2. For generalized Jeffrey Fluid:

$$\underline{\underline{\tau}} + \lambda_1\left[\overset{\Delta}{\underline{\underline{\tau}}} - a\left(\underline{\underline{\tau}}\cdot\underline{\underline{\dot\gamma}} + \underline{\underline{\dot\gamma}}\cdot\underline{\underline{\tau}}\right)\right] = 2\eta_0\left[\underline{\underline{\dot\gamma}} + \lambda_2\overset{\Delta}{\underline{\underline{\dot\gamma}}}\right]$$

where $\overset{\Delta}{(\,)} = \frac{D(\,)}{Dt} + \underline{\underline{\omega}}\cdot(\,) - (\,)\cdot\underline{\underline{\omega}}$ $\quad \nabla(\,) = \begin{pmatrix} \frac{\partial(\,)}{\partial x_1} & \frac{\partial(\,)}{\partial x_2} & \cdots \\ \vdots & & \end{pmatrix}$

For shear flow $\begin{cases} u = \dot\gamma y \\ v = 0 \end{cases}$ $\quad \nabla u = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot\gamma$

$$\underline{\underline{\dot\gamma}} = \tfrac{1}{2}\left(\nabla u + \nabla u^T\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{\dot\gamma}{2} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\alpha$$

$$\underline{\underline{\omega}} = \tfrac{1}{2}\left(\nabla u - \nabla u^T\right) = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{\dot\gamma}{2} \equiv \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\alpha$$

Subs: (steady; fully developed)

$$\tau_{ij} + \lambda_1\Big[\cancel{\dot\tau_{ij}} + v_k\cancel{\tau_{ij,k}} + \omega_{ik}\tau_{kj} - \tau_{ik}\omega_{kj} - a\left(\tau_{ik}\dot\gamma_{kj} + \dot\gamma_{ik}\tau_{kj}\right)\Big]$$
$$= 2\eta_0\Big(\dot\gamma_{ij} + \lambda_2\Big(\cancel{\frac{D\dot\gamma_{ij}}{Dt}} + \omega_{ik}\dot\gamma_{kj} - \dot\gamma_{ik}\omega_{kj}\Big)\Big]$$

note $\tau_{12} = \tau_{21}$, & $\tau_{13} = \tau_{23} = \tau_{33} = 0$

the non-trivial terms are:

$$\tau_{11} + \lambda_1\Big(-\alpha\tau_{21} \mp \alpha\tau_{12} - a\left(\alpha\tau_{12} + \alpha\tau_{21}\right)\Big] = 2\eta_0\left(0 \pm \lambda_2\left(\alpha^2 + \alpha^2\right)\right)$$
$$-2\alpha\tau_{12} - 2a\alpha\tau_{12}$$

$$\tau_{11} \pm 2\lambda_1\alpha\tau_{12}(1+a) = -4\eta_0\lambda_2\alpha^2$$

(1) $\cdots\cdots\; \tau_{11} \pm \lambda_1\dot\gamma\tau_{12}(1+a) = -\eta_0\lambda_2\dot\gamma^2$

$$\tau_{12} + \lambda_1 \left[-\alpha \tau_{22} \mp \alpha \tau_{11} - a\left(\alpha \tau_{11} + \alpha \tau_{22}\right) \right] = 2\eta_0 \left[\alpha + \lambda_2 \left(0 - 0 \right) \right]$$

② $\tau_{12} + \lambda_1 \left[+\alpha \tau_{11}(1 \pm a) \pm \alpha \tau_{22}(1+a) \right] = 2\eta_0 \mp \alpha$

$$\tau_{22} + \lambda_1 \left[+\tau_{12}\alpha + \tau_{21}\alpha - a\left(\tau_{21}\alpha + \tau_{12}\alpha\right) \right] = 2\eta_0 \left[0 + \lambda_2 \left(+\alpha^2 + \alpha^2 \right) \right]$$

③ $\tau_{22} + \lambda_1 \dot{\gamma} \tau_{12}(1 \pm a) = +\eta_0 \lambda_2 \dot{\gamma}^2$

Subs ①, ③ into ②

$$\tau_{12} + \left(\lambda_1 \dot{\gamma} \tau_{12}(1 \mp a) - \eta_0 \lambda_2 \dot{\gamma}^2 \right) \frac{\lambda_1 \dot{\gamma}}{2} \frac{(1-a)}{(1+a)} - \frac{\lambda_1 \dot{\gamma}}{2}(1 \mp a)\left(-\lambda_1 \dot{\gamma} \tau_{12}(1-a) + \eta_0 \lambda_2 \dot{\gamma}^2 \right) = \eta_0 \dot{\gamma}$$

~~$= \eta_0 \lambda_2 \dot{\gamma}^2$~~

$$\Rightarrow \tau_{12} \left[1 + \lambda_1^2 \dot{\gamma}^2 (1-a^2) \right] - \eta_0 \lambda_1 \lambda_2 \dot{\gamma}^3 + \eta_0 \lambda_1 \lambda_2 \dot{\gamma}^3 = \eta_0 \dot{\gamma}$$

$$\Rightarrow \text{④} \quad \tau_{12} = \frac{\eta_0 \dot{\gamma}\left(1 + \lambda_1 \lambda_2 \dot{\gamma}^2\right)}{1 + \lambda_1^2 \dot{\gamma}^2 (1-a^2)}$$

⑤ $\tau_{11} - \tau_{22} = -2\eta_0 \lambda_2 \dot{\gamma}^2 + 2\lambda_1 \dot{\gamma} \tau_{12}$

$$= -2\eta_0 \lambda_2 \dot{\gamma}^2 + \frac{2\eta_0 \lambda_1 \dot{\gamma}^2 \left(1 + \lambda_1 \lambda_2 \dot{\gamma}^2\right)}{1 + \lambda_1^2 \dot{\gamma}^2 (1-a^2)}$$

$$= \frac{2\eta_0 \dot{\gamma}^2 (\lambda_1 - \lambda_2) + 2\eta_0 \lambda_1^2 \lambda_2 \dot{\gamma}^4 a^2}{1 + \lambda_1^2 \dot{\gamma}^2 (1-a^2)}$$

⑥ $\tau_{22} - \tau_{33} = \eta_0 \lambda_2 \dot{\gamma}^2 - \lambda_1 \dot{\gamma}(1-a) \frac{\eta_0 \dot{\gamma}\left(1 + \lambda_1 \lambda_2 \dot{\gamma}^2\right)}{1 + \lambda_1^2 \dot{\gamma}^2 (1-a^2)}$

$$= \frac{\eta_0 \dot{\gamma}^2 \left[\lambda_2 - \lambda_1 (1-a)\right] + \eta_0 \lambda_1^2 \lambda_2 \dot{\gamma}^4 (a - a^2)}{1 + \lambda_1^2 \dot{\gamma}^2 (1-a^2)}$$