OVERVIEW

STRUCTURAL MODELS: FROM THE TIMELESS TO THE TIMELY

The Timeless…

Euler and Königsberg

Long ago (as early as 1636), citizens of Königsberg (modern-day Kaliningrad) challenged themselves to take a walk around the city so that they crossed each of the seven bridges over the River Pregel exactly once and returned to their point of departure (Figure O.1). Soon, they discovered that their attempts were in vain; however, the problem is sufficiently complicated that it is not clear from empirical trials *why* the suggested stroll is impossible. One attempt at solution is highlighted in red (Figure O.2). In this case, one is trapped on an island and cannot cross bridge 7 without crossing an already-crossed bridge.

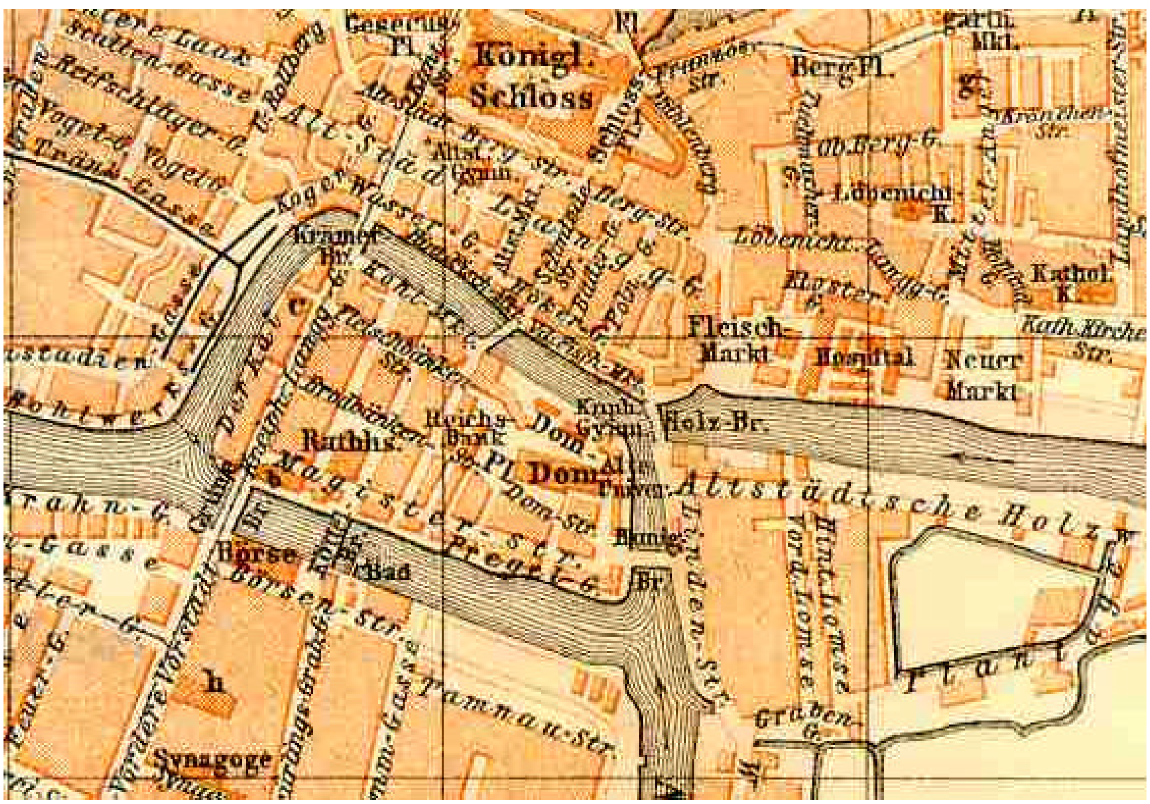


Figure O.1. Map of a portion of Königsberg from 1890 [we seek a better historical map for the final product.]

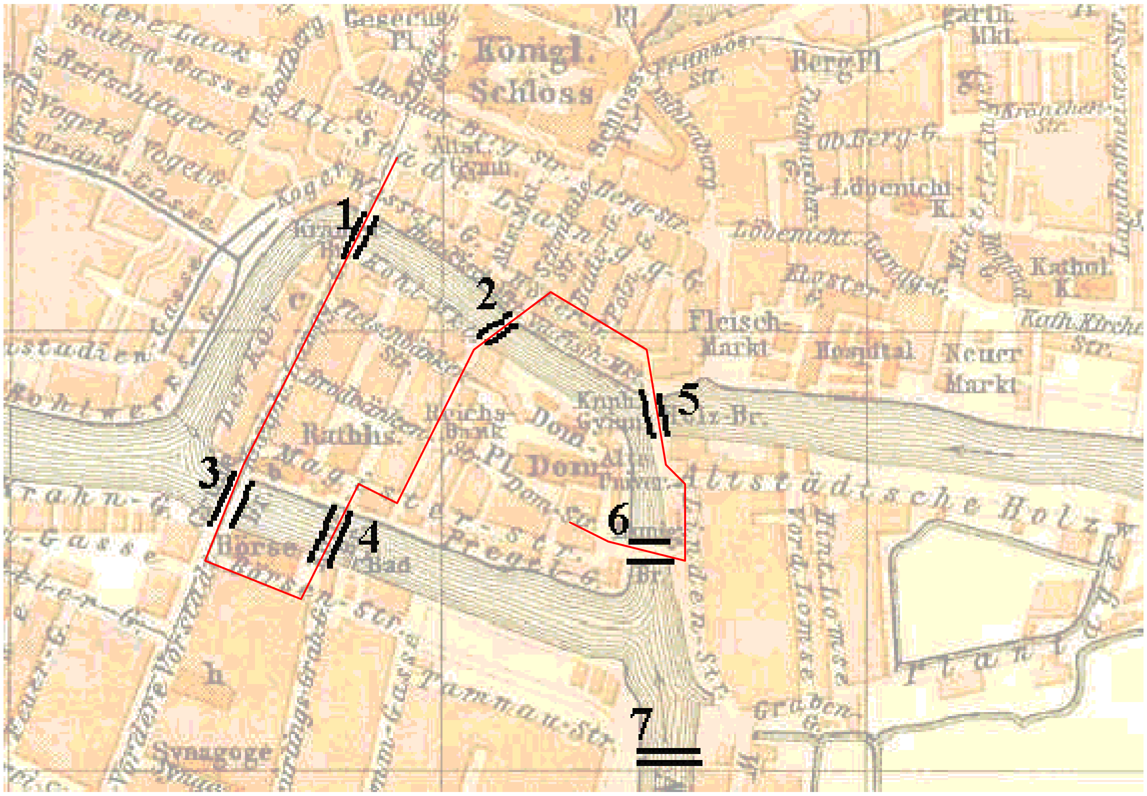


Figure O.2. The Seven Bridges of Königsberg highlighted on the map; one attempt at crossing each exactly once is shown.

When the apparently complicated connection patterns were partitioned into the two categories "regions" and "bridges" (Figure O.3), and then the regions represented as nodes and the bridges as edges (Figure O.4), a graphical model that captured only the structural elements of the problem emerged. Thus, it became possible for Leonhard Euler to show why such a walk was impossible.

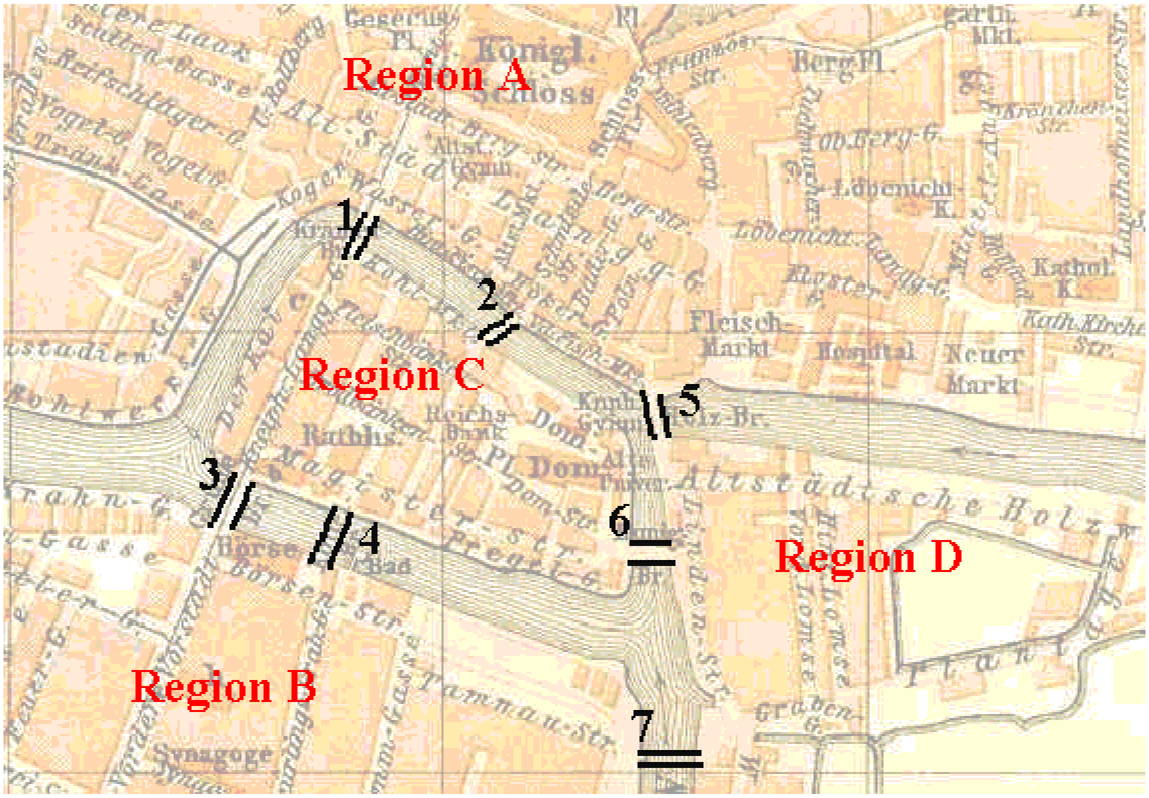


Figure O.3. Regions of the city are represented as A, B, C, and D. Bridges are assigned numerals from 1 to 7.

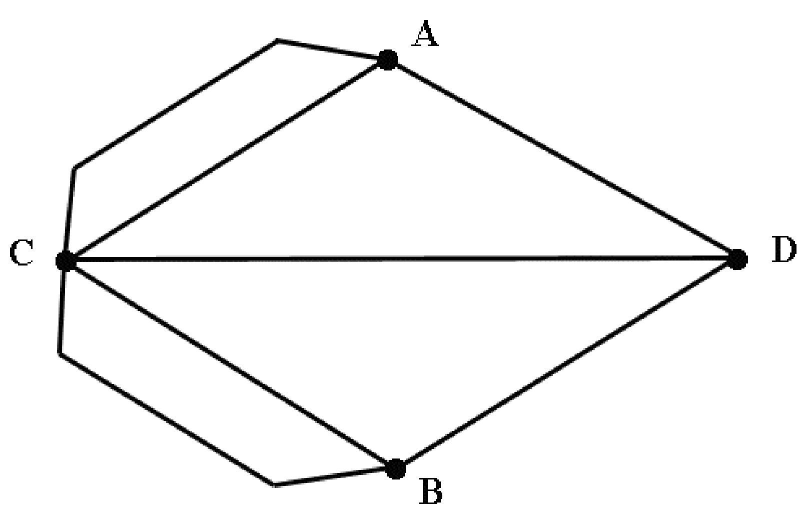


Figure O.4. Seven Bridges of Königsberg: structural model (graph) based on geographic map. Only the nodes (representing regions on the map) are labelled, A, B, C, and D. Edges represent associated bridges.

A trip of the sort desired by the townspeople is a sequence of nodes where each adjoining pair of nodes is joined by a different edge, and all seven edges are included. For example, A, D, B, C, A is a sequence starting and ending at A (returning to the point of departure), but only 4 edges, AD, DB, one BC, one BA, of the seven are included. In any such a sequence of nodes and edges, at each node an edge comes in and an edge goes out. Thus the total number of edges at each node must be even (this number is called the *degree* of that node). In the graph of the Königsberg bridges, not just one but <u>all</u> of the nodes have odd degree (Figure O.4). So a traversal is impossible. In fact, Euler proved that a graph that comes in one piece (*connected* graph) has a traversal of the type mentioned (*Eulerian trail*) precisely when the degree of every node is even.

When a graph is used to represent the structural elements of a real-world situation, as it was above, we refer to it as a *structural model*. Often, it is useful to look at some of the deeper issues represented in the process of creating models, for it is there that one can find parallels to guide applications in various directions. In the case of Königsberg, the representation of the geographic situation as a graph enabled Euler to solve the immediate superficial problem of a walk through a particular town. As suggested above, in the comments about Eulerian trails, this representation also enabled him to extend the initial idea to solve an entire class of mathematical problems.

Often, with a bit of imagination, one can use mathematics to capture geographical structure and use geographical analysis to further mathematical inquiry. Such was the case for Königsberg; another classical case involved Hamilton's interest in traversability.

Hamilton and the Around-the-World Puzzle

Hamilton was interested in a different type of graphical traversal (Harary, 1969). Instead of having every edge occur exactly once, he wanted each node represented exactly once along the traversal. For instance, in the graphical representation of the cube below (Figure O.5), the sequence A, B, C, D, E, F, G, H, A, would produce a *Hamiltonian circuit*.

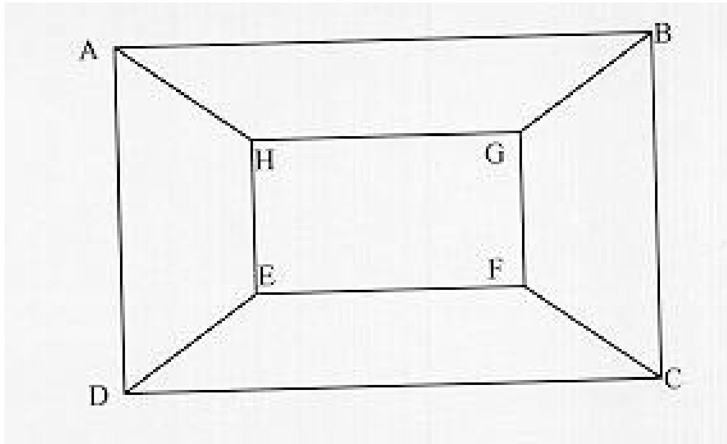


Figure O.5. Cube represented as a graph. The sequence A, B, C, D, E, F, G, H, A, produces a *Hamiltonian circuit*.

Hamilton himself presented this problem in the form of a puzzle called the Around-the-World Puzzle, a graphical representation of the dodecahedron with city names attached to each node. See if you can solve Hamilton's puzzle, using the graph of the dodecahedron in Figure O.6.

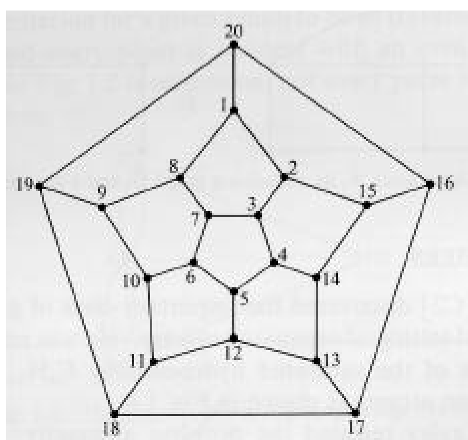


Figure O.6. Dodecahedron represented as a graph. Each node represents a city. Is this graph Hamiltonian? It is; see if you can find a sequence of edges meeting the initial criteria.

It can be quite difficult to determine exactly when a graph is or is not Hamiltonian. Here are a few easy examples of graphs that are not Hamiltonian (Figure O.7). To traverse all nodes, and return to the origin, requires passing through a node twice.

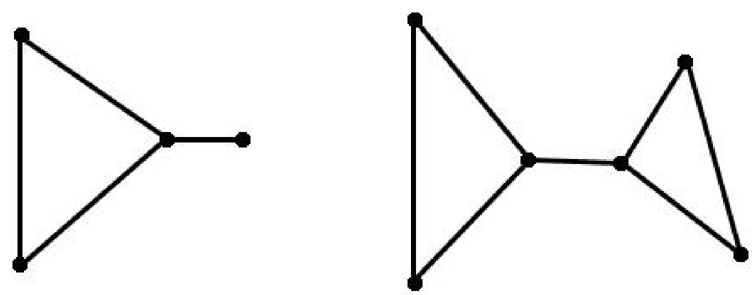


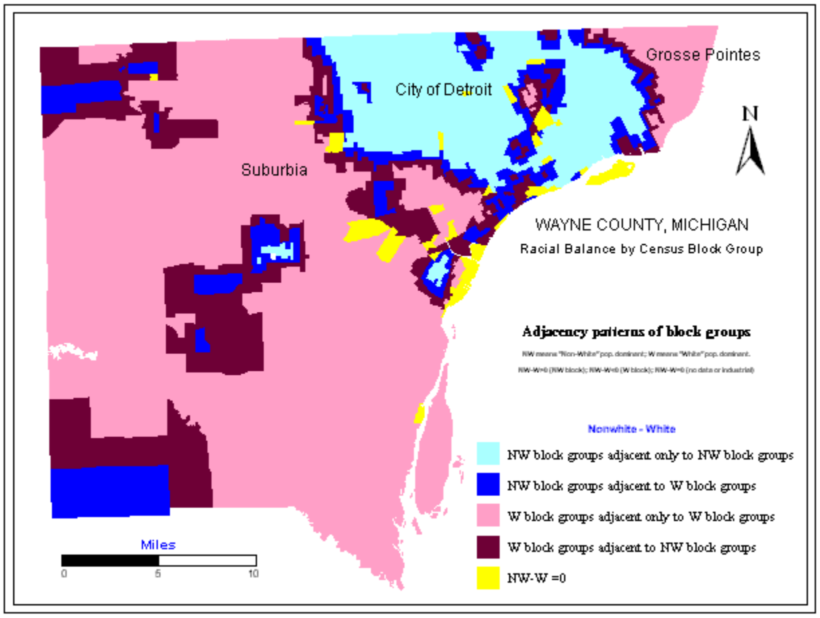
Figure O.7. Graphs that are not Hamiltonian; it is not possible to traverse each node exactly once and return to the origin.

Hamilton shows us that there is more going on behind a simple global issue, such as traversing a set of cities, than meets the eye. Hamilton and Euler saw "geo-graphs" as an initial stimulus for far-reaching research. Careful initial alignment of the geographical and mathematical elements can penetrate in many different directions. It is such alignments and their consequences that are the thrust of this book.

The Timely…

Adjacency is a concept that is both graphical and geographical. In this section, we introduce a sample that draws on concepts that we leave as undefined for the example, although they will be defined later (Arlinghaus and Arlinghaus, 1997). Often, a few nice maps serve as motivation to pique one's interest to delve into more notationally complex matters later in a book: that is our hope with this example.

Figure O.8. Positive spatial autocorrelation: directly adjacent block-group neighbors. Source of data and selected polygons: United States Bureau of the Census, 1990. Source of base maps: Atlas GIS, v.3.03, Environmental Systems Research Institute (ESRI). Maps reflect accuracy standards imposed by ESRI and the Census.



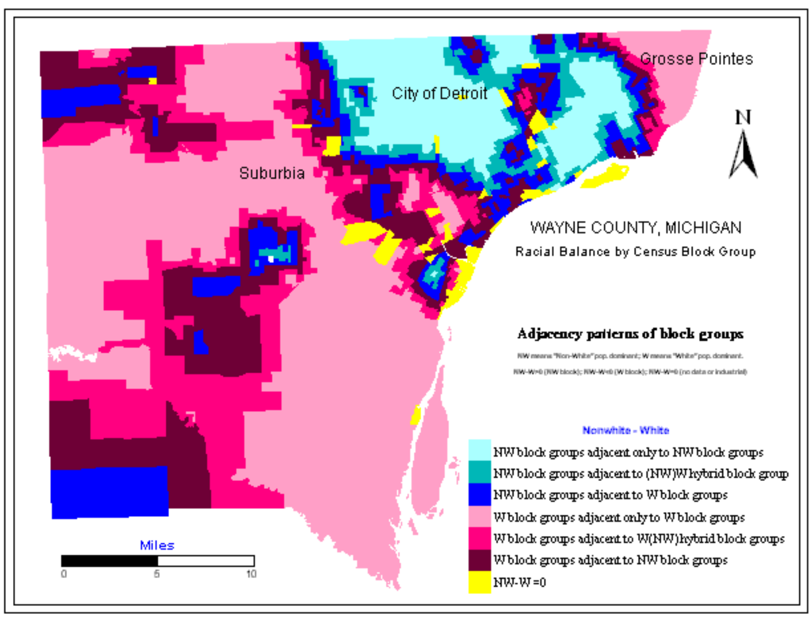


Figure O.9. Positive spatial autocorrelation: second-level adjacency of neighboring block-groups. Source of data and selected polygons: United States Bureau of the Census, 1990. Source of base maps: Atlas GIS, v.3.03, Environmental Systems Research Institute (ESRI). Maps reflect accuracy standards imposed by ESRI and the Census.

Spatial autocorrelation measures, to some extent, the influence of neighboring regions on each other: it is a measure of clustering of geographic regions based on some characteristic shared, or not shared, by pairs of adjacent regions. Given a map, with regions colored by some variable, two regions are said to be adjacent (neighbors) if they have a common boundary that includes a line segment (touching at a point, only, does not constitute adjacency). With positive spatial autocorrelation, similar values of the variable are clustered in space. With negative spatial autocorrelation, dissimilar values of the variable are clustered in space. No spatial autocorrelation indicates a random pattern of clustering in space.

The enclosed maps of Wayne County, Michigan, Figures O.8 and O.9 (also in the Frontispiece), show conditions of highly positive spatial autocorrelation. Census block groups that are predominantly populated by white population tend to be directly adjacent to block groups of similar character; census block groups that are predominantly populated by non-white population tend to be directly adjacent to block groups of similar character (Figure O.8). Thus, Detroit is seen as having predominantly non-white population while the Grosse Pointes and western and southwestern suburbia are largely white. The racial polarization between white and non-white is as evident on the map as it is to county residents.

Indeed, a similar situation exists for second-order adjacency patterns (neighbors) as well (Figure O.9). The buffer zones surrounding Detroit appear almost as topographic ridges in the adjacency pattern of racial relations. These ideas that are evident on the map can be captured more formally, as we shall see later, using graph-theoretic techniques.

Graphical and Geo-graphical intersections, such as those suggested by these classical and modern examples, will form the backbone of this book. Some examples will aim at careful alignment of basic material; others will extend the alignment into more complex directions. All will make ample use of graphic materials, from historical to contemporary maps. We hope that the reader will gain as much enjoyment from discovering the sort of understanding that structural models can offer to spatial analysis (be it mathematical or geographical) as we have.

References

Vasiliev in Arlinghaus, S.; Griffith D.; Arlinghaus, W., Drake, W., Nystuen, J. *CRC Practical Handbook of Spatial Statistics*. CRC Press. 1996.

Arlinghaus, S. and Arlinghaus W. A graph-theoretic view of the join-count statistic. *Solstice: An Electronic Journal of Geography and Mathematics*. 1997, Vol. VI, No. 2. http://www.imagenet.org

Harary, F. *Graph Theory*. Reading, MA: Addison-Wesley, 1969.