}{t^2 \mathbb{P}(|\sum_{j=1}^n a_j U_j| \geq 1) + \frac{2t^4+2t^2+1}{4} \exp(2-2t^2)}$$
$$\geq \frac{\gamma_4^2}{t^2 \gamma_4 + \frac{2t^4+2t^2+1}{4} \exp(2-2t^2)} ,$$

with $\gamma_4 := 0.1407$. Choosing $t = 1.77$ yields

$$A_{\mathbb{C}}(a,1) \geq 0.03699 > \frac{1}{28}$$

which proves Theorem 1.2 also in the complex case of the polydisc sections.

Remark 6.1. The estimates of Theorem 1.2 cannot be improved by more than a factor of $\simeq 5.2$ in the real case and by a factor of $\simeq 7.3$ in the complex case. Indeed, consider the diagonal directions. Let $a^n := \frac{1}{\sqrt{n}}(1, \cdots, 1) \in \mathbb{R}^n$, $|a^n| = 1$. For $n = 2, 3$ the vectors $a^2 \in \mathbb{R}^2$ and $a^3 \in \mathbb{R}^3$ yield the minimal values of hyperplane sections $A_{\mathbb{R}}(a,1)$, $|a| = 1$ in Q_2 and in Q_3,

$$A_{\mathbb{R}}(a^2,1) = \sqrt{2} - 1 \simeq 0.4142 > A_{\mathbb{R}}(a^3,1) = \frac{6\sqrt{3}-9}{4} \simeq 0.3481,$$

cf. König, Koldobsky [KK1]. It is unclear whether $A(a^n,1)$ provides the minimal value of hyperplane section volumes in Q_n for $n > 3$. Actually, the sequence $(A(a^n,1))_{n=2}^{\infty}$ is decreasing with

$$\lim_{n\to\infty} A_{\mathbb{R}}(a^n,1) = \frac{2}{\pi} \int_0^\infty \exp(-\frac{s^2}{6}) \ \cos(s) \ ds = \sqrt{\frac{6}{\pi e^3}} \simeq 0.3084 \ .$$

Therefore no improvement of the lower bound beyond $\sqrt{\frac{6}{\pi e^3}} \simeq 5.2 \cdot 0.05974$ is possible in the real case. In the complex case

$$\lim_{n\to\infty} A_{\mathbb{C}}(a^n,1) = \frac{1}{2} \int_0^\infty \exp(-\frac{s^2}{8}) \ J_0(s) \ s \ ds = \frac{2}{e^2} \simeq 0.2707 \ .$$

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