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Beginner's Notes on

Fluid Dynamics

"The Road to Tsunamis"

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Contents

Foreword

This volume is a compilation of my notes on the fundamentals of fluid dynamics and serves as an introduction to tsunamis. With that in mind, I have written the following as/for a seismology student. I am not a fluid dynamicist, but am interested in playing around with fluid dynamics equations and concepts in order to apply them to tsunamis.

While this text is systematically structured as an intermediate level course in fluid dynamics for a geoscience student with an interest in tsunamis, it, by no means, serves as a complete reference. For instance, I have kept out most of the introductory stuff on vector & tensor calculus – as they are included in many fluid mechanic textbooks – and skipped over many of the otherwise time consuming and perhaps distracting examples. I believe such material ought to be addressed in lectures and/or in exercises, but not in the text.

I have allocated the final sections to introductions to computational fluid dynamics (CFD) and tsunamis. I have also included a quick and dirty introduction to a benchmarked tsunami simulation softwares (MOST) as I think this is a more hands-on way to get familiar with many of the concepts.

This volume is inspired, in part, by my notes from Neelesh Patankar's lectures at Northwestern University as well as my interesting conversations and correspondences with Emile Okal, Costas Synolakis & Nikos Kalligeris.

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1   Temporal and Spatial Scales

"Scales" are perhaps the most confusing and at the same time useful concepts in fluid mechanics. There are usually two classes of "micro" and "macro" scales and while it seems like an obvious statement, – in fact, in many cases it is – it is crucial to choose a proper scale before starting to address a fluid mechanics problem.

1.1   Micro Scale

Fluids can be thought of as "bodies" of tiny particles or molecules. Let's begin with an example: when dealing with *air* as a fluid *which can flow*, one would be tempted to think about it as individual N_2, O_2,... particles with typical diameters on the order of 1Å at a typical distance of ~ 40Å from each other.[1] Also, while liquid water has an atomic size of $\sim 3-4$Å, the intermolecular distance of H_2O is $\sim 4-5$Å.[2] The typical amount of time needed to distinguish between various atomic motions is $\sim 10^{-15}$s or 1 fs. This means, "meaningful" motions at the micro-scale have to occur on the order of angstroms and femtometers (see Fig. 1.1).

1.2   Macro Scale

Obviously, we need to deal with larger sizes and times too. For instance, we think about and/or interact with wind, streams, showers, and waves (and

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[1]While hydrogen atoms have a diameter of ~ 1.1Å, protons have a diameter of $\sim 1.7 \times 10^{-15}$m or ~ 1.7 fm.

[2]Just in order to get some sense of all these numbers, human hair have typical thicknesses of $\sim 50-100\mu$m (1 μm $= 10^{-6}$m).

even larger things such as storms, floods, hurricanes, tsunamis, etc) everyday. Here, the sizes are on the order of meters and kilometers and times (from daily experience) are on the order of seconds, minutes and hours (see Fig. 1.1).

Micro   Macro

size ~ 10^{-9} m $\longleftrightarrow$ size ~ 1 m
time ~ 10^{-15} s   time ~ 1 s
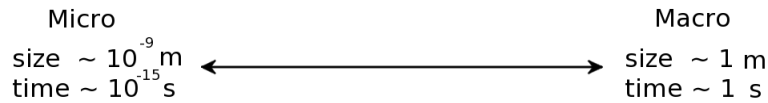
Figure 1.1: Scales in fluid mechanics.

1.3   Continuum Approximation: L_c

For the scope of this text, we describe fluids *not at atmic or molecular scale*, but in a **continuum** which is an *experimental* threshold as we will discuss below.

Consider Fig. 1.2 where a gas is passing around a sphere in a bulged pipe. In this setup, various parameters of the flow will change throughout its path. But before any attempt to address this issue, we have to decide which aspect of the problem we want to solve. Do we want to think about the variations in flow velocity or do we want to study the variations in its density? Do we want to solve for particle (molecule) distances (and therefore flow density) over time and in space or do we assume to have this information beforehand? These questions and many similar ones should be resolved before setting out to solve a problem.

Fortunately, such an insight is usually easy to gain. In order to do this, people have come up with the idea of "local density" which is the average density at a given point – what we ideally try to shrink in size to become infinitesimal. If we consider a fixed point in space and count the number of particles closer than a very small distance of r_0 to it and then keep doing this
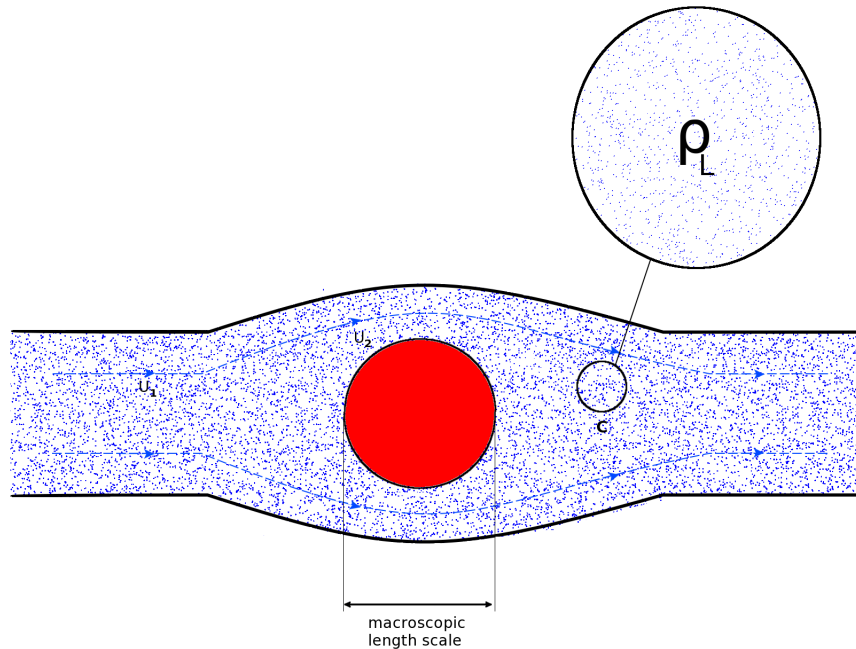
Figure 1.2: A fluid passing around a sphere in a bulged pipe. In general, $U_1 \neq U_2$ which for a compressible flow, makes the local densities at the two points different.

by incrementally adding to r_0, the result will look like what is shown in Fig. 1.3.

As you can see in Fig. 1.3, at small radii, both curves fluctuate and only reach a constant level at higher values of radius.[3] *We can define a dimension/volume, L_c, big enough to accommodate a given number of molecules in a way that the local density would be constant for that size of volume everywhere in the fluid.* Such a volume totally depends on the type of fluid. For example, while the fluctuations of local density in Fig. 3(b) for a fluid with 15,000 molecules in the differential volume (e.g., nitrogen) will be $\sim 0.8\%$, it will be $\sim 25\%$ for a fluid with 15 molecules per differential volume (e.g., water).

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[3]The different behaviors of the two curves in Fig. 3(a) and 3(b) has to do with the different molecular structures of solids and liquids, as in solids, molecules are parts of a lattice, whereas in liquids their positions are fairly random.

Another noteworthy aspect that we will not cover here is that the shapes of these curves will vary to some extent over the transition from 2-D to 3-D.
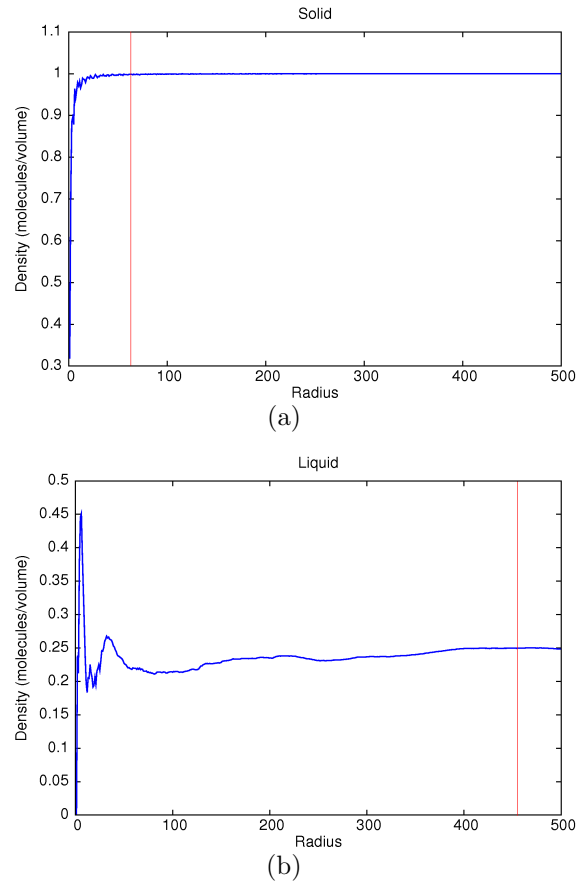
Figure 1.3: Density curves for a (a) 2-D solid and (b) a 2-D liquid, stated as molecules per unit volume. Red lines denote L_c.

Consequently, for a container of nitrogen gas, $L_c \approx 75 - 100$ nm, and for the same volume of water, $L_c \approx 10$ nm. In other words, dealing with liquids such as water on the macro scale is much easier than gases since they start to leave the choppy behavior and begin to behave themselves at smaller sizes.

In a nutshell, in fluid mechanics and in the domain of this text, we will mostly be dealing with *continua* which are liquids at length scales larger than a few nanometers ($L_c \gg 10^{-8}$ m).

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2 Material Derivative

2.1 Introduction

In any given medium, Newton's Second Law must be applied to *the same* body of material. That is to say that the material for which the equations of motion and acting forces are written must remain the same throughout the motion. This seems to be a trivial statement, but in fact is very important in dealing with non-rigid material such as fluids. In solids, applying Newton's second law is usually straightforward, since the target volume usually does not undergo internal evolution, or at least making such an assumption does not undermine the solution too much. This is mainly due to the fact that the subject molecules are trapped in a strong-enough potential field and tend to stay there unless an opposing strong-enough force dictates otherwise. Usually, the oscillatory or vibrating force around the potential well which the molecules are trapped in is the characteristic of solids. In fluids, however, the molecules have more freedom of movement and they not only vibrate around an equilibrium, but also, most dominantly, they can move around the fluid body in random Brownian motions, and this is why the density-volume curves for liquids and solids differ so drastically at small volumes (Figs. 3(a) and 3(b)).

In fluids, even if the mass element is seemingly at rest, it evolves in time due to thermal diffusion. In other words, the molecules in A are not those in B in Fig. 2.1, although it is assumed to be the same volume traveling along with the flow. As a result, the basic requirement of Newton's second law is not satisfied for fixed fluid mass and/or volume.
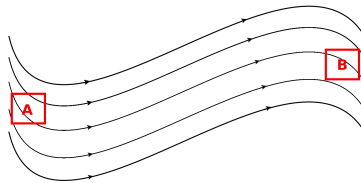
Figure 2.1: Volume of fluid has traveled from A along with the flow to reach B. It is the same volume, but it has changed and evolved over the path. The molecules in A are not the same as those in B.

2.2 Material Domain

The change in the content of any given fixed volume (originally intended to keep the mass preserved) raises the question if Newton's second law could be applied, *in its current form*, to fluids. The solution to this problem lies in the definition of the "fixed volume." If we could somehow define an imaginary volume the content of which would be constant throughout its motion, then we will be able to apply the second law without any apparent trouble.

A *material domain* is a domain which gets deformed to ensure that every particle on the surface of the volume will move by $\vec{u}(\vec{x}, t)$ which is a nonuniform velocity field throughout the volume over the small time fragment dt. The result would be that the mass in the volume will remain constant and the molecules inside the boundaries of the volume will remain the same throughout the path traveled by the volume. The price to pay, is that differential equations will no longer hold the way they do under usual circumstances (e.g., in solids) and we can no longer equate the differential acceleration times mass to force simply as $m(d\vec{u}/dt)$. However, if the size of this volume was to approach zero and take the form of a differential point called, the *material point*, then we can define a spacial derivative.

Material derivative follows the material point to see how it has changed over time. For instance, for a *material volume*, considering Fig. 2.1, we can write

$$\frac{u_B - u_A}{t_B - t_A} = \frac{u(A, t + \Delta t) - u(A, t)}{(t + dt) - dt} = a \equiv \text{acceleration} \qquad (2.1)$$

where $u(x, t)$ is the velocity of the flow particle as function of space and time. This kind of derivative can be applied to any given characteristic, F, of the material volume. In other words,

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$$\frac{DF}{Dt} = \lim_{\Delta t \to 0} \frac{F(A, t + \Delta t) - F(A, t)}{\Delta t} \tag{2.2}$$

where $\frac{D}{D_t}$ (or D_t) is called the *material derivative*.

2.2.1   Flow Field Specification

What we do in dealing with various types of flows in the real world is measure the characteristics of moving fluids, or *flows*, at fixed points since in reality we are unable to follow infinitesimally small volumes of fluid particles over time. This is called the transition from the *Lagrangian* point to the *Eulerian* point. In the Lagrangian expression of flow field we deal with individual material points, while in the Eulerian expression we calculate and measure various characteristics at fixed points in space and time. These are two different ways to specify flow fields but they can be converted to one another.

If we consider the changes in the characteristic F of a flow, using the definition of gradient, we can write,

$$DF = \frac{\partial F}{\partial t}Dt + \frac{\partial F}{\partial x_i}dx_i = \frac{\partial F}{\partial t}Dt + \frac{\partial F}{\partial x_i}u_i Dt \tag{2.3}$$

where x_i is the i-th coordinate of space. Dividing both sides by Dt we get,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_i} u_i$$

or

$$D_t F = \partial_t F + \partial_i F u_i \tag{2.4}$$

or in the Gibbs notation,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \vec{u}.\nabla F \tag{2.5}$$

Eqs. (2.5) and (2.4) serve as definitions of material derivative. The first and second terms on the right are called "local" and "convection" accelerations. While *local acceleration* deals with temporal variations of a given quantity (here velocity) at a constant position, *convection acceleration* expresses the variations in that quantity throughout the fluid.

2.2.2 Flow Acceleration

Using the idea of Eqs. (2.1) and (2.5), we can calculate acceleration as,

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}.\nabla \vec{u} \tag{2.6}$$

or in the tensor notation,

$$D_t u_i = \partial_t u_i + u_j u_{i,j} = \partial_t u_i + u_j \partial_j u_i \tag{2.7}$$

where $\vec{u}$ is a function of both space and time. Eq. (2.7) is the material derivative of velocity and relates acceleration of a fluid material point to its Eulerian definition.

As a result, we can interpret experimental data to individual material points or, more importantly, predict positions of fluid particles at given fixed points by the equations of motion we derived here using the definition of material points. Material derivative can be considered as an operator as in Eq. (2.8), acting on any given vector such as $\vec{a}$.[4].

$$D_t = \partial_t + u_j \partial_j \tag{2.8}$$

$$D_t a_i = \partial_t a_i + u_j \partial_j a_i \tag{2.9}$$

One final point to remember when using the concept of material derivative is that, it is no different from the normal way of computing derivative in the sense that is using the idea of infinitesimal changes, however, one should keep in mind that the two of the above are not interchangeable, but can be converted to one another using Eq. (2.5).

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[4]Pay attention to the i and j indices.