# Notes on Tsunamis

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# Contents

# Dispersion of Gravity Waves

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## 1 Introduction

Here we will try to obtain an equation for the dispersion of gravity waves, such as tsunamis. In gravity waves, the main driving force is gravity. That is to say, once a disturbance occurs in a fluid as a result of some external force (e.g. wind, tides, earthquakes, landslides, etc), it will propagate in the fluid with gravity as the main acting force. In order to find out how these waves behave in the spatial and/or temporal frequency domains, we will need to the behavior of their velocities (as the main evolution measure in fluids). To this, we will define a velocity potential (just to make life easier and make the equations look nicer) and using that, we will use the separation of variables method to solve the resulting differential equation.

## 2 Main Equations

As the first step, we will consider the Navier-Stokes equation for an incompressible, Newtonian flow,

$$\rho\frac{D\vec{u}}{Dt} = \rho\frac{\partial\vec{u}}{\partial t} + \rho\vec{u}.\nabla\vec{u} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{u} \tag{1}$$

where $\rho$ and $\mu$ are density and viscosity of the flow respectively, $\vec{g}$ is the acceleration due to gravity, $P$ is pressure, and $\vec{u}$ is the velocity of the flow. By neglecting the viscous term and ignoring *rotation* (i.e. $\vec{\nabla} \times \vec{u} = 0$), we can use Eq. (2) to modify Eq. (1) into Eq. (3):

$$\vec{u} \times \vec{\nabla} \times \vec{u} = \frac{1}{2}\vec{\nabla}(u)^2 + \vec{u}.\vec{\nabla}\vec{u} \tag{2}$$

$$\rho\frac{\partial\varphi}{\partial t} + \frac{1}{2}\rho u^2 = -\nabla P + \nabla(\rho g z) \tag{3}$$

Here, we have assumed a velocity field potential, $\varphi$, as

$$\varphi = \vec{\nabla}\vec{u} \tag{4}$$

and have tried to incorporate the gravity force in form of a potential as well. By rearranging Eq. (3) and factoring out the gradient, and then integrating in space, we get,

$$\nabla(\frac{\partial\varphi}{\partial t} + \frac{1}{2}u^2 + \frac{P}{\rho} + gz) = 0 \tag{5}$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}u^2 + \frac{P}{\rho} + gz = C(t) \tag{6}$$

where we have divided everything by $\rho$ for future convenience.

Now, if we try to approximate the solution by ignoring the second-order term and setting the constant $C(t)$ equal to zero, we obtain,

$$\frac{\partial\varphi}{\partial t} = -\frac{P}{\rho} - gz \tag{7}$$

**Boundary Conditions**

To find the value of $P$, we can use a boundary condition. We know that on the surface, pressure is a constant and since by knowing the value of pressure at any given point in a fluid, we can derive its value at any other point, we drop it out as a constant, since it is a constant on the surface.

$$\frac{\partial\varphi}{\partial t} = -gz + \text{constant} \tag{8}$$

Another boundary condition is that at the bottom of the sea, the normal component of velocity should vanish and thus, by considering the potential, $\varphi$, we have

$$\frac{\partial\varphi}{\partial z} = -\vec{u_n} = 0 \tag{9}$$

The time derivative of Eq. (8) is

$$\frac{\partial^2 \varphi}{\partial t^2} = -g\frac{\partial z}{\partial t} \tag{10}$$

which combining (9) will be

$$\frac{\partial^2 \varphi}{\partial t^2} = -g\vec{u_n} = -g\frac{\partial \varphi}{\partial z} \tag{11}$$

Eq. (11) is much more good-looking than the original Eqs. (6) or (7). To solve this, we will use the separation of variables method by assuming (an indeed bold but useful assumption!),

$$\varphi(x, y, z, t) = \psi(x, y)U(z)e^{i\omega t} \tag{12}$$

which if plugged into (11) will give

$$\omega^2 U(z) = g\frac{\partial U(z)}{\partial z} \tag{13}$$

Now, again by inserting Eq. (11) into the equation for conservation of mass (or the continuity equation), $\vec{\nabla} \times \vec{u} = 0$ which translates into the potential form as,

$$\nabla^2 \varphi = 0 \tag{14}$$

and dividing both sides by $\varphi$, we get

$$\frac{1}{\psi(x, y)}\Big[\frac{\partial^2 \psi(x, y)}{\partial x^2} + \frac{\partial^2 \psi(x, y)}{\partial y^2}\Big] + \frac{1}{U(z)}\frac{\partial^2 U(z)}{\partial z^2} = 0 \tag{15}$$

Now, since the first term in (15) is a function of only $x$ and $y$ and the second term is a function of only $z$, then their sum can only vanish if they are constants, one being the negative of the other. For convenience, we set the first term equal to $k^2$ and the second equal to $-k^2$ and thus getting two differential equations,

$$\frac{\partial^2\psi(x,y)}{\partial x^2}+\frac{\partial^2\psi(x,y)}{\partial y^2}=-k^2\psi(x,y) \tag{16}$$

$$\frac{\partial^2 U(z)}{\partial z^2}=k^2U(z) \tag{17}$$

which are wave equations, the solution to first one in a sinusoid and a hyperbolic function satisfies the second one.

$$\psi(x,y)=Ae^{i(k_x x+k_y y)}U(z)=\cosh k(z+z_0) \tag{18}$$

where $A$ and $z_0$ are integration constants. $k$ in (18) is the wave vector, pointing in the direction of propagation and has the size of,

$$|k|=\frac{2\pi}{\lambda} \tag{19}$$

Now, using Eqs. (9) and (12), we have

$$\frac{\partial\varphi}{\partial z}=\psi(x,y)e^{i\omega t}\frac{d}{dz}\Big[\cosh k(z+z_0)\Big]=0 \tag{20}$$

$$\frac{\partial\varphi}{\partial z}=\psi(x,y)e^{i\omega t}k\sinh k(z+z_0)=0 \tag{21}$$

which will vanish at the bottom of the sea ($z=-h$) if,

$$z_0=h \tag{22}$$

and so,

$$U(z)=\cosh k(z+h) \tag{23}$$

By inserting (23) into (13),

$$\omega^2 \cosh k(z+h) = kg \sinh k(z+h) \tag{24}$$

and solving for $\omega$ we get,

$$\omega(k) = \sqrt{gk \tanh k(z+h)} \tag{25}$$

Eventually, to get the frequency dependence of velocity, we use the definition of wave velocity as frequency over wave number,

$$c^2 = \frac{\omega^2}{k^2} = \frac{gk \tanh(2\pi/\lambda)(z+h)}{(2\pi/\lambda)^2} \tag{26}$$

which for the "unperturbed" surface $z = 0$ gives

$$c = \sqrt{\frac{g\lambda}{2\pi} \tanh(\frac{2\pi}{\lambda})h} \tag{27}$$

**Approximate Solutions**

Eq. (27) is the dispersion equation for gravity waves, including tsunamis. Three cases, regarding $\lambda$ and $h$ can rise,

**A.** $h \approx \lambda$
Eq. (27) holds and dispersion occurs.

**B.** $h >> \lambda$
As the argument of the hyperbolic tangent becomes very large, the function goess to 1 and thus,

$$c \approx \sqrt{\frac{g\lambda}{2\pi}} \tag{28}$$

**C.** $h << \lambda$
Known as the "shallow water approximation" will cause the argument of the hyperbolic tangent function to become very small and thus the function can be approximated by its argument. As a result, Eq. (27) will become,

$$c \approx \sqrt{gh} \tag{29}$$

which is the velocity of non-disperse gravity waves (e.g. non-dispersive tsunamis) in the ocean.

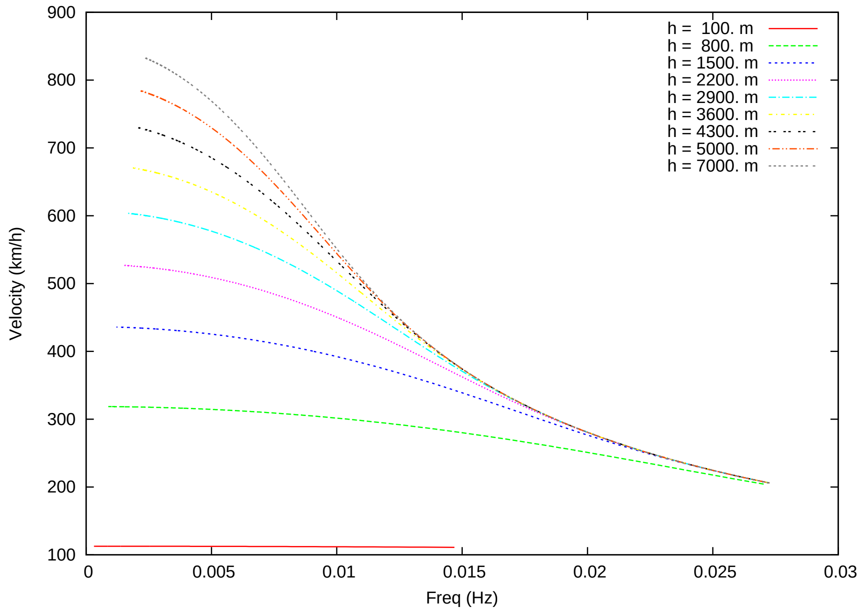
## 2.1 Final Remarks

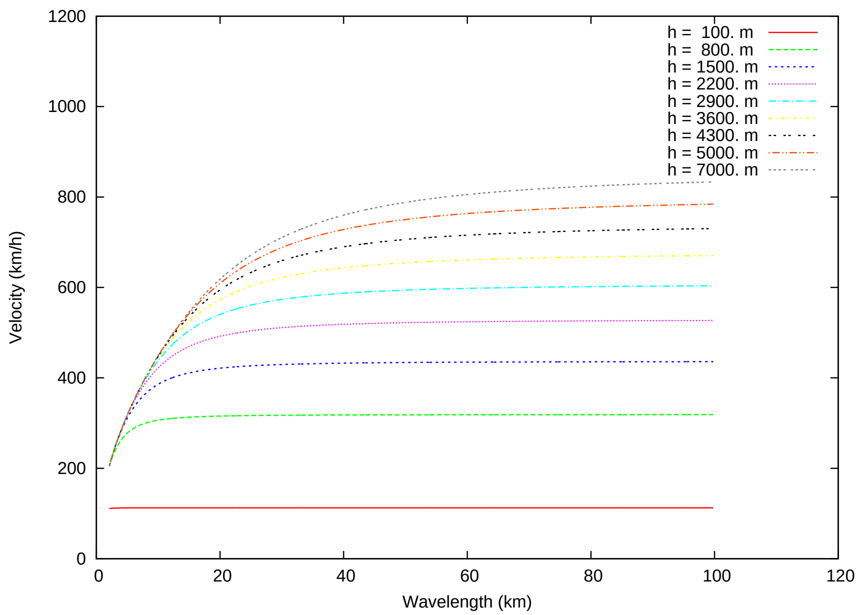
1. Tsunami waves as gravity waves have velocities as stated by Eq. (29).

2. As can be seen from Eqs. (27) and (29), the velocity at which gravity waves travel, is a function of ($x$,$y$) since in an ocean, $h(x,y)$. A direct result of this, is that the velocity of gravity waves, such as tsunamis, travel is uniform at all depths in a single point. Tsunamis move like a wall of energy.

3. Using average values of ocean depth, $h \approx 4000$ m, and $g \approx 10$ m/s$^2$, we get $c \approx 200$ m/s or $c \approx 720$ km/h which is the speed of a jet airliner.

4. The following curves are plotted using Eq. (27) for nine different depth values.[1]




[1]The curves are made with the *tsudisp* routine, written in fortran.

# Material Derivative

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## 3 Introduction

In any given medium, Newton's Second Law must be applied to the same body of material. That is to say that the material for which the equations of motion and acting forces are written must be the same. This seems to be a trivial statement, but in fact is very important in dealing with non-rigid material such as fluids. In solids, applying Newton's Law is usually straightforward, since the target volume does not undergo internal evolution, or at least making such an assumption does not undermine the solution too much. This is mainly due to the fact that the subject molecules are trapped in a strong-enough potential field and tend to stay there unless an opposing strong-enough force dictates otherwise. Usually, the oscillatory or vibrating force around the potential well the molecules are trapped in is the characteristic of solids. In fluids, however, the molecules have more freedom of movement and they not only vibrate around an equilibrium, but also, most dominantly, they can move around the fluid body in random Brownian motions, and this is why the density-volume curves for liquids and solids differ so drastically at small volumes (Figs. 1 and 2).

In fluids, even if the mass element is seemingly at rest, it evolves in time due to thermal diffusion. In other words, the molecules in $A$ are not those in $B$ in Fig. 3, although it is assumed to be the same volume traveling along with the flow. As a result, the basic requirement of Newton's Second Law is not satisfied with fixed mass and/or volume.

## 4 Material Domain

The change in the content of any given fixed volume (originally intended to keep the mass preserved) rises the question if Newton's Second Law could be applied, as it is, to fluids. The solution to this problem lies in the definition of the "fixed volume." If we could somehow define an imaginary volume the content of which would be constant through its motion, then we will be able to apply the Second Law without any apparent trouble.

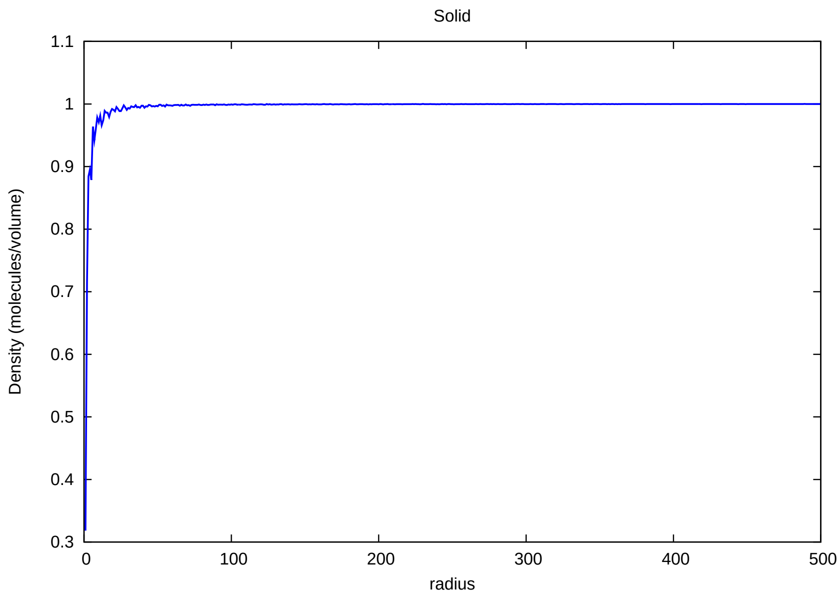


**Figure 1:** Density curve for a 2-D solid, stated as molecules per unit volume.

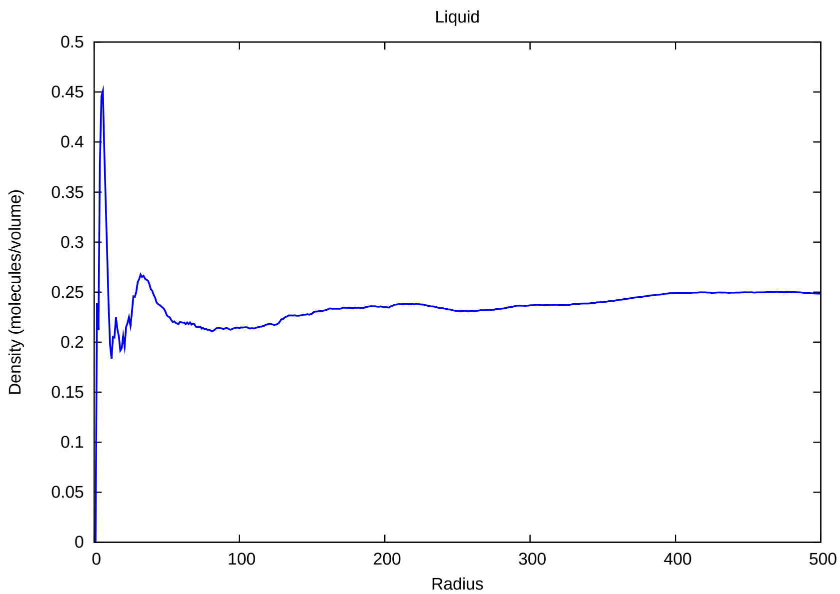


**Figure 2:** Density curve for a 2-D liquid, stated as molecules per unit volume.

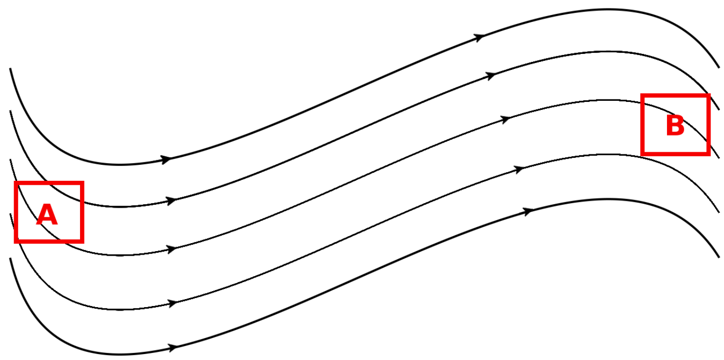


**Figure 3:** Volume of fluid has traveled from $A$ along with the flow to reach $B$. It is the same volume, but it has changed and evolved over the path. The molecules in $A$ are not the same as those in $B$.

A *material domain* is a domain which gets deformed to ensure that every particle on the surface of the volume will move by $\vec{u}(\vec{x}, t)$ which is a nonuniform velocity field throughout the volume over the small time fragment $dt$. The result would be that the mass in the volume will remain constant and the molecules inside the boundaries of the volume will remain the same throughout the path traveled by the volume. The price to pay, is that differential equations will no longer hold the way they do under usual circumstances (e,g, in solids) and can no longer equate the differential acceleration times mass to force simply as $m(du/dt)$. However, if the size of this volume was to approach to zero and would take the form of a differential point called, material point, we can define a spacial derivative.

*Material derivative* follows the material point to see how it has changed over time. For instance, considering Fig. 3, we can write

$$\frac{u_B - u_A}{t_B - t_A} = \frac{u(A, t + \Delta t) - u(A, t)}{(t + dt) - dt} = a \equiv \text{acceleration} \tag{1}$$

where $u(x, t)$ is the velocity of the flow particle as function of space and time. This kind of derivative can be applied to any given characteristic, $F$, of the material volume. In other words,

$$\frac{DF}{Dt} = \lim_{\Delta t \to 0} \frac{F(A, t + \Delta t) - F(A, t)}{\Delta t} \tag{2}$$

where $\frac{D}{D_t}$ is called the *material derivative*.

## 4.1 Flow Field Specification

What we do in dealing with various types of flows in the real world is measuring characteristics of the moving fluid at fixed points since in reality we are unable to follow infinitesimally small volumes of fluid particles over time. This is called the transition from the *Lagrangian* point to the Eulerian point. In Lagrangian specification of flow field we are dealing with individual material points, as opposed to the Eulerian specification where we calculate and measure various characteristics at fixed points in space and time. These two are different ways to specify flow fields but can be converted to one another.

If we consider the changes to the characteristic $F$ of a flow, using the definition of gradient, we can write,

$$DF = \frac{\partial F}{\partial t} Dt + \frac{\partial F}{\partial x_i} dx_i = \frac{\partial F}{\partial t} Dt + \frac{\partial F}{\partial x_i} u_i Dt \tag{3}$$

where $x_i$ is the $i$-th coordinate of space. Dividing both sides by $Dt$ we get,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_i} u_i \tag{4}$$

or in the Gibbs notation,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \vec{u}.\nabla F \tag{5}$$

## 4.2 Flow Acceleration

Using the idea of Eqs. (1) and (5), we can calculate acceleration as,

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}.\nabla \vec{u} \tag{6}$$

or in the tensor notation,

$$D_t u_i = \partial_t u_i + u_j u_{i,j} = \partial_t u_i + u_j \partial_j u_i \tag{7}$$

where $\vec{u}$ is a function of both space and time. Eq. (7) is the material derivative of velocity and relates acceleration of a fluid material point to its Eulerian definition. As a result we can interpret experimental data to individual material points or more importantly, predict positions of fluid particles at given fixed points by the equations of motion we derive using the definition of material points. Material derivative can be considered as an operator as in Eq. (8), acting on any given vector such as $\vec{a}$.[2].

$$D_t = \partial_t + u_j \partial_j \tag{8}$$

$$D_t a_i = \partial_t a_i + u_j \partial_j a_i \tag{9}$$

One final point to remember when using the concept of material derivative is that, it is no different from the normal way of computing derivative in the sense that is using the idea of infinitesimal changes, however, one should keep in mind the two of the above are not interchangeable, but can be converted to one another using Eq. (5).

[2]Pay attention to the $i$ and $j$ indices.