Portfolio Rebalancing in General Equilibrium [*]

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Abstract

This paper develops an overlapping generations model of optimal rebalancing where agents differ in age and risk tolerance. Equilibrium rebalancing is driven by a leverage effect that influences levered and unlevered agents in opposite directions, an aggregate risk tolerance effect that depends on the distribution of wealth, and an intertemporal hedging effect. After a negative macroeconomic shock, relatively risk tolerant investors sell risky assets while more risk averse investors buy them. Owing to interactions of leverage and changing wealth, however, all agents have higher exposure to aggregate risk after a negative macroeconomic shock and lower exposure after a positive shock.

Keywords: household finance, portfolio choice, heterogeneity in risk tolerance and age

JEL codes: E21, E44, G11

1   Introduction

Understanding how different people trade in response to large market movements, or how they rebalance their portfolios, is central to finance and macroeconomics. Studying rebalancing in a meaningful way requires a model that incorporates investor heterogeneity because, if all investors are identical, then all will hold the same portfolio and there will be no trade. We study optimal rebalancing behavior with a simple general equilibrium overlapping generations model that features heterogeneity in both age and risk tolerance. The model is focused on asset allocation, with one risky asset in fixed supply and a risk-free asset in zero net supply. We find that both general equilibrium and lifecycle considerations have large effects on optimal rebalancing and thus should be inherent parts of asset allocation models. Optimal rebalancing is fairly complex, with trading driven by leverage, aggregate risk tolerance, and intertemporal hedging effects.

Understanding the effects that drive optimal rebalancing behavior is particularly important for two reasons. First, as more empirical work is done on the rebalancing behavior of individuals it is important to have a benchmark for equilibrium behavior in an economy in which investors behave optimally. While our model is quite simple, it helps to provide such a benchmark. Second, to meet the needs of investors with different horizons and preferences, the mutual fund industry has created a number of products called "lifecycle" or "target-date" funds that are designed to serve as the whole portfolio for a large group of investors over their lifespans. To the extent that these funds follow simple linear rules of thumb for allocating assets across time, their allocations will be both suboptimal and generally infea-

sible if universally adhered to because simple rules ignore general equilibrium constraints. Our model helps to illustrate how "universal target-date" funds can be structured. Universal target-date funds follow an optimal strategy over time and are feasible in general equilibrium.

Empirical work on observed rebalancing is still relatively rare. Its primary finding is that most investors appear to display inertia in their asset allocations.[1] To motivate our analysis along these lines, we plot nonparametric estimates of stock share as a function of age from three waves of the Survey of Consumer Finances around the recent financial crisis. The left panel of Figure 1 depicts stock shares in the 2004 and 2007 waves, between which stock prices rose about 30 percent. The right panel covers 2007 and 2010, between which prices fell about 28 percent. Both plots also give the "passive" share implied by market returns and initial shares. Average rebalancing by age is therefore given by the difference between the passive curve and the actual curve for the second wave in each plot.[2]

Several interesting features are immediately apparent when looking at the plots. First, older investors have consistently higher stock shares in these data, which may be due to cohort effects such as those described in Malmendier and Nagel (2011). Second, there is substantial rebalancing in the data. While many individual investors may display inertia in their asset allocations, in aggregate investors do appear to react to market movements. Third, most age groups do not simply revert to their initial share. Rather, older investors,

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[1]Calvet, Campbell and Sodini (2009) use four years of Swedish data to show that households only partially rebalance their portfolios, or they only partially reverse the passive changes in their asset allocations. Brunnermeier and Nagel (2008) find similar results using data from the Panel Study of Income Dynamics.

[2]The curves are estimated with simple nonparametric regressions using Epanechnikov kernels. The passive curves are plotted assuming the stock returns reported in the text and total average non-stock returns of 6% between 2004 and 2007 and 4% between 2007 and 2010. To be included in the sample each individual must report at least $10,000 of financial wealth and some positive stock holding. There are 1389, 1260, and 1313 observations in 2004, 2007 and 2010, respectively.

Figure 1: Rebalancing, SCF Data

(a) From 2004 to 2007

(b) From 2007 to 2010
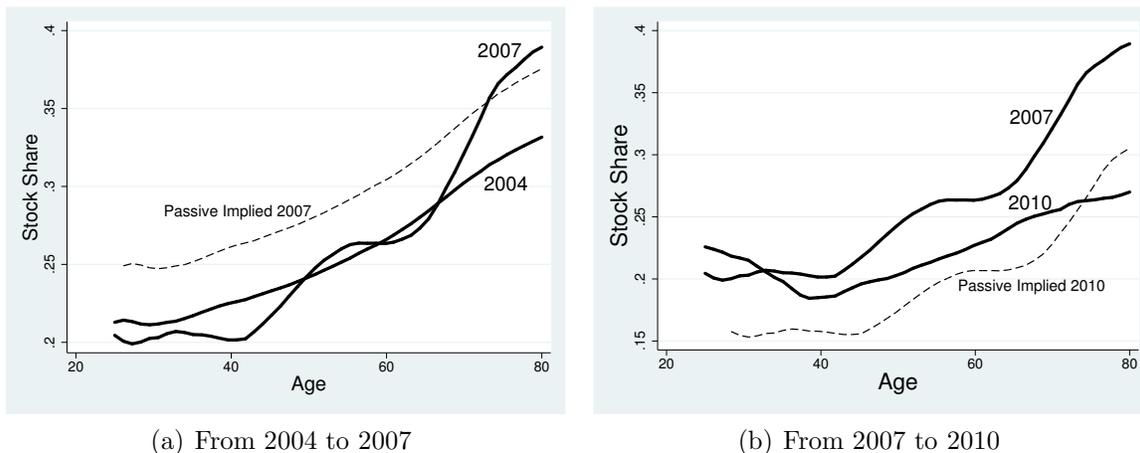
who on average hold more stock to begin with, buy more stock than required to return to their initial positions when the market rises. They also sell more stock than required to return to their initial position when the market falls. Younger investors display the opposite pattern, selling more than required to return to their initial position when prices rise, and buying shares as markets fall. In other words, rather than the stock share by age function either reverting back to the initial share or shifting to the passive share, it appears to pivot in each wave. While our model will not capture every feature of these plots, it highlights the importance of the general equilibrium constraints that the plots suggest, and it allows older investors to behave differently than younger investors, as they appear to do in the data.

Though our model is intentionally simple, it sheds light on the counter-intuitive implications of general equilibrium. Our first set of implications is for the direction of rebalancing in equilibrium. We show that after a negative macroeconomic (and return) shock, more risk averse investors buy risky assets while less risk averse investors sell. After a positive shock the opposite happens. This finding is consistent with the pivots in the plots above. Our second

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set of implications is for portfolio shares in equilibrium. After a negative shock all investors hold a higher risky asset share than they held before the shock. Risk tolerant investors hold levered positions that makes their risky asset share rise when prices fall. However, falling prices reduce their relative wealth, so aggregate risk tolerance falls and expected returns rise. Because expected returns are higher after prices fall, agents' desired risky asset shares after a negative shock are higher than before the shock. Thus, while the risk tolerant investors sell shares after a negative shock, their selling only partially offsets the increase in their risky asset shares. Again, after a positive shock the opposite happens. Finally, we show that the effect of intertemporal hedging is quite small in our general equilibrium setting.

Section 2 describes intuitively the three effects that drive our equilibrium outcomes: the leverage, aggregate risk tolerance, and intertemporal hedging effects. Section 3 discusses how our work relates to other research. In Section 4 we describe the overlapping generations model in detail. The model has a partially closed-form solution, which makes the intuition more transparent and substantially eases the solution of the model. Section 5 presents our quantitative solution of the model. Section 6 presents our conclusions.

2   Preview

Most of the earliest models of asset allocation in general equilibrium feature homogeneous agents, with the implication that agents will never trade or rebalance, as in Lucas (1978). We refer to such an implication as the buy and hold rule. Other models, when considered in partial equilibrium, imply that agents should always revert to their initial stock share, as

in Merton (1971). We refer to this implication as the constant share rule. Optimal portfolio rebalancing deviates from these two very different rules for at least three reasons in our model and more generally. First, there is a *leverage effect.* Since the risk-free asset is in zero net supply in our model, there will always be some investors that hold a levered position in the risky asset. For those with a levered position, their passive risky asset share actually declines as the market rises, and vice versa. Levered investors need to buy when the market rises and sell when it falls in order to keep their risky-asset shares constant, which is the opposite of standard rebalancing advice. Second, there is an *aggregate risk tolerance effect.* As the wealth of different types of agents shifts, aggregate risk tolerance also shifts, which in turn affects equilibrium prices and expected returns. Movements in aggregate risk tolerance induce people to trade in response to market movements in a way that does not maintain a constant share. In our specification, the aggregate risk tolerance effect is substantial. Third, there is an *intertemporal hedging effect.* Since future expected returns are time-varying, young agents have an incentive to hedge changes in their investment opportunities when they are middle aged. In principle, intertemporal hedging can have a significant effect on asset demand, but for reasons we will explain, in our specification it is quite difficult to generate enough mean reversion in asset returns to generate a substantial intertemporal hedging effect. We discuss hedging in Section 4.

A simple example can help clarify the leverage effect. Suppose an investor wants to maintain a levered portfolio with a share of 200% in the risky asset. With a $100 stake to begin with, the investor borrows $100 to add to that stake and buys $200 worth of the risky

asset. If the risky asset suddenly doubles in value, the investor then has \$400 worth of the risky asset and an unchanged \$100 of debt, for a net financial wealth of \$300. The constant share rule generally has investors sell assets when their prices rise. After this increase in the price of the risky asset, however, the share of the investor's net financial wealth in the risky asset is only $400/300 = 133\%$, since the investor's net financial wealth has risen even more than the value of the risky asset. Therefore, if the goal is to maintain a 200% risky asset share, the investor needs to borrow more to buy additional shares. Hence, levered investors rebalance in a direction that is the opposite of unlevered investors. The general equilibrium constraint that each buyer must trade with a seller is partially met by having levered investors trade with unlevered investors.

The aggregate risk tolerance effect is a consequence of general equilibrium adding up. It arises because the share of the risky asset held by an individual is a function of that individual's risk tolerance relative to aggregate risk tolerance; and aggregate risk tolerance must move in general equilibrium when there are shocks that affect the distribution of wealth. We will now show that this feature of general equilibrium makes it impossible for all investors to maintain a constant portfolio share. Returns adjust to make people willing to adjust their portfolio shares in a way that allows the adding-up constraint to be satisfied.

To see this, consider the aggregation of an economy characterized by individuals who have portfolio shares determined as in Merton (1971). That is, an individual i's share of investable wealth $\mathcal{W}_i$ held in the risky asset would be

$$\theta_i \equiv \frac{PS_i}{\mathcal{W}_i} = \tau_i \frac{\mathcal{S}}{\sigma}, \tag{1}$$

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where P is the price of the risky asset and S_i is the amount of the risky asset. The only heterogeneity across individuals here comes from individual risk tolerance τ_i and the only heterogeneity over time comes from a time-varying expected Sharpe ratio (expected excess return over volatility) $\mathcal{S}$ or from volatility σ. After aggregating this expression and imposing adding up constraints, we can derive an expression for the risky asset share that does not depend on the distribution of the asset's return:

$$\theta_i = \tau_i \left[\sum_i \frac{\mathcal{W}_i \tau_i}{\mathcal{W}} \right]^{-1} = \frac{\tau_i}{\mathcal{T}}, \tag{2}$$

where $\mathcal{T} = \sum_i \frac{\mathcal{W}_i \tau_i}{\mathcal{W}}$ is aggregate risk tolerance.[3]

This equation implies that neither the buy and hold rule nor the constant share rule makes sense in general equilibrium. As prices change, wealth effects cause aggregate risk tolerance to change. This induces rebalancing for all types of investors. This result also appears in Dumas (1989), Wang (1996), and other papers. In our model we make precise the exact nature of this rebalancing.

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[3]Recall that the risk free asset is in zero net supply, so the total value of all investable wealth must equal the total value of the risky asset. Multiplying (1) through by each type's wealth and aggregating across types yields

$$\sum_i \theta_i \mathcal{W}_i = \sum_i P S_i = \left[\sum_i \mathcal{W}_i \tau_i \right] \frac{\mathcal{S}}{\sigma} = \mathcal{W}. \tag{3}$$

Dividing equation (3) through by aggregate wealth and comparing the resulting expression to equation (1) allow us to derive (2)

3 Relation to the Literature

There is a great deal of research about asset allocation and portfolio choice in finance and economics. These topics have become particularly interesting recently as economists have begun to study household finance in detail following Campbell (2006). Three papers that discuss optimal trading behavior are Dumas (1989), Longstaff and Wang (2012), and Wang (1996). Dumas (1989) solves for the equilibrium wealth sharing rule in a dynamic equilibrium model with more and less risk tolerant investors. This paper nicely illustrates the importance of the distribution of wealth as a state variable. Longstaff and Wang (2012) discuss the leverage effect that we describe. Like our model, these papers find that more risk tolerant investors sell after negative market returns while less risk tolerant investors buy. All three of these papers feature continuous time models, in which prices evolve according to Brownian motion processes and agents trade continuously. As Longstaff and Wang (2012) explain, more risk tolerant investors trade in a dynamic way to create long positions in call options, while less risk tolerant investors create short call positions. As we discuss below, that paper's main contribution is about the size of the credit market. Dumas (1989) shows that more risk tolerant investors would like to hold perpetual call options. Since investors are continuously hedging to create options positions, these models generally imply infinite trading volume. Our model differs from these by featuring lifecycle effects, discrete time, and relatively large price movements, which will be more realistic for many investors. None of these papers focuses on rebalancing behavior.

Two papers that show how optimal portfolio choice changes when some of the assumptions

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of classic models are weakened are Liu (2007) and Kim and Omberg (1996). Kim and Omberg (1996) show that optimal behavior can be quite complex when investors are not myopic, with some investors hedging while others speculate. Liu (2007) shows that the properties of dynamic portfolio choice models can be quite different from those of static choice models.

Relatively little of the research on portfolio choice, however, considers the general equilibrium constraints that we apply in our model. Three recent papers that feature general equilibrium are Cochrane, Longstaff, and Santa-Clara (2008), Garleanu and Panageas (2015), and Chan and Kogan (2002). Cochrane, Longstaff, and Santa-Clara (2008) has heterogeneous assets rather than heterogeneous agents, so it is not about trade. Garleanu and Panageas (2015), like our paper, models investor heterogeneity in risk preference and age in an overlapping generations model. Chan and Kogan (2002) examines the aggregate risk tolerance effect in detail. All of these papers focus on the behavior of asset prices, showing that general equilibrium can generate return predictability that is consistent with empirical evidence. None of these papers explores the implications of market returns for either trading or the portfolio choice of different types of individuals, which is the focus of our paper.

The focus of our paper is much closer to the focus of Campbell and Viceira (2002). They find an important intertemporal hedging component of portfolio choice. In our model the leverage effect and the time-varying aggregate risk tolerance effect are much more important. The intertemporal hedging effect in our model is intentionally weak because we do not build in features that might lead to hedging such as mean reversion in fundamentals or noise traders. We have experimented with several different sets of parameter values to see if we

can make the intertemporal hedging effect larger in our context, but we have not been able to find a scenario in which it matters very much. Approaches such as Campbell and Viceira that use actual asset returns, which might reflect mispricing, have greater scope for finding intertemporal hedging.

There is a large and important literature about labor income risk and portfolio choice, including Bodie, Merton, and Samuelson (1992), Jaganathan and Kocherlakota (1996), Cocco, Gomes, and Maenhout (2005), and Gomes and Michaelides (2005). In a recent paper that is related to our work, Glover, Heathcote, Krueger, and Rios-Rull (2014) find that young people can benefit from severe recessions if old people are forced to sell assets to finance their consumption, causing market prices to fall more than wages. We purposely abstract from human capital considerations by assuming that agents only earn labor income before they have an opportunity to rebalance. Our aim is thus to focus on the general equilibrium effects while suppressing the important, but complex effects of human capital.

The leverage effect that we find is of course related to the effect discussed by Black (1976). Geanakoplos (2010) has recently developed a model emphasizing how shocks to valuation shift aggregate risk tolerance. Risk tolerant investors, who take highly levered risky positions, may go bankrupt after bad shocks, thereby decreasing aggregate risk tolerance and further depressing prices. This analysis hence features both the leverage and aggregate risk tolerance effects. The details of his model are quite different from ours. Moreover, the focus of his analysis is defaults, which will not occur in our setting. Nevertheless, both his model and our model feature time varying aggregate risk tolerance driven by changes in the distribution

of wealth.

As mentioned in the introduction, Calvet, Campbell, and Sodini (2009) and Brunnermeier and Nagel (2008) examine portfolio rebalancing with administrative holdings data and survey data, respectively. Both papers find that people do not completely rebalance, or that investors do not completely reverse the passive changes in their asset allocations. This finding is consistent with optimal rebalancing according to our model. Moreover, the leverage effect implies that investors with high initial shares will change their allocations in the opposite direction from that predicted by full rebalancing, or, in other words, that risk tolerant investors will chase returns. In fact, regressions in the online appendix of Calvet, Campbell, and Sodini (2009) indicate that the coefficients on passive changes in allocation are increasing in initial stock share. This is exactly what we would expect to find.[4]

When academic researchers think about asset allocation, they generally include all risky assets (including all types of corporate bonds) in their definition of the risky mutual fund that all investors should hold. In our model, regardless of how one might divide assets into safe or risky categories, we define the "risky asset" or the "tree" to be the set of all assets in positive net supply. We define the risk-free asset to be in zero net supply. The levered position in trees in our model encompasses both stock holdings and the risky part of the corporate bonds.

By contrast, when practitioners think about portfolio choice they generally treat stocks, bonds, and cash-like securities as separate asset categories in their allocation and rebalancing

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[4]Since Calvet, Campbell, and Sodini (2009) have detailed data on investors' actual holdings and returns, some portion of the returns their investors experience are idiosyncratic. This rationalizes the fact that the coefficients in the online appendix of Calvet, Campbell, and Sodini (2009) increase in initial share but they do not become positive for investors with high initial shares.

recommendations, as Canner, Mankiw, and Weil (1997) point out. To understand the size of the leverage effect, it is important to remember that stock is a levered investment in underlying assets. Thus, for example, if the corporate sector is financed 60% by stock and 40% by debt, an investor with more than 60% in stock is actually levered relative to the economy as a whole. If some corporate debt can be considered essentially risk-free then it should be possible to express such an investor's position as a portfolio of a levered position in the underlying assets of the economy and some borrowing at the risk-free rate. The leverage effect will be important for all investors that hold a levered position in the underlying assets of the economy—which is everyone who has an above-average degree of leverage (though typically not everyone who has an above-median degree of leverage).

4   Model

We include in the model only the complications necessary to study rebalancing meaningfully. The economy has an infinite horizon and overlapping generations. There is heterogeneity in preferences to generate trade within generations. Specifically, there are two types of agents with different degrees of risk tolerance. The level of risk tolerance is randomly assigned at the beginning of the agent's life and remains unchanged throughout life. Each agent lives for three equal periods: young (Y), middle-aged (M), and old (O). Agents work in a representative firm and receive labor income when they are young and live off their savings when they are middle-aged and old. All agents start life with zero wealth and do not leave assets to future generations; that is, they have no bequest motive. There is no population

growth, so there is always an equal mass across the three cohorts. In this section, we provide the details of how we model and specify the technology, the asset markets, and the agents' preferences and their resulting decision problems.

4.1 Technology and Production

Our setup has a character very similar to that of an endowment economy. There is a fixed supply of productive assets or "trees." Hence, we have a Lucas (1978) endowment economy, though unlike Lucas, we will have trade because not all agents are identical, and the endowment includes labor as well as "trees." The young supply labor inelastically. Trees, combined with labor, produce output that is divided into labor income and dividend income. The labor income stream serves to provide resources to each generation. The dividend income provides a return to risky saving. This setup allows us to provide resources to each generation without inheritance. As noted above, by providing labor income only at the start of the life, we intentionally abstract from the interaction of human capital with the demand for risky assets.

The production function is Cobb-Douglas,

$$Y_t = \frac{1}{\alpha} D_t K_t^\alpha L_t^{1-\alpha}, \tag{4}$$

where K_t is the fixed quantity of trees, L_t is labor, α is capital's share in production, and D_t is the stochastic level of technology. We normalize the quantity of trees K_t to be one. The quantity of labor is also normalized to be one. Therefore, total output is $Y_t = \frac{D_t}{\alpha}$, total

labor income is $\frac{(1-\alpha)D_t}{\alpha}$, and total dividend income is D_t. That is, given our normalization total dividends equal the stochastic level of technology. All labor income goes to the young, since they are the only ones supplying labor. Dividend income will be distributed to the middle-aged and old agents depending on their holdings of the risky tree.

In order not to build in mean reversion mechanically, we assume that the shock to dividends G_{t+1} is i.i.d, so

$$D_{t+1} = D_t G_{t+1}. \tag{5}$$

In our calculations, we parameterize the dividend growth as having finite support $\{G_n\}_{n=1}^N$, where G_n occurs with probability π_n each period. The dividend growth shocks are realized at the beginning of each period before any decisions are made.

4.2   Assets

Before turning to agents' decision problems, it is helpful to define assets and returns. In our economy, there are two assets available to agents: one risk-free discount bond and one risky tree security. The risk-free bond is in zero net supply, and the risky tree has a fixed supply of one. We parameterize the dividend growth process to have two states. These two assets provide a complete market.

One unit of a bond purchased at period t pays 1 unit of consumption in period $t + 1$ regardless of the state of the economy then. A bond purchased at time t has a price of $1/R_t$, so

$$R_t = \text{the gross risk-free interest rate} = \frac{1}{\text{price of bond}}. \tag{6}$$

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One unit of the tree has a price P_t in period t and pays $D_{t+1} + P_{t+1}$ if sold in period $t + 1$.

Thus, the gross tree return is

$$R_t^{\text{Tree}} = \frac{P_{t+1} + D_{t+1}}{P_t}, \qquad (7)$$

and the excess tree return is

$$Z_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t R_t}. \qquad (8)$$

4.3 Agents' Decision Problems

Since this paper focuses on asset allocation, we specify preferences so that there is meaningful heterogeneity in risk tolerance, but the consumption-saving tradeoff is simple. We use the Epstein-Zin-Weil preferences with a unit elasticity of intertemporal substitution (Epstein and Zin 1989, Weil 1990). The utility of agents born in date t with risk tolerance $\tau_j = \frac{1}{\gamma_j}$ is given by

$$\rho_Y \ln(C_{Yt}^j) + \frac{1 - \rho_Y}{1 - \gamma_j} \ln E_t \exp\left((1 - \gamma_j)\rho_M \ln(C_{Mt+1}^j) + (1 - \rho_M) \ln\left(E_{t+1}(C_{Ot+2}^j)^{1-\gamma_j}\right)\right), \quad (9)$$

where C_{Yt}^j, $C_{M,t+1}^j$ and $C_{O,t+2}^j$ denote consumption at young, middle, and old ages, and ρ_Y and ρ_M govern time preference when young and when middle-aged. The parameters ρ_Y and ρ_M will turn out to be the average propensities to consume. With an underlying discount rate equal to δ, we have

$$\rho_Y = \frac{1}{1 + \delta + \delta^2} \qquad (10)$$

and

$$\rho_M = \frac{1}{1 + \delta}. \tag{11}$$

The degree of risk tolerance can be either high ($j = H$) or low ($j = L$). Because this utility has a recursive form, we work backwards.

Old Agents

The problem for old agents at date t is trivial. They simply consume all of their wealth, so total consumption of the old C_{Ot}^j equals the total wealth of the old W_{Ot}^j. Note that capitalized W denotes wealth per person.

Middle-aged Agents

A middle-aged agent at date t chooses consumption when middle-aged and a probability distribution of consumption when old by choosing bond holding B_{Mt} and tree holding S_{Mt} to maximize intertemporal utility

$$\rho_M \ln(C_{Mt}^j) + (1 - \rho_M) \ln \left(E_t (C_{Ot+1}^j)^{1-\gamma_j} \right)^{\frac{1}{1-\gamma_j}}, \tag{12}$$

subject to the budget constraints

$$C_{Mt}^j + B_{Mt}^j/R_t + P_t S_{Mt}^j = W_{Mt}^j, \tag{13}$$

$$C_{Ot+1}^j = W_{Ot+1}^j = B_{Mt}^j + S_{Mt}^j (P_{t+1} + D_{t+1}). \tag{14}$$

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Recall that the middle-aged agents have no labor income, so they must live off of their wealth W_{Mt}^j.

Consider the portfolio allocation of this middle-aged agent where θ is the share of savings (invested wealth) in the risky asset. The optimal portfolio decision is given by

$$\theta_{Mt}^j = \arg \max_{\theta \in \Theta} \left\{ E_t \left\{ (1-\theta)R_t + \theta R_t Z_{t+1} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}} . \tag{15}$$

It is convenient to define the *certainty equivalent total return*

$$\phi_{Mt}^j = \left\{ E_t \left\{ (1-\theta_{Mt}^j)R_t + \theta_{Mt}^j R_t Z_{t+1} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}} . \tag{16}$$

Given the homotheticity of preferences and the absence of labor income after youth, the optimal portfolio share θ_{Mt}^j that yields ϕ_{Mt}^j also maximizes expected utility from consumption when old. Thus, the middle-aged agent's problem is equivalent to solving the following decision problem:

$$\max_{C_{Mt}^j, \theta \in \Theta} \rho_M \ln(C_{Mt}^j) + (1-\rho_M) \ln(W_{Mt}^j - C_{Mt}^j) + (1-\rho_M) \ln \left\{ E_t \left\{ (1-\theta)R_t + \theta R_t Z_{t+1} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}} , \tag{17}$$

which has the familiar special structure found with an intertemporal elasticity of substitution equal to one. Since only the first two terms depend on C_{Mt}^j, the first order condition for C_{Mt}^j is

$$\frac{\rho_M}{C_{Mt}^j} = \frac{1-\rho_M}{W_{Mt}^j - C_{Mt}^j}, \tag{18}$$

which has the solution

$$C_{Mt}^j = \rho_M W_{Mt}^j. \tag{19}$$

That is, log utility in the intertemporal dimension leads to a simple consumption function and makes saving and portfolio choices independent. Thus, the maximized utility of the middle aged is given by

$$V_{Mt}^j(W_{Mt}^j) = \rho_M \ln(\rho_M W_{Mt}^j) + (1 - \rho_M) \ln\left((1 - \rho_M)W_{Mt}^j \phi_{Mt}^j\right). \tag{20}$$

Bond holdings B_{Mt}^j are given by $(1 - \theta_{Mt}^j)(1 - \rho_M)W_{Mt}^j R_t$ and tree holdings S_{Mt}^j are given by $\theta_{Mt}^j(1 - \rho_M)W_{Mt}^j/P_t$.

Young Agents

With no initial wealth, the young have only their labor income for consumption and saving. Labor income of both types of young agent in period t is

$$W_{Yt}^j = W_{Yt} = (1 - \alpha)D_t/\alpha. \tag{21}$$

A young agent of type j decides on consumption and bond and tree holdings (B_{Yt}^j, S_{Yt}^j) (that together determine middle-aged wealth $W_{M,t+1}^j$) to maximize the utility

$$\rho_Y \ln(C_{Yt}^j) + \frac{1 - \rho_Y}{1 - \gamma_j} \ln\left(E_t \exp((1 - \gamma_j)V_{M,t+1}^j(W_{M,t+1}^j))\right) \tag{22}$$

subject to the budget constraints

$$C_{Yt}^j + B_{Yt}^j/R_t + P_t S_{Yt}^j = W_{Yt}^j \tag{23}$$

and

$$W_{M,t+1}^j = B_{Yt}^j + S_{Yt}^j(P_{t+1} + D_{t+1}). \tag{24}$$

Substituting the maximized middle-aged utility given by equation (20) into (22) yields the following decision problem for a young agent:

$$\max_{C_{Yt}^j, \theta \in \Theta} \quad \rho_Y \ln(C_{Yt}^j) + (1 - \rho_Y) \ln(W_{Yt}^j - C_{Yt}^j) \tag{25}$$

$$+(1 - \rho_Y) \ln \left\{ \left[E_t \left((1 - \theta) R_t + \theta R_t Z_{t+1} \right) (\phi_{M,t+1}^j)^{1-\rho_M} \right)^{1-\gamma_j} \right]^{\frac{1}{1-\gamma_j}} \right\}$$

$$+(1 - \rho_Y) \left[\rho_M \ln(\rho_M) + (1 - \rho_M) \ln(1 - \rho_M) \right].$$

Only the first two terms depend on C_{Yt}^j. Again because of log utility, optimal consumption is very simple. It is given by

$$C_{Yt}^j = \rho_Y W_{Yt}^j. \tag{26}$$

Since only the third term of equation (25) depends on θ, the optimal tree portfolio share θ_{Yt}^j for the young maximizes the certainty equivalent, time-aggregated total return function for the young, ϕ_{Yt}^j. That is,

$$\theta_{Yt}^j = \arg\max_{\theta \in \Theta} \left\{ E_t \left\{ [(1 - \theta) R_t + \theta R_t Z_{t+1}] (\phi_{M,t+1}^j)^{1-\rho_M} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}} \tag{27}$$

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and

$$\phi_{Yt}^j = \left\{ E_t \left\{ [(1 - \theta_{Yt}^j)R_t + \theta_{Yt}^j R_t Z_{t+1}](\phi_{M,t+1}^j)^{1-\rho_M} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}} . \tag{28}$$

Comparing the middle-aged portfolio problem in (16) with the young portfolio problem in (28), we find that the young solve a more complicated problem given their longer life span. Middle-aged agents with only one more period of life simply maximize the certainty equivalent return of their portfolio investment next period, as in equation (16). Young agents with two more periods to live need to pay attention to the covariance between portfolio returns from youth to middle-age and the certainty equivalent of the returns going forward beyond middle age.

Value of a Universal Target-Date Fund as Numeraire

Since future expected returns are time-varying, the multi-period decision problem of the young creates an opportunity for intertemporal hedging (Merton 1973). To sharpen the interpretation of intertemporal hedging, define

$$Q_{M,t+1}^j = \frac{1}{\rho_M^{\rho_M}(1 - \rho_M)^{1-\rho_M}(\phi_{M,t+1}^j)^{1-\rho_M}}, \tag{29}$$

which prices the variable annuity or "universal target date fund" for a given age and risk aversion. It yields, at lowest cost, a lifetime utility equal to the lifetime utility from one unit of consumption at period $t+1$ and one unit of consumption at period $t+2$. That is, the agent will be indifferent between being limited to consuming one unit of consumption

at both middle and old age and having wealth of $Q^j_{M,t+1}$ to deploy optimally.[5] The sense in

which the universal target date fund is "universal" is that it remains the optimal thing to do

even if everyone else has a universal target date fund in equilibrium. Given this definition

of the universal target date fund, equation (28) can be rewritten as follows:

$$\phi^j_{Yt} = \rho_M^{-\rho_M}(1-\rho_M)^{\rho_M-1}\max_{\theta\in\Theta}\left\{E_t\left\{\frac{(1-\theta)R_t+\theta R_t Z_{t+1}}{Q^j_{M,t+1}}\right\}^{1-\gamma_j}\right\}^{\frac{1}{1-\gamma_j}}. \qquad (30)$$

Thus, a young agent maximizes the simple certainty equivalent of his or her portfolio's return

relative to the return on the universal target date fund. In other words, this universal target

date fund paying out in periods $t+1$ and $t+2$ serves as a numeraire for returns from t to

$t+1$.[6]

---

[5]To see this, note that by (12), consumption of one at both middle and old age yields a lifetime utility of zero for the middle-aged agent. Unit middle-aged holdings of $Q_{M,t+1}$ must therefore make the middle-aged lifetime utility given by (17) equal to zero once the optimal consumption and portfolio share are substituted in, so
$$\rho_M\ln(\rho_M Q^j_{M,t+1})+(1-\rho_M)\ln((1-\rho_M)Q^j_{M,t+1})+(1-\rho_M)\ln(\phi^j_{M,t+1})=0.$$
The value of $Q^j_{M,t+1}$ given in (29) is the solution to this equation. The universal target date fund itself implicitly replicates the optimal portfolio strategy for the agent.

[6]Note that, because of its definition as the amount of wealth yielding lifetime utility equal to the lifetime utility from consumption constant at one, equation (20) can be rewritten
$$V^j_{Mt}(W^j_{Mt})=\ln(W^j_{Mt})-\ln(Q^j_{Mt}).$$
Defining the corresponding price of the universal target date fund in youth that can give lifetime utility equivalent to the lifetime utility of one unit of consumption in each of the three periods of life, $[Q^j_{Yt}]^{-1} = \rho_Y^{\rho_Y}(1-\rho_Y)^{1-\rho_Y}[\rho_M^{\rho_M}(1-\rho_M)^{1-\rho_M}]^{1-\rho_Y}(\phi^j_{Yt})^{1-\rho_Y}$, the maximized utility for a young agent that results from substituting in the optimal decisions can be similarly written as
$$V^j_{Yt}(W^j_{Yt})=\ln(W^j_{Yt})-\ln(Q^j_{Yt}).$$
Thus, the price of the universal target-date fund fully captures the dependence of the value function on the investment opportunity set. Note that, unless $\gamma=1$, wealth and Q are no longer additively separable after the application of the appropriate curvature for risk preferences.

4.4 Market Clearing

In our economy, the safe discount bonds are in zero net supply, and the risky tree has an inelastic supply normalized to one. We now substitute the asset demands we derived in the previous subsection into the asset market clearing conditions. Recall that the individual-level risky tree (S) and safe bond (B) demands of the young are given by $S^j_{Yt} = (1-\rho_Y)W^j_{Yt}\theta^j_{Yt}/P_t$ and $B^j_{Yt} = (1 - \rho_Y)W^j_{Yt}R_t(1 - \theta^j_{Yt})$. The asset demands of the middle-aged are given by $S^j_{Mt} = (1 - \rho_M)W^j_{Mt}\theta^j_{Mt}/P_t$ and $B^j_{Mt} = (1 - \rho_M)W^j_{Mt}(1 - \theta^j_{Mt})R_t$. The old have no asset demand. Summing over the type j and the generations, the asset market clearing conditions can be written as

$$1 = \frac{1}{P_t} \sum_{j=H,L} \psi_j \left[(1 - \rho_M)W^j_{Mt}\theta^j_{Mt} + (1 - \rho_Y)W^j_{Yt}\theta^j_{Yt} \right] \tag{31}$$

and

$$0 = \sum_{j=H,L} \psi_j \left[(1 - \rho_M)W^j_{Mt}(1 - \theta^j_{Mt}) + (1 - \rho_Y)W^j_{Yt}(1 - \theta^j_{Yt}) \right], \tag{32}$$

where

$$\psi_j = \text{the constant fraction of agents of type } j.$$

Note that R_t is determined implicitly in equation (32) because the θ are functions of R_t.

In addition to the asset market clearing conditions, the goods market clearing condition is also instructive. It is

$$\frac{D_t}{\alpha} = \frac{\rho_Y(1 - \alpha)D_t}{\alpha} + \rho_M(\psi_H W^H_{Mt} + \psi_L W^L_{Mt}) + W_{Ot}, \tag{33}$$

where the wealth level of the old W_{Ot} is given by the sum across the two types. Note that the propensity to consume out of wealth is 1 for the old, so there is no parameter multiplying W_{Ot}. Since the asset demands for old agents are equal to zero regardless of the level of risk aversion, we do not need to keep track separately of the wealth of each old group.

The total financial wealth of the economy is the sum of the value of the tree P_t and the dividend D_t, since the risk-free asset is in zero net supply. At the beginning of the period all financial wealth is held by the middle-aged and the old. Hence, the total wealth of the old is

$$W_{Ot} = P_t + D_t - \psi_H W_{Mt}^H - \psi_L W_{Mt}^L. \tag{34}$$

Substituting this expression into equation (33) and solving for P_t gives rise to an equation for P_t based only on contemporaneous variables:

$$P_t = (1 - \rho_Y)\frac{1-\alpha}{\alpha}D_t + (1 - \rho_M)(\psi_H W_{Mt}^H + \psi_L W_{Mt}^L). \tag{35}$$

The right-hand side of (35) can be interpreted simply as the sum of savings supplies, and therefore total asset demands, of the young and middle aged, determined by their propensities to save and initial levels of wealth.

Normalizing all the variables in the above equation by dividend D_t gives the expression for the price-dividend ratio p_t,

$$p_t = (1 - \rho_Y)\frac{1-\alpha}{\alpha} + (1 - \rho_M)(\psi_H w_{Mt}^H + \psi_L w_{Mt}^L), \tag{36}$$

where lower case letters denote the corresponding variables divided by the dividend D_t.

Thus, the price-dividend ratio p_t is linear in the total (per-dividend) middle-aged wealth $\psi_H w_{Mt}^H + \psi_L w_{Mt}^L$. The first term on the right-hand side of (36) corresponds to the demand for saving of the young, which is a constant share of dividends and is therefore a constant in the per-dividend expression.

Having an analytic expression for the price-dividend ratio both adds transparency to our analysis and simplifies the solution of the model. Analytic expressions for portfolio shares and expected returns for the risky and risk-free assets are not available. In Section 5, we will turn to numerical solutions of the model. Unlike the price-dividend ratio, which only depends on total middle-aged wealth, portfolio shares and expected returns will depend on the distribution of wealth among the middle-aged agents. This distribution depends on the history of shocks. For example, as we will discuss explicitly in Section 5, a good dividend growth shock raises the share of wealth held by the middle-aged risk-tolerant agents who invested heavily in the risky asset when young. This increased wealth share of the risk-tolerant agents in turn affects asset demands and expected returns.

4.5 Equilibrium Solution

4.5.1 State Variables

Consider the state variable vector at the moment prices are determined. At that moment, the wealth of each group of agents is known. Because all preferences are recursive, nothing from the past directly enters into the preferences induced over actions now and in the fu-

ture. The sum of human wealth and non-human wealth for each agent, together with the stochastic process for returns from that moment on fully determines what actions maximize an agent's utility. Each agent's decision problem has a perfect scale symmetry because of the homotheticity of preferences together with linear budget constraints. This scale symmetry aggregates perfectly. Thus, only wealth relative to D_t matters for behavior. Furthermore, the scale symmetry means that behavior can be smoothly aggregated across agents within a type, so that, in effect, there is a representative agent of each type. Thus, despite the complexity of the agents' asset allocation choices (particularly the choices of the young, who have to worry about intertemporal hedging), the set of possible equilibria is determined entirely by forward-looking considerations. These decisions are conditioned on the wealth of each type at that moment when prices and therefore pre-consumption wealth are known.

These considerations indicate that the dynamic stochastic equilibrium from any point on only needs to be a function of the vector of normalized wealth levels for each type of agent. (Recall that the normalization is dividing through by the dividend, D.) What is less obvious is that, once all old agents are considered as belonging to one type, since their risk aversion no longer matters for their behavior, the vector of possible normalized wealth levels spans only a two-dimensional space. For convenience, represent a point on this two-dimensional space by the normalized wealth of the daring and cautious middle-aged agents. By (36), the sum of middle-aged wealth, together with the constant value of the wealth of the young in relation to the dividend determines the price of the tree and hence the value of all wealth. Of the nonhuman wealth owned at the beginning of the period, everything not in the hands of

middle-aged agents must be in the hands of the old, since the absence of inheritance means that the young begin their first period of economic life with zero nonhuman wealth. Thus, the wealth of the old can be deduced from the wealth of the two middle-aged types. And as mentioned already, the normalized human wealth of the young is constant at $\frac{1-\alpha}{\alpha}$. So we can take the state variable vector to be the two-dimensional vector of normalized wealth levels of the middle-aged types where wealth is measured prior to consumption.

Because the forward-looking behavior of each type (other than the old, who are simple) makes the dynamic stochastic general equilibrium complex, it is not possible to give a closed-form expression for the Markov transition from one period to the next. But again, since each representative agent's range of behavior only depends on that type's wealth and forward-looking expectations for returns given different possible states of the economy, no lagged variable needs to enter into the determination of the equilibrium. Taking as given the outcome of the complex decision problem of agents in one period (including the complexity of needing to figure out the entire forward-looking dynamic stochastic general equilibrium), it is possible to give a closed form solution for the price of the tree in the next period. We discuss this in the next subsection.

4.5.2 Markov Representation

To show that our model is Markov, we make two points. First, the optimal portfolios are entirely forward-looking. Individual agents only care about the intertemporal probabilistic pattern of returns over their lifetime. The expected return distribution is entirely determined by the state vector of the aggregate normalized wealth levels of the two types of middle-aged

agents. Second, given the forward-looking portfolio behavior of agents, what happens in the next period is fully determined.

Once agents have made their decisions at time t, the key facts that go forward to the next period are the assets owned by the daring and cautious middle-aged agents ($j = \{L, H\}$). The middle-aged wealth next period is given by the portfolio choice of the young in the current period, prices and returns next period. Specifically, consider a young household of type $j = \{H, L\}$ with savings of $(1 - \rho_Y)\frac{1-\alpha}{\alpha}$. By investing optimally a share θ^j_{Yt} of her savings in the risky asset, her wealth at the beginning of the next period (turning middle-aged) after the dividend growth rate shock is given by

$$w^j_{M,t+1} = \left[(1 - \theta^j_{Yt})\frac{R_t}{G_{t+1}} + \theta^j_{Yt}\frac{1 + p_{t+1}}{p_t}\right](1 - \rho_Y)\frac{1 - \alpha}{\alpha}. \tag{37}$$

In the square brackets of the right-hand side, the first term is the return on the part of savings in bonds and the second term is the return on the part of savings in the tree. The total returns from both bonds and tree imply their wealth when middle-aged.

To derive the law of motion of wealth, we use equation (36) to eliminate the future price p_{t+1} from equation (37), and obtain

$$w^H_{M,t+1} = \frac{A_2 B_1 - (A_1 + \frac{1}{\psi_L})B_2}{A_1 B_1 - (A_1 + \frac{1}{\psi_L})(B_1 + \frac{1}{\psi_H})}\frac{1}{\psi_H}, \tag{38}$$

$$w^L_{M,t+1} = \frac{A_2(B_1 + \frac{1}{\psi_H}) - A_1 B_2}{(A_1 + \frac{1}{\psi_L})(B_1 + \frac{1}{\psi_H}) - A_1 B_1}\frac{1}{\psi_L}, \tag{39}$$

where

$$A_1 = -\frac{(1-\rho_M)(1-\rho_Y)^{\frac{1-\alpha}{\alpha}}\theta_{Yt}^L}{p_t}$$

$$A_2 = (1-\rho_Y)\frac{1-\alpha}{\alpha}\left[\frac{(1-\theta_{Yt}^L)R_t}{G_{t+1}} + \frac{\theta_{Yt}^L\left(1+(1-\rho_M)\frac{1-\alpha}{\alpha}\right)}{p_t}\right],$$

$$B_1 = -\frac{(1-\rho_M)(1-\rho_Y)^{\frac{1-\alpha}{\alpha}}\theta_{Yt}^H}{p_t},$$

$$B_2 = (1-\rho_Y)\frac{1-\alpha}{\alpha}\left[\frac{(1-\theta_{Yt}^H)R_t}{G_{t+1}} + \frac{\theta_{Yt}^H\left(1+(1-\rho_M)\frac{1-\alpha}{\alpha}\right)}{p_t}\right].$$

Note that A_1, A_2, B_1, and B_2 are time varying. See Appendix A.1 for how they are determined.

We now present a complete characterization of the solution (see Section 4.5.3 and Appendix A.1) and demonstrate that there exists a stationary Markov equilibrium of the model (see Section 4.5.4 and Appendix A.2). We discuss the computation of the equilibrium in Appendix A.3.

The proof of the existence of the equilibrium follows directly from applying the results of Duffie, Geanokoplos, Mas-Collel, and McLennan (1994a) including an unpublished proof graciously supplied by Andrew McLennan (Duffie, Geanokoplos, Mas-Collel, and McLennan, 1994b). Appendix A.2 extends that proof by showing that it applies to our three-period model rather than their two-period model and that its logic applies to the Epstein-Zinn preferences that we assume.

4.5.3 Definition of Equilibrium

The model and its solution in the paper are stated in terms of variables indexed by time, notation which is natural for the analysis of shocks that lead to rebalancing. In this section, we turn to a proof of the existence of the stationary Markov equilibrium. Since the equilibrium is in terms of the time invariant functions, we use **bold** to denote the equilibrium functions associated with these variables. Consistent with the notation tradition of the literature on Markov equilibria, the prime ($'$) denotes the next period variables or functions.

We review the relevant notation. The idiosyncratic dividend growth rate shock G has a finite support $\{G_n\}_{n=1}^N$ with probability π_n for each G_n. The middle-aged post-shock wealth $w = (w_M^L, w_M^H)$ is the state variable of the economy. We will use lower-case letters of variables to indicate the values of the variables divided by same-dated dividends: $x = X/D$. ρ_Y and ρ_M denote the marginal propensities to consume for the young and the middle aged, respectively. ψ_j denotes the constant fraction of agents of type j in each cohort. γ_j is the risk aversion parameter of type j. α denotes the capital share, so the labor income of the young is $w_Y^j = w_Y = \frac{1-\alpha}{\alpha}$ for both $j = H, L$.

We first define the recursive equilibrium by collecting the equations that fully characterize the equilibrium, and then discuss these equations in more detail. Equation numbers in the square brackets cross-reference the equations for the variables. (Note that the cross-referenced equations are not normalized by dividends [capital letters instead of lower case letters].) We denote the equations defining the stationary Markov equilibrium in terms of functions as SM.1, SM.2, etc. to distinguish them from the equations elsewhere in the text,

which are expressed in terms of variables.

The recursive equilibrium consists of price functions $\{\boldsymbol{p}(w), \boldsymbol{R}(w), \boldsymbol{Z}(w, G')\}$, allocations $\{\boldsymbol{c_Y^j}(w), \boldsymbol{c_M^j}(w), \boldsymbol{c_O}(w), \boldsymbol{\theta_Y^j}(w), \boldsymbol{\theta_M^j}(w)\}_{j=H,L}$, associated return functions $\{\boldsymbol{\phi_M^j}(w)\}_{j=H,L}$, and the law of motion of wealth $\boldsymbol{w'}(w, G') = (\boldsymbol{w_M^{L\prime}}(w, G'), \boldsymbol{w_M^{H\prime}}(w, G'))$, such that

1. Given prices and laws of motion of wealth, the allocations solve the household problems:

$$\boldsymbol{c_Y^j}(w) = \rho_Y w_Y, \quad [26] \tag{SM.1}$$

$$\boldsymbol{c_M^j}(w) = \rho_M w_M^j, \quad [19] \tag{SM.2}$$

$$\boldsymbol{c_O}(w) = 1 + \boldsymbol{p}(w) - (\psi_H w_M^H + \psi_L w_M^L), \quad [34] \tag{SM.3}$$

$$\boldsymbol{Z}(w, G') = \frac{(\boldsymbol{p}(\boldsymbol{w'}(w, G')) + 1)G'}{\boldsymbol{R}(w)\boldsymbol{p}(w)}, \quad [8] \tag{SM.4}$$

$$\boldsymbol{\theta_M^j}(w) = \arg\max_{\theta \in \Theta} \left(E_{G'} \left[\{1 - \theta + \theta \boldsymbol{Z}(w, G')\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}, \quad [15] \tag{SM.5}$$

$$\boldsymbol{\phi_M^j}(w) = \boldsymbol{R}(w) \left(E_{G'} \left[\left\{ 1 - \boldsymbol{\theta_M^j}(w) + \boldsymbol{\theta_M^j}(w) \boldsymbol{Z}(w, G') \right\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}, \quad [16] \tag{SM.6}$$

$$\boldsymbol{\theta_Y^j}(w) = \arg\max_{\theta \in \Theta} \left(E_{G'} \left[\left\{ [1 - \theta + \theta \boldsymbol{Z}(w, G')] \boldsymbol{\phi_M^j}(\boldsymbol{w'}(w, G'))^{1-\rho_M} \right\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}. \quad [27] \tag{SM.7}$$

2. Prices clear goods markets:

$$\boldsymbol{p}(w) = (1 - \rho_Y)w_Y + (1 - \rho_M)(\psi_H w_M^H + \psi_L w_M^L), \quad [36] \qquad \text{(SM.8)}$$

and bond markets:

$$0 = \sum_{j=H,L} \psi_j \left\{ (1 - \rho_M)w_M^j(1 - \boldsymbol{\theta_M^j}(w)) + (1 - \rho_Y)w_Y(1 - \boldsymbol{\theta_Y^j}(w)) \right\}. \quad [32] \text{ (SM.9)}$$

Note that R_t is determined implicitly in the bond market equation through the dependency of the portfolio shares on it.

3. Wealth evolves consistent with portfolio allocations and prices:

$$\boldsymbol{w_M^{H\prime}}(w, G') = \left[(1 - \boldsymbol{\theta_Y^H}(w))\frac{\boldsymbol{R}(w)}{G'} + \boldsymbol{\theta_Y^H}(w)\frac{1 + \boldsymbol{p}(\boldsymbol{w'}(w, G'))}{\boldsymbol{p}(w)} \right] (1 - \rho_Y)w_Y. \quad [38] \text{ (SM.10)}$$

$$\boldsymbol{w_M^{L\prime}}(w, G') = \left[(1 - \boldsymbol{\theta_Y^L}(w))\frac{\boldsymbol{R}(w)}{G'} + \boldsymbol{\theta_Y^L}(w)\frac{1 + \boldsymbol{p}(\boldsymbol{w'}(w, G'))}{\boldsymbol{p}(w)} \right] (1 - \rho_Y)w_Y, \quad [39] \text{ (SM.11)}$$

See Appendix A.1 for further discussion of these equations and the determination of their solution.

4.5.4 Existence of Equilibrium

PROPOSITION. There exists a recursive equilibrium that satisfies equations (SM.1)-(SM.11).

Proof. See Appendix A.2.

Duffie, Geanokoplos, Mas-Colell, and McLennan (1994) have a result (Proposition 1.2) under quite general conditions that if an equilibrium always exists for any finite-horizon, then an ergodic Markov equilibrium also exists in the infinite horizon. The proof in Appendix A.2 shows that their general proof applies to our specific setting.

4.5.5 Alternative State Variable Representation

The combination of the price-dividend ratio and aggregate risk tolerance is sufficient to fully characterize the state of the model economy at any point in time. In terms of our model, aggregate risk tolerance is

$$
\mathcal{T}_t = \frac{\mathcal{W}_{Yt}^H \tau_H + \mathcal{W}_{Mt}^H \tau_H + \mathcal{W}_{Yt}^L \tau_L + \mathcal{W}_{Mt}^L \tau_L}{\mathcal{W}_t},
\tag{40}
$$

where the savings or invested wealth of each type $j = \{H, L\}$ and $a = \{Y, M\}$ is $\mathcal{W}_{at}^j = (1 - \rho_a)\psi_j w_{at}^j D_t$ and the total invested wealth across all types is $\mathcal{W}_t = \mathcal{W}_{Yt}^H + \mathcal{W}_{Mt}^H + \mathcal{W}_{Yt}^L + \mathcal{W}_{Mt}^L$.[7] Both the price dividend ratio and aggregate risk tolerance are simple functions of (w_{Mt}^H, w_{Mt}^L).[8] One could in fact view $(p_t, \mathcal{T}_t)$ as the state vector for the economy as an alternative to the equivalent description of the state by (w_{Mt}^H, w_{Mt}^L). We study the dynamics

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[7]Note that the relevant wealth for this aggregation in our model is savings or invested wealth, i.e., wealth after consumption. In the continuous-time Merton model, there is no consumption during the investment period, so the distinction between before- and after-consumption wealth is absent.

[8]Recall (36) and rewrite (40) as

$$
\mathcal{T}_t = \frac{(1 - \rho_Y)[(1 - \alpha)/\alpha][\psi_H \tau_H + \psi_L \tau_L] + (1 - \rho_M)[\psi_H w_{Mt}^H \tau_H + \psi_L w_{Mt}^L \tau_L]}{(1 - \rho_Y)[(1 - \alpha)/\alpha] + (1 - \rho_M)[\psi_H w_{Mt}^H + \psi_L w_{Mt}^L]}.
$$

These two equations are easily invertible (indeed, given p_t and $\mathcal{T}_t$ they translate into a pair of linear equations in w_{Mt}^H and w_{Mt}^L). Thus, there is a one-to-one mapping between (w_{Mt}^H, w_{Mt}^L) and the vector $(p_t, \mathcal{T}_t)$.

of both variables below.

5   Quantitative Results

In this section, we examine numerical solutions of a particular parameterization of our model focusing on the optimal portfolio allocations and rebalancing behavior of households in general equilibrium. Portfolios vary across agents with different levels of risk tolerance. In particular, the optimal portfolio share is larger than the market portfolio share (100%) for the high-risk-tolerance agents and lower than the market portfolio share for the low-risk-tolerance agents. The portfolio tree shares of all types comove in response to the dividend growth rate shock: declining after a good shock and rising after a bad shock. The underlying wealth dynamics ensure that such comovements in the portfolio shares are consistent with general equilibrium.

5.1   Parameterization and Solution

Table 1 summarizes the parameter values used in the quantitative analysis. A model period corresponds to 20 years in the data. The 20-year discount rate of 0.67 corresponds to an annual discount rate of 0.98. Consequently, ρ_M is 0.75 and ρ_Y is 0.69. Capital's share is 0.33. The cross-sectional heterogeneity in risk tolerance is set to match the findings in Kimball, Sahm, and Shapiro (2008). They estimate the mean risk tolerance to be 0.21 and the standard deviation to be 0.17. The distribution they estimate has skewness of 1.84. We match these three moments by giving 92% of agents a low risk tolerance τ_L of 0.156 and

8% of agents a high risk tolerance τ_H of 0.797. We refer to the low-risk-tolerance agents as *cautious agents* and the high-risk-tolerance agents as *daring agents*.

The dividend growth shock process is assumed to be i.i.d with two states of equal probability. In the bad state, the detrended gross growth rate is $G_1 = 0.67 \approx (\frac{1.005}{1.025})^{20}$ and in the good state, it is $G_2 = 1.5 \approx (\frac{1.045}{1.025})^{20}$. With our 20-year time horizon, these parameters imply that the dividend growth rate is 2.5% per year in expected terms, 0.5% per year in the bad state and 4.5% per year in the good state. The dividend growth rate shocks capture generational risk. The mean scenario mimics the experience of the United States, the good scenario mimics that of South Korea, and the bad scenario mimics that of the Japan in the last two decades. Since all key variables below are expressed on a per-dividend basis and the elasticity of intertemporal substitution is equal to 1, the trend growth rate of 2.5% per year does not affect any of our analysis.

Table 1: Benchmark Parameter Values

Capital's Share	α	0.33
Discount Rate	δ	0.33
Middle-aged Average Propensity to Consume	ρ_M	0.75
Young Average Propensity to Consume	ρ_Y	0.69
Low Risk Tolerance	τ_L	0.156
High Risk Tolerance	τ_H	0.797
Fraction of Low Risk Tolerance Agents	ψ_L	0.92
Fraction of High Risk Tolerance Agents	ψ_H	0.08
i.i.d. Dividend Growth Shock Process		
Detrended Growth Rate	(G_1, G_2)	$(0.67, 1.50)$
Probability	(π_1, π_2)	$(0.50, 0.50)$

As noted above, our model does not have an analytical solution, so we solve the model numerically. To make the model stationary, we normalize relevant variables with same-dated

dividends. Given six types of agents and two types of assets in our model, it might seem that the dimension of the state space would be large. As shown in Section 4.5, however, we are able to reduce the state space to two endogenous variables: the wealth levels of the middle-aged cautious and middle-aged daring agents (w_{Mt}^H, w_{Mt}^L). We are able to omit states governing the dividend growth rate from the state space due to the i.i.d. assumption. We discretize the state space in our numerical solution, but we allow the decision variables to be continuous. We use the recursive dynamic technique to solve the model. For the detailed computational algorithm see Appendix A.

We now present the quantitative behavior of our model economy, focusing particularly on asset holdings. Our findings on price and return dynamics are not the main contribution of this paper, so we discuss them in Appendix B.

5.2   The Dynamics of Risk Tolerance

Understanding the endogenous wealth dynamics across agents is central to understanding the dynamics of asset holdings. The history of shocks matters because it affects the distribution of wealth and therefore aggregate risk tolerance. We define aggregate risk tolerance in Equation (40). We display impulse response functions, starting from the average steady state and then show the average response of the economy after a good or bad shock in period 1. Figure 2 shows how aggregate risk tolerance responds to both positive and negative shocks. The plot is symmetric because it gives deviations from the mean and good shocks and bad shocks are equally likely.

Figure 2: Aggregate Risk Tolerance
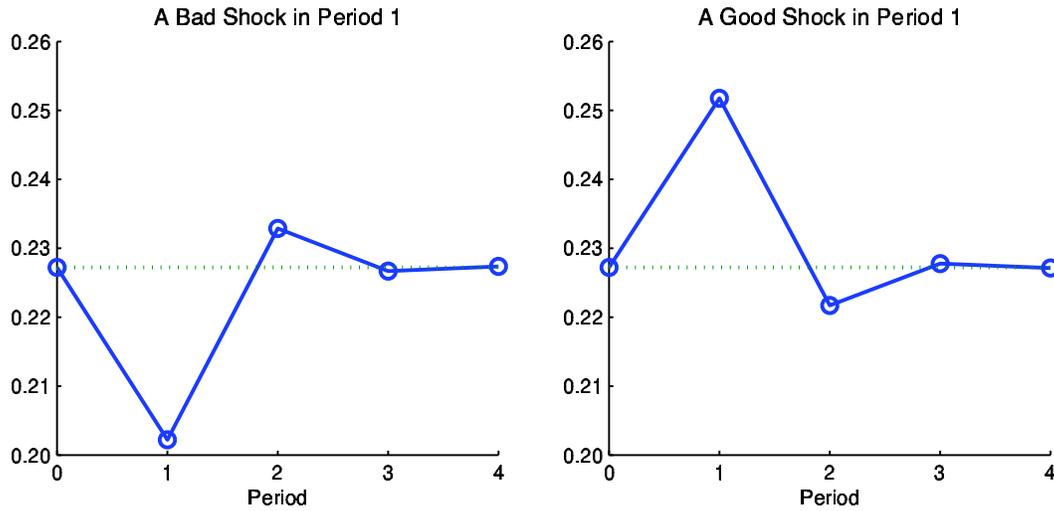
The impulse response shown in Figure 2 shows aggregate risk tolerance rises as a result of a good shock and falls as a result of a bad shock. While the long-run average aggregate risk tolerance is about 0.227, it falls to 0.202 after a bad shock and rises to 0.25 after a good shock. This is because a good shock corresponds to a positive return and an increase in the wealth of daring agents. The shock in period 1 matters less for risk tolerance in period 2 because one of the affected cohorts dies, and another affected cohort reaches the simplicity of old age. Yet on average the period 1 movement in risk tolerance rebounds so that it is more than reversed in period 2.

What explains the dependence of aggregate risk tolerance in period 2 on the shock in period 1? A good shock at time 1 puts a large share of invested wealth in the hands of daring middle-aged agents, who then hold a large fraction of all available shares of the risky asset, leaving less for their more cautious peers and for the young cohort as a whole. Thus, the overall holdings of the young cohort are less aggressive when they enter economic life on the

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heels of a good dividend-growth-rate shock than when they enter economic life on the heels of a bad dividend-growth-rate shock. This is particularly relevant for daring young agents, as it means that the fraction of all invested wealth in middle age that they hold is less than it would otherwise be. Thus, after a good shock in period 1 the wealth of daring middle aged agents in period 2 is unusually low, and therefore the aggregate risk tolerance in that state of the world is also unusually low. This explanation is symmetric, so that after a bad shock in period 1 risk tolerance is unusually high in period 2. Similar reasoning can explain how shocks affect risk tolerance in later periods. In principle each shock affects all future values of aggregate risk tolerance, but in practice the effect of any shock tends to average out after 3 periods.

Figures 3 plots the impulse responses for wealth shares after bad and good shocks and Figure 4 shows how the aggregate wealth shares of different types of agents sum to one. Consistent with the rest of our results, the wealth share of the middle aged daring goes down after a bad shock and up after a good shock. The wealth share of the middle aged cautious shows the opposite pattern, and the wealth share of the young does not vary much with shocks.

5.3   Portfolio Choice and Rebalancing

Most modern discussions of asset pricing in general equilibrium focus on the properties of asset prices and returns. Asset holdings are left implicit. Since we are particularly concerned with implications for individual asset demands, we look directly at portfolio choice and

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Figure 3: Wealth Shares by Type
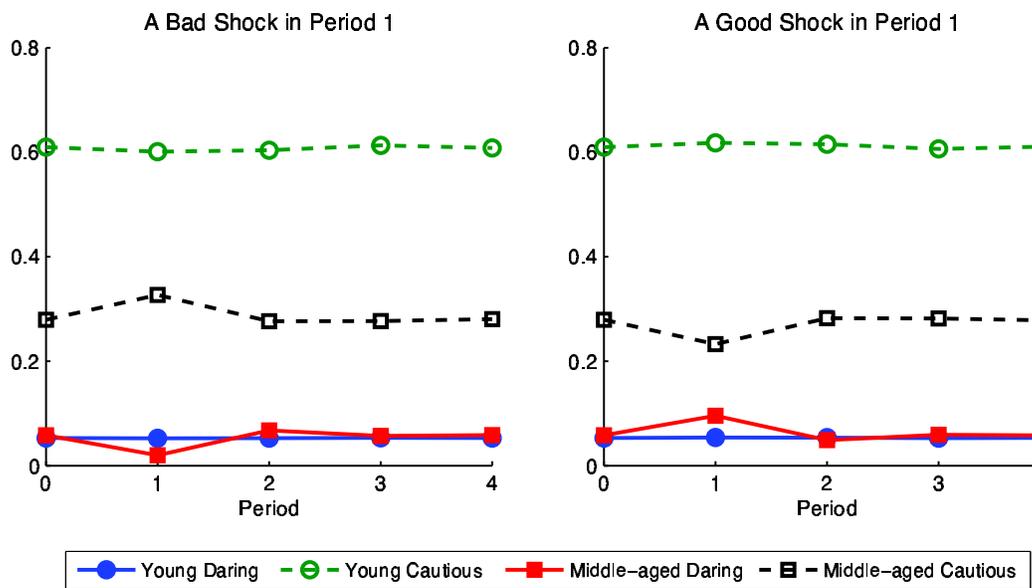
Figure 4: Aggregating Wealth Shares by Type
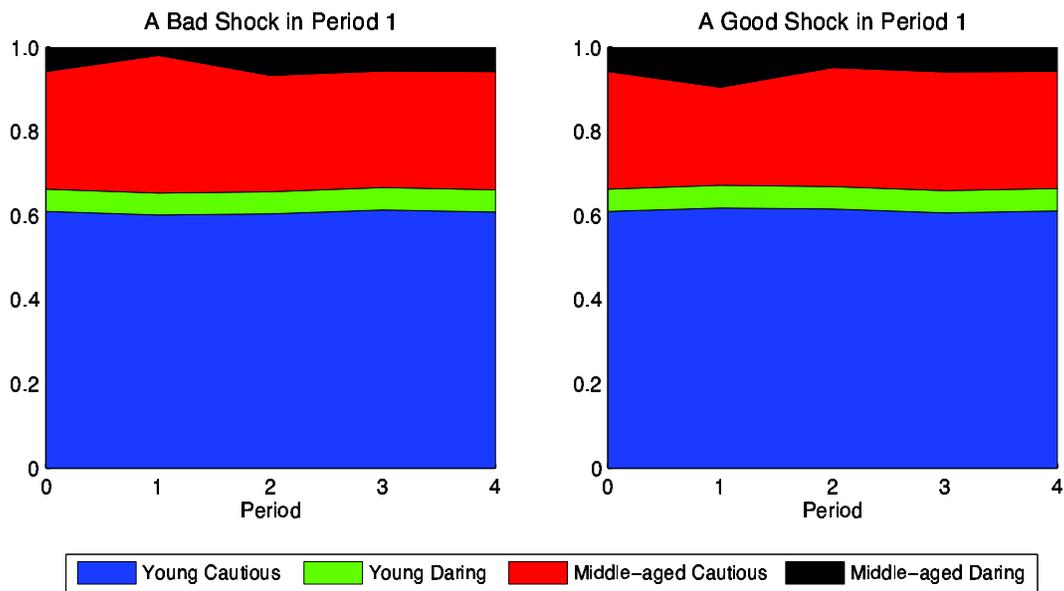
rebalancing in response to shocks in this subsection. Figure 5 plots the impulse response of the equilibrium tree holdings of different types of agents to both bad and good shocks. Figure 6 plots the same thing in a cumulative fashion to show explicitly that the holdings always sum to one.

Figure 5: Equilibrium Tree Holdings
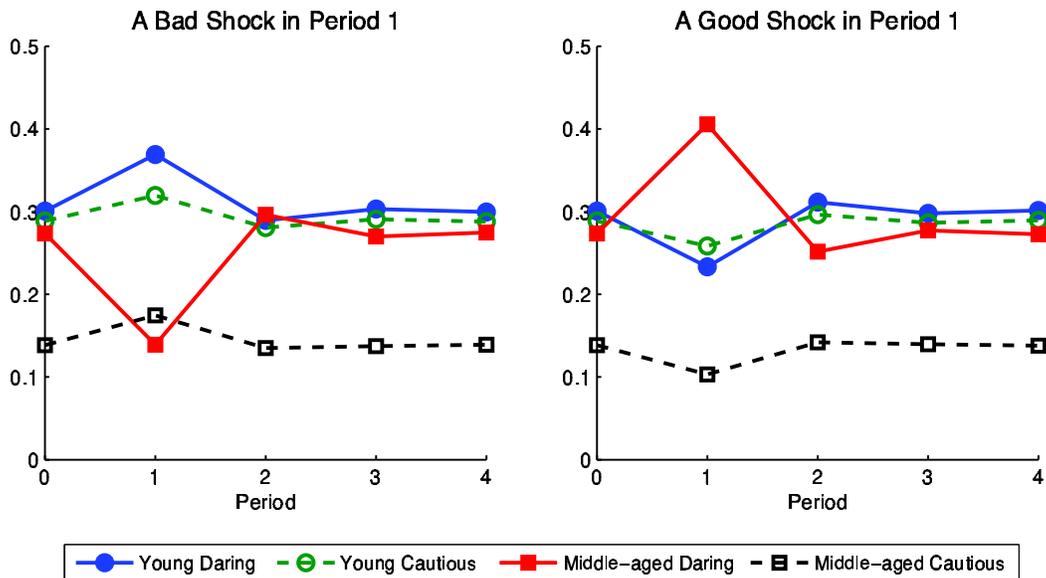
Like the impulse response plot of aggregate risk tolerance in Figure 2, the period 0 value in these plots is the long-run mean tree holding for each group. Also like Figure 2, these figures are symmetric in the shock, in that the impulse response for positive shocks is the opposite relative to that mean of the impulse response for the negative shocks. Both figures show that tree holdings respond substantially to good and bad shocks. After a bad shock, the tree holdings of the young agents of both types increase. The holdings of daring middle aged investors fall substantially while those of cautious middle-aged investors rise. After a good shock the holdings of daring middle-aged investors rise substantially and those of the

Figure 6: Tree Holdings in Aggregate
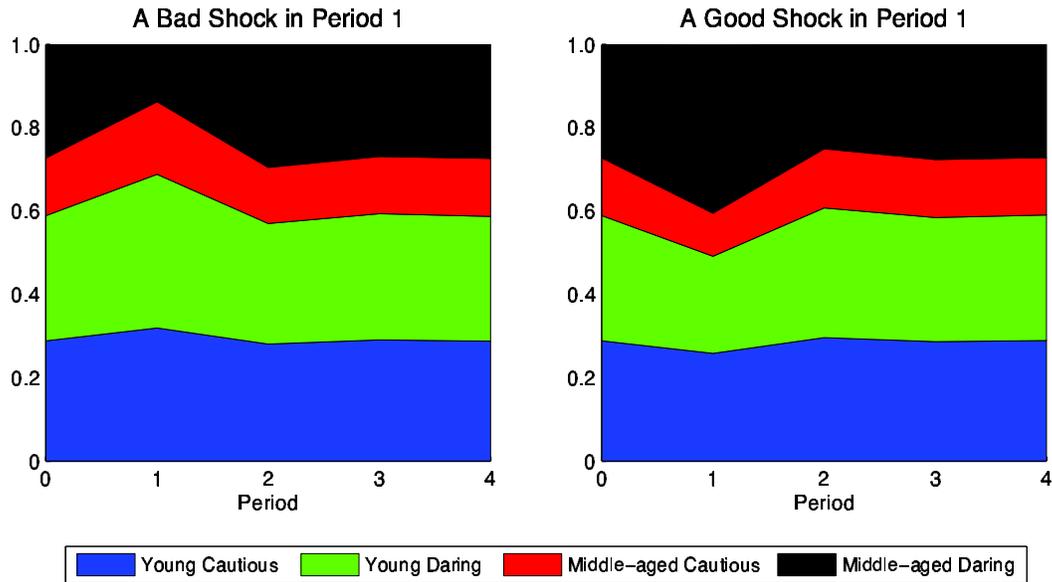
other types fall. By period 2 these effects are largely reversed.

The effects in Figures 5 and 6 are driven by the same type of logic used to explain aggregate risk tolerance above. After a bad shock the wealth of middle-aged daring investors has declined substantially and they hold less of the risky asset. Because the risky asset is relatively cheap, other investor types (young daring, young cautious, middle-aged cautious) are happy to hold more. The middle-aged cautious investors are rebalancing in the direction (but not the exact extent) a constant share rule would imply. The logic after a good shock is the reverse.

The findings of these figures are developed further in Table 2. The first three columns of Table 2 report the portfolio share of each type of agent. The statistics in the table help us to isolate three important effects: leverage, aggregate risk tolerance, and intertemporal hedging.

First, *leverage* arises endogenously, with the daring agents being highly levered and with the cautious agents holding a risky, but unlevered portfolio. As shown in the first column of Table 2, the cautious agents hold only about 48% of their savings (invested wealth) in the risky asset and the rest in the risk-free bonds. By contrast, the daring agents invest on average 568% of their invested wealth in the risky asset by selling the risk-free bonds to the cautious agents. This difference in portfolio shares across types enables shocks to change the distribution of wealth across types.

Table 2: Portfolio Choice, Savings Weight and Asset Demand by Type (Percent)

G_t	Portfolio Share θ			Savings Weight $\frac{W_i}{W}$		Tree Amount $\frac{\theta W_i}{W}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	48	53	43	92.8	85.0	49	36
Young	47	53	42	60.1	61.7	32	26
Middle	49	53	44	32.7	23.3	17	10
Daring	568	704	432	7.2	15.0	51	65
Young	570	706	433	5.2	5.4	37	23
Middle	566	701	432	2.0	9.6	14	42
Total	90	105	74	100.0	100.0	100	100
Young	89	105	73	65.3	67.1	69	48
Middle	90	105	75	34.7	32.9	31	52

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Second, as the second and third columns of Table 2 show, one of the most striking implications of the model is that all types have a higher mean portfolio tree share after a bad shock than after a good shock due to the *aggregate risk tolerance* effect. A bad shock pushes down aggregate risk tolerance and raises the expected excess return. As a result, all types of agents choose a larger portfolio share. After a good shock, aggregate risk

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tolerance rises, the expected excess return declines, and all agents decrease their portfolio shares. It may seem surprising that all agents are able to move their portfolio shares in the same direction in general equilibrium. Section 5.5 below shows that changes in the wealth distribution combined with the effects of leverage, in particular, changes in the share of all invested wealth in the hands of middle-aged daring agents, allow all portfolio shares to move in the same direction.

Third, Table 2 indicates that for each level of risk tolerance the portfolio share of the young is similar to that of the middle-aged. Recall that we abstract from the dynamic interaction between human capital and portfolio choice by having labor income only at the beginning of the life. The comparison between young and middle-aged portfolios for a given level of risk tolerance thus isolates the pure intertemporal hedging effect. Table 2 indicates that the intertemporal hedging effect is small quantitatively. Qualitatively, the cautious young have a lower portfolio share than the cautious middle-aged, while the daring young have a higher portfolio share than the daring middle-aged. Section 5.7 discusses these implications in detail.[9]

The average young portfolio share has important implications for the dynamics of the total middle-aged wealth and aggregate risk tolerance next period. Recall that the cautious agents account for 92% of the population and the daring agents account for the other 8%. The average young portfolio share is 105% ($= 0.92 \times 53 + 0.08 \times 704$) after a bad shock and 74% ($= 0.92 \times 43 + 0.08 \times 433$) after a good shock. Thus, the young hold a slightly

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[9]Our results do not appear to be special to the parameterization that we have assumed. In Appendix C we both halve and double the variation in risk tolerance and the magnitude of economic shocks. The results we obtain are qualitatively the same.

more daring portfolio (105%) than the market portfolio (100%) after a good shock, but a much more conservative portfolio (74%) after a bad shock. These patterns for the portfolio share of the young contribute to the history dependence of aggregate risk tolerance that we discussed in the previous subsection.

5.4   Prices and Returns

In this section we describe the prices and returns implied by the model in order to make the intuition behind our rebalancing results clear. Table 3 presents the summary statistics based on a model simulation of 10,000 periods, which is a sufficiently long sample to eliminate the simulation uncertainty. Returns are expressed as annual rates. The log risk free rate has a mean of 1.9% and a standard deviation of 22 basis points. The log realized tree return is on average 3.63% with a standard deviation of 2%. Thus, the model economy generates a 1.7% risk premium with a standard deviation of 2%, resulting in an average Sharpe Ratio of 0.85. Although our model is not specifically designed to deliver a large risk premium, it goes a substantial fraction of the way toward explaining the historical premium. The model's Sharpe Ratio is quite high because the return volatility implied by the model is low. This results from our choice to match the volatility of macroeconomic shocks rather than market shocks.

Similarly, the model's expected returns and price-dividend ratio are much less volatile than in historical data. For instance, the price-dividend ratio is on average 19, with a standard deviation of 0.54. Because shocks to the fundamentals are i.i.d. in the growth

Table 3: Summary Statistics on Prices, Annualized

Variable	Mean	Standard Deviation	Correlation with G	Auto Correlation
ln(Risk Free Rate)	1.92%	0.22%	0.808	−0.059
ln(Realized Tree Return)	3.63%	2.00%	0.996	0.028
ln(Realized Excess Return)	1.70%	2.00%	0.997	−0.057
Expected ln(Tree Return)	3.61%	0.16%	0.285	−0.353
Expected ln(Excess Return)	1.68%	0.13%	−0.968	−0.172
Tree Return Volatility	2.00%	0.05%	−0.985	−0.114
Sharpe Ratio	0.85	0.05	−0.955	−0.203
Tree Price-Dividend Ratio	18.94	0.54	−0.471	−0.246

rate, movements in the realized return on the risky asset move essentially one-for-one with the dividend shock. Moreover, the i.i.d. growth shock means that any forecastability of returns comes from general equilibrium effects, not from the fundamentals *per se*. It turns out that these general equilibrium effects have a contribution to returns that is relatively small compared to sheer good and bad luck, so there is relatively little serial correlation in the realized risky return and the risk-free rate.

The expected excess return is critical for portfolio choice. It is strongly negatively correlated with the dividend growth rate shock and is somewhat mean reverting. Mechanically, these results are coming from movements in the risk-free rate, which is highly correlated with the shock to the dividend growth rate. Intuitively, the risk-free rate is moving because the shock to dividends changes the wealth of daring agents and therefore shifts the aggregate risk tolerance. We discuss the dynamics of the price dividend ratio in more detail in Appendix B.

We plot the Sharpe ratio of the risky asset in Figure 7. Consistent with the findings in Table 3, the Sharpe ratio increases after a bad shock and decreases after a good shock.

Figure 7: Sharpe Ratio
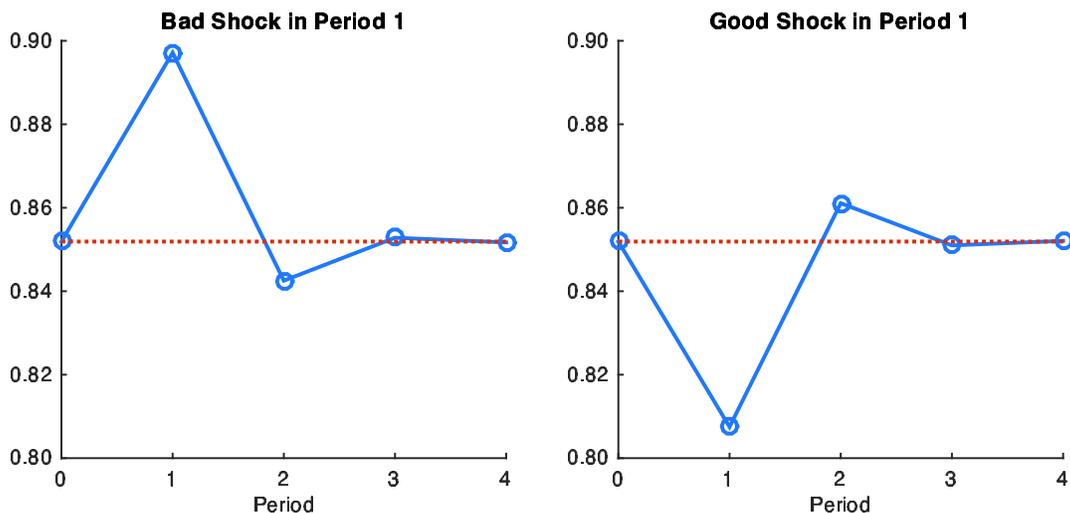
5.5   Correlated Movements in Portfolio Shares

It may seem surprising that all types of agents move their portfolios in the same direction in a general equilibrium. The key forces behind this finding are the changes in the total wealth and therefore in the invested wealth levels of the middle-aged agents that generate changes in the tree holdings $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$ of each type to satisfy the adding-up constraint

$$1 = \frac{\mathcal{W}_{Yt}^L}{\mathcal{W}_t}\theta_{Yt}^L + \frac{\mathcal{W}_{Yt}^H}{\mathcal{W}_t}\theta_{Yt}^H + \frac{\mathcal{W}_{Mt}^L}{\mathcal{W}_t}\theta_{Mt}^L + \frac{\mathcal{W}_{Mt}^H}{\mathcal{W}_t}\theta_{Mt}^H. \tag{41}$$

To understand the implications of this constraint, look first at the "savings weights," the weights for invested wealth or after-consumption wealth $\frac{\mathcal{W}_i}{\mathcal{W}}$, reported in the middle columns of Table 2. The cautious agents have a larger savings weight than the daring because the cautious have a larger mass. The savings weight of the cautious is, however, lower after a good shock (85%) than after a bad shock (92.8%), so the aggregate risk tolerance increases after a

good shock. Both the daring and cautious young agents have a slightly larger savings weight after a good shock than after a bad shock. The cautious middle-aged have a much smaller savings weight after a good shock (23.3%) than after a bad shock (32.7%). In contrast, the daring middle-aged have a much larger savings weight after a good shock (9.6%) than after a bad shock (2%).

Now, examine the quantity of trees held by each type of agent in the last two columns of Table 2. The quantity of trees held by each type equals that type's portfolio share multiplied by the savings weight of each type. Even though the economy has relatively fewer daring agents, the tree holdings of the daring agents are comparable in total size to those of the cautious agents due to their heavily levered position. After a good shock, the young agents have slightly higher savings weights but much lower portfolio shares, leading to much lower total tree holdings. After a good shock, the middle-aged cautious agents have a much lower savings weight and thus an even lower quantity of tree holdings given their decreased portfolio share. On the other hand, the middle-aged daring agents have a much higher savings weight after a good shock, almost four times higher. The increase in their savings weight dominates the decline in their portfolio share, implying a larger total quantity of tree holdings by the middle-aged daring agents.

This effect can be illustrated with the impulse response plot in Figure 8. This figure plots the reaction of each investor type to bad and good shocks by comparing their portfolio shares to their long-run mean portfolio shares. The shares of the young daring and middle-aged daring are so similar that the lines are very difficult to distinguish in the plot. The figure

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clearly shows that after a good shock, all portfolio shares drop. After a bad shock they rise.

Figure 8: Portfolio Shares relative to Mean
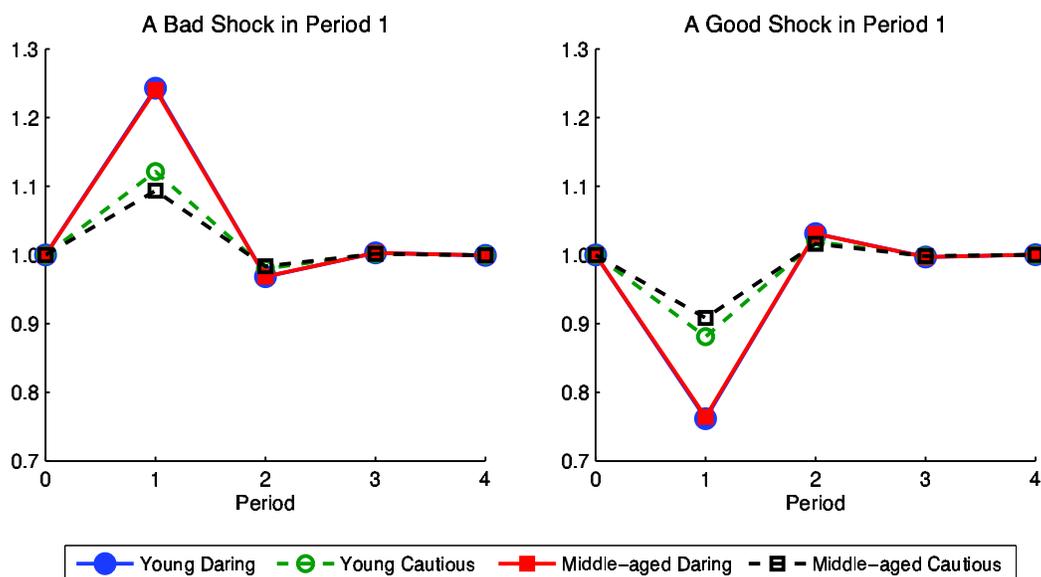
In our parameterization all types of agents in the economy adjust their portfolio shares in the same direction in response to the dividend growth shocks. In particular, all increase their portfolio shares after a bad shock and decrease them after a good shock. This is also true for the all of the dramatically different alternative parameterizations that we consider in Appendix C, in which we double and halve the size of the shock and double and halve the distribution of risk tolerance. It may be possible, however, to find counterexamples. Although it need not always be that everyone's shares move in the same direction, we consider this the natural case since the expected return is the main determinant of the optimal shares for people to hold. While intertemporal hedging also matters in determining optimal holdings, the variation in the expected return is much larger than the variation in

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hedging so its effect dominates. We discuss the magnitude of hedging effects below.

Even though all agents move their portfolio shares in the same direction, some must sell while others buy in order to attain their new positions. The changes in the underlying wealth distribution make such portfolio behavior consistent with general equilibrium.

5.6 Rebalancing by Cohort

So far we have been focusing on the portfolio dynamics of each type of agent at a point in time. Now we track the portfolio rebalancing of each cohort over time. When young, the agents choose an optimal portfolio share after the dividend shock. At the beginning of middle age, before any rebalancing, the value of their tree holdings as a share of their middle-aged wealth changes passively in response to the realized dividend shock and the realized tree return. We refer to this portfolio share as the realized portfolio share.[10] The agents then choose the optimal portfolio share for their middle age (governing wealth accumulation between then and old age). We present the sequence of the three portfolio shares in Table 4. Because we are tracking cohorts (or individuals), we group equilibrium observations by the history of shocks rather than reporting an impulse response as we have done previously. G_Y and G_M denote the shocks that the agents experience when young and when middle-aged, respectively.

Let's first examine the passive movements of the portfolio share of each cohort. Since the cautious young agents are not levered, the realized middle-age portfolio share rises sub-

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[10]This realized portfolio share implicitly assumes that consumption is financed by asset sales in proportion to asset shares, including proportionate paying off of debt for agents who have negative bond holdings.

Table 4: Rebalancing: Portfolio Share, by Cohort (Percent)

G_Y, G_M	Cautious Agents			Daring Agents		
	Young	Realized	Middle-Aged	Young	Realized	Middle-Aged
Bad, Bad	53.1	52.0	53.4	706	988	700
Good, Bad	41.8	39.9	53.4	433	594	702
Bad, Good	53.1	71.0	42.7	706	166	395
Good, Good	41.8	59.0	45.9	433	161	467

Note: G_Y and G_M denote the shocks that agents experience when young and when middle-aged, respectively.

stantially after a good shock at their middle age, and declines slightly after a bad shock. By contrast, since the daring young agents are highly levered, their realized middle-age portfolio share rises after a bad shock, but declines after a good shock. In addition, the magnitude of the changes in their portfolio share is quite large. For instance, the realized portfolio share rises from 706% to 988% after a history of two bad shocks, and declines from 706% to 166% after a bad shock when young followed by a good shock at middle age.

If the constant share rule were more or less correct, the agents would undo the movements in the realized middle-aged portfolio shares and return to the portfolio share when young. In terms of the direction of the adjustment at middle age, the constant share rule is consistent with the behavior of the cautious agents. The cautious agents do not, however, return to the same portfolio share when young. In fact, the new optimal level is typically quite different from the optimal share when young. The difference in the optimal shares between the two periods is particularly large when the shock they experience when young is different from the shock at middle age.

In addition to the quantitative differences, there are two types of qualitative differences between the optimal rebalancing shown in Table 4 and the constant share rule. First, even

returning to the same tree share requires opposite adjustments for people who seek to be more levered than the economy as a whole as compared to agents who seek to be less levered than the economy as a whole. Second, for the daring agents, the direction of their adjustments at middle age is sometimes opposite to the direction that would restore their tree share to what they had earlier. In particular, after a bad shock when young followed by a good shock at middle age, the daring middle-aged agents further increase their portfolio share above the realized portfolio share.

5.7   Intertemporal Hedging

Intertemporal hedging in our model arises when the portfolio shares shift consistently by age to take into account any predictability in returns. Referring back to Table 2, there is little difference in the portfolio choice between the young and the middle-aged. Given that our generational model produces only limited predictability in returns, it is not surprising that we see little intertemporal hedging. On the other hand, the qualitative implications of intertemporal hedging are still interesting and deserve some discussion.

To understand these patterns of intertemporal hedging, recall the portfolio choice problem of the young in equation (28) and of the middle-aged in equation (16). Unlike the middle-aged, who care about simple real returns, the young care about returns relative to the returns of the variable annuity, as indicated by (30). Table 5 presents the annuity price conditional on the current and past shocks.

For the cautious agents, the variable annuity price is higher after a bad shock than after

Table 5: Period-t Variable Annuity Price

	Cautious		Daring	
G_t	Bad	Good	Bad	Good
Mean	0.894	0.882	0.785	0.790
G_{t-1}				
Bad	0.891	0.889	0.762	0.802
Good	0.897	0.876	0.767	0.779

a good shock. The combination of a low risk-free rate and lower expected tree returns late in life after a bad shock makes it harder to finance a comfortable retirement (see Table 3). This unfavorable movement in the variable annuity price after a bad shock increases the effective risk of holding tree shares when young. By contrast, this variable annuity price is irrelevant for the middle-aged cautious agents; they care only about one-period returns. Thus, the young cautious agents invest less in the tree than the middle-aged cautious agents. The annuity price variation in period t is larger when the dividend shock is good in period $t-1$. Thus, this finding is reinforced when the cautious agents experienced a good shock when young.

For the daring agents, however, the variable annuity price is lower after a bad shock than after a good shock, since they tend to be so highly levered that the increase in the equity premium after a bad shock makes it easier for them to finance a comfortable retirement. This variable annuity price pattern helps the daring agents smooth consumption at their middle age, and thus the young daring agents invest more in the risky asset than the middle-aged daring agents. The variation in the period-t annuity price is larger when the dividend shock is bad (which yields a greater covariance of the equity premium with the later shock) than when the dividend growth shock is good. Thus, quantitatively, the young daring agents invest

more in risky assets than the middle-aged daring agents, especially when they experience a bad shock in their youth.

6 Conclusion

We develop a model to illustrate how optimizing agents of different ages and risk preferences rebalance in equilibrium. The dynamics of portfolios differ starkly from benchmark models where agents either hold constant portfolio shares or buy and hold the market portfolio. The model includes agents with different risk tolerance, so there is meaningful trade, and three periods of life, so there is meaningful rebalancing over time. The model has a partial analytic solution. In particular, the state of the economy can be summarized by the distribution of wealth of middle-aged investors. This analytic solution both helps make the findings of the paper transparent and simplifies the solution of the full dynamic equilibrium model.

There are several effects that are key to understanding rebalancing in equilibrium. First, there is a leverage effect. In the model, leverage arises endogenously because of the heterogeneity in risk preferences. Shocks to the valuation of risky assets cause levered agents to rebalance in the opposite direction from unlevered agents. Second, there is an aggregate risk tolerance effect. Shocks to the valuation of risky assets affect agents' wealth differently depending on their initial position in the risky asset. Such shifts in the distribution of wealth change aggregate risk tolerance and therefore the demand for assets going forward. Note that the leverage and aggregate risk tolerance effects are connected because leverage magnifies how valuation shocks change the distribution of wealth. For example, after a bad shock

leverage increases. The levered agents sell to reduce their leverage and the unlevered agents buy. But the increase in leverage is so great that even after rebalancing levered agents have higher exposure to risky assets. Therefore, all agents become more exposed to risk after a bad shock, and the opposite is true after a good shock

Third, there is an intertemporal hedging effect whereby equilibrium time variation in expected returns affects the portfolio choices of young investors as they exploit mean reversion in asset returns. This intertemporal hedging effect arising from mean reversion is what is often highlighted in analyses emphasizing time- or age-variation in portfolio shares. While it is present in our model, it is quantitatively weak by design. The model is solved under a specification with i.i.d. growth rates, so mean reversion in returns is not built into the specification. Other analyses of intertemporal hedging in the literature are based on empirical properties of actual returns, which might include mean reversion arising from mispricing. Our model has all agents frictionlessly optimizing, hence many sources of mean reversion that may be present in the data are intentionally not present in the model in order to highlight the implications of equilibrium for rebalancing.

Some key implications of the model are broadly consistent with recent rebalancing experience by age in data from the Survey of Consumer Finances. Determining the implications of our model for standard rebalancing data requires some thought. Our model features the underlying fundamentals: one risky asset and risk-free borrowing and lending. Data from markets feature stocks, bonds and other types of risky assets. Of course, these financial assets are claims on the fundamentals that also depend on the corporate capital structure.

In particular, leverage in the corporate sector has important implications for the risk exposure of stockholders. For example, consider an investor with 100 percent stocks when the capital structure in the economy is half equity and half debt. In terms of our model, that investor will have a risky asset share of two. Because of the leverage effect described in the model, as stock prices rise a 100 percent stock investor will have a lower risky tree share before rebalancing because her wealth grows faster than the value of her equity. What is going on is that when the value of all capital in the economy increases, stocks become less levered relative to the economy as a whole. Thus, to maintain the same leverage relative to the economy as a whole, an investor who began with 100 percent in stocks would need to buy more stocks after this market uptick. However, a finding of the paper is that because of general equilibrium forces, an investor will not rebalance to the same tree share as before the market uptick, but to a somewhat lower tree share, tempering somewhat the need to buy more stock. This example highlights how intuition about portfolios in terms of stocks must be interpreted with caution because of the underlying leverage of the corporate sector.

Looking back at the implied rebalancing from Survey of Consumer Finances data reported in Figure 1, several features of the recent history are consistent with the model but inconsistent with both the constant share and buy and hold rules. In the SCF data some of the older investors behave like the daring agents in the model, perhaps because of cohort effects. In Panel A, which reflects recent positive returns, older investors are actively buying to increase their stock share above the passive implied level for 2007. The reason they need to buy is that, because they are levered, when there is a good shock their exposure

to risk decreases as wealth increases more than the value of their equity holdings. Younger investors are either actively selling or simply not buying as much. Younger investors act like the cautious agents in our model, avoiding large risky asset holdings after a positive shock. In Panel B, which reflects recent negative returns, the patterns are reversed. Older investors sell more than the passive implied quantity of assets, and younger investors buy more than what would leave them at their passive level. These patterns are broadly consistent with the model, but inconsistent with a constant share or buy and hold rule.

Our framework is useful for understanding how portfolios would respond to a stock market crash in an economy where all investors were behaving optimally. Contrary to both the constant share and buy and hold rules, after a market crash our investors increase their share in risky assets relative to the pre-crash levels, though by different amounts depending on their risk tolerance.

Finally, our analysis points to a principle that must hold much more generally than the particular model we study here: funds that serve a nontrivial fraction of investors will have to take into account general equilibrium constraints to be feasible and optimal. In particular, many existing target-date funds, which follow simple linear allocation rules regardless of economic conditions, will not be optimal for most people. Moreover, as target-date funds become ubiquitous, they will become infeasible since they will not find counterparties to trade with as they rebalance. Our particular model points to how "universal target-date funds," or funds that are optimal for all investors given risk tolerance and age, should adjust their target asset allocation in response to economic shocks.

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Appendix A: Stationary Markov Equilibrium

A.1 Definition of Equilibrium

This appendix defines the equilibrium and discusses its solution. For convenience, Section 4.5.3 is repeated here. As discussed in Section 4.5.3, the time-invariant equilibrium functions associated with the model's variables are denoted in **bold**. Consistent with the notation tradition of the literature on Markov equilibria, the prime ($'$) denotes the next period variables or functions.

The idiosyncratic dividend growth rate shock G has a finite support $\{G_n\}_{n=1}^N$ with probability π_n for each G_n. The middle-aged post-shock wealth $w = (w_M^L, w_M^H)$ is the state variable of the economy. We will use lower-case letters of variables to indicate the values of the variables divided by same-dated dividends: $x = X/D$. ρ_Y and ρ_M denote the marginal propensities to consume for the young and the middle aged, respectively. ψ_j denotes the constant fraction of agents of type j in each cohort. γ_j is the risk aversion parameter of type j. α denotes the capital share, so the labor income of the young is $w_Y^j = w_Y = \frac{1-\alpha}{\alpha}$ for both $j = H, L$.

We first define the recursive equilibrium by collecting the equations that fully characterize the equilibrium, and then discuss these equations in more detail. Equation numbers in the square brackets denote the corresponding equations in the main text. (Note that the cross-referenced equations are not normalized by dividends [capital letters instead of lower case letters].)

The recursive equilibrium consists of price **functions** $\{\boldsymbol{p}(w), \boldsymbol{R}(w), \boldsymbol{Z}(w, G')\}$, allocations $\{\boldsymbol{c}_Y^j(w), \boldsymbol{c}_M^j(w), \boldsymbol{c}_O(w), \boldsymbol{\theta}_Y^j(w), \boldsymbol{\theta}_M^j(w)\}_{j=H,L}$, associated return functions $\{\boldsymbol{\phi}_M^j(w)\}_{j=H,L}$, and the law of motion of wealth $\boldsymbol{w}'(w, G') = (\boldsymbol{w}_M^{L\prime}(w, G'), \boldsymbol{w}_M^{H\prime}(w, G'))$, such that

1. Given prices and laws of motion of wealth, the allocations solve the household problems:

$$\boldsymbol{c}_Y^j(w) = \rho_Y w_Y, \quad [26] \tag{SM.1}$$

$$\boldsymbol{c}_M^j(w) = \rho_M w_M^j, \quad [19] \tag{SM.2}$$

$$\boldsymbol{c}_O(w) = 1 + \boldsymbol{p}(w) - (\psi_H w_M^H + \psi_L w_M^L), \quad [34] \tag{SM.3}$$

$$\boldsymbol{Z}(w, G') = \frac{(\boldsymbol{p}(\boldsymbol{w}'(w, G')) + 1)G'}{\boldsymbol{R}(w)\boldsymbol{p}(w)}, \quad [8] \tag{SM.4}$$

$$\boldsymbol{\theta}_M^j(w) = \arg\max_{\theta \in \Theta} \left(E_{G'} \left[\{1 - \theta + \theta \boldsymbol{Z}(w, G')\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}, \quad [15] \tag{SM.5}$$

$$\boldsymbol{\phi}_M^j(w) = \boldsymbol{R}(w) \left(E_{G'} \left[\left\{ 1 - \boldsymbol{\theta}_M^j(w) + \boldsymbol{\theta}_M^j(w)\boldsymbol{Z}(w, G') \right\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}, \quad [16] \tag{SM.6}$$

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$$\boldsymbol{\theta}_Y^j(w) = \arg\max_{\theta \in \Theta} \left(E_{G'} \left[\left\{ [1 - \theta + \theta \boldsymbol{Z}(w, G')] \phi_M^j(\boldsymbol{w'}(w, G'))^{1-\rho_M} \right\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}. \quad [27] \quad \text{(SM.7)}$$

2. Prices clear goods markets:

$$\boldsymbol{p}(w) = (1 - \rho_Y)w_Y + (1 - \rho_M)(\psi_H w_M^H + \psi_L w_M^L), \quad [36] \quad \text{(SM.8)}$$

and bond markets:

$$0 = \sum_{j=H,L} \psi_j \left\{ (1 - \rho_M)w_M^j(1 - \boldsymbol{\theta}_M^j(w)) + (1 - \rho_Y)w_Y(1 - \boldsymbol{\theta}_Y^j(w)) \right\}. \quad [32] \quad \text{(SM.9)}$$

Note that R_t is determined implicitly in the bond market equation through the dependency of the portfolio shares on it.

3. Wealth evolves consistent with portfolio allocations and prices:

$$\boldsymbol{w}_M^{H\prime}(w, G') = \left[(1 - \boldsymbol{\theta}_Y^H(w)) \frac{\boldsymbol{R}(w)}{G'} + \boldsymbol{\theta}_Y^H(w) \frac{1 + \boldsymbol{p}(\boldsymbol{w'}(w, G'))}{\boldsymbol{p}(w)} \right] (1 - \rho_Y)w_Y. \quad [38] \quad \text{(SM.10)}$$

$$\boldsymbol{w}_M^{L\prime}(w, G') = \left[(1 - \boldsymbol{\theta}_Y^L(w)) \frac{\boldsymbol{R}(w)}{G'} + \boldsymbol{\theta}_Y^L(w) \frac{1 + \boldsymbol{p}(\boldsymbol{w'}(w, G'))}{\boldsymbol{p}(w)} \right] (1 - \rho_Y)w_Y, \quad [39] \quad \text{(SM.11)}$$

The first three consumption equations are relatively straightforward. Equations (SM.1) and (SM.2) specify optimal consumption of young and middle-aged households, derived from their savings-consumption tradeoffs under inter-temporal log utility, and old households consume everything as in equation (SM.3).

The next four equations are related to the portfolio problems. To streamline the portfolio problems of the young and the middle-aged, we define two return functions in equations (SM.4) and (SM.6). The excess return function $\boldsymbol{Z}(w, G')$ in equation (SM.4) specifies the excess return of risky tree investment conditional on the dividend growth shock G' next period. The total return function $\phi_M^j(w)$ in equation (SM.6) gives the total return on savings under the optimal portfolio choice for the middle-aged, which are in the utility form given the Epstein-Zin-Weil preferences. Equation (SM.5) gives the optimal risky asset share for the middle-aged households. Equation (SM.7) gives the optimal risky asset shares of the young.

The market clearing conditions specify the equilibrium price functions. The goods-market clearing condition in equation (SM.8) is actually savings supply equals savings demand, and specifies the tree price as a function of the sum of middle aged wealth, and the interest rate function $\boldsymbol{R}(w)$ is implicitly given by the bond market clearing condition in equation (SM.9), where optimal portfolio shares are functions of the interest rate R.

Let's discuss in greater detail the law of motion for the middle-aged wealth vector. The next-period middle-aged wealth vector is given by the portfolio choice of the young in the current period, prices, and returns conditional on the dividend growth shock G' next period. Specifically, consider a young household of type $j = \{H, L\}$ with savings of $(1 - \rho_Y)w_Y$. By investing optimally a share $\boldsymbol{\theta}_Y^j(w)$ of her savings in the risky asset, her returns at the beginning of the next period (turning middle-aged) after the dividend growth rate shock G' are given by equations (SM.11) and (SM.10). In the square brackets of the right-hand side, the first term is the return on the part of savings in bonds and the second term is the return on the part of savings in the tree. The total returns from both bonds and tree imply their wealth when middle-aged for each possible dividend growth rate shock.

To derive the law of motion of wealth in a way that makes the Markov nature more transparent, plug in the following relation into equations (SM.11) and (SM.10)

$$\boldsymbol{p}(\boldsymbol{w}'(w, G')) = (1 - \rho_M)\left(w_Y + \psi_H \boldsymbol{w}_M^{H\prime}(w, G') + \psi_L \boldsymbol{w}_M^{L\prime}(w, G')\right).$$

We derive

$$
\left(\frac{1}{\psi_L} - \frac{(1 - \rho_M)(1 - \rho_Y)w_Y \boldsymbol{\theta}_Y^L(w)}{\boldsymbol{p}(w)}\right)\psi_L \boldsymbol{w}_M^{L\prime}(w, G') - \frac{(1 - \rho_M)(1 - \rho_Y)w_Y \boldsymbol{\theta}_Y^L(w)}{\boldsymbol{p}(w)}\psi_H \boldsymbol{w}_M^{H\prime}(w, G')
$$
$$
= (1 - \rho_Y)w_Y \left[\frac{(1 - \boldsymbol{\theta}_Y^L(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta}_Y^L(w)(1 + (1 - \rho_M)w_Y)}{\boldsymbol{p}(w)}\right],
$$

$$
-\frac{(1 - \rho_M)(1 - \rho_Y)w_Y \boldsymbol{\theta}_Y^H(w)}{\boldsymbol{p}(w)}\psi_L \boldsymbol{w}_M^{L\prime}(w, G') + \left(\frac{1}{\psi_H} - \frac{(1 - \rho_M)(1 - \rho_Y)w_Y \boldsymbol{\theta}_Y^H(w)}{\boldsymbol{p}(w)}\right)\psi_H \boldsymbol{w}_M^{H\prime}(w, G')
$$
$$
= (1 - \rho_Y)w_Y \left[\frac{(1 - \boldsymbol{\theta}_Y^H(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta}_Y^H(w)(1 + (1 - \rho_M)w_Y)}{\boldsymbol{p}(w)}\right].
$$

Thus, we have

$$(A_1 + \frac{1}{\psi_L})\psi_L \boldsymbol{w}_M^{L\prime}(w, G') + A_1 \psi_H \boldsymbol{w}_M^{H\prime}(w, G') = A_2,$$

$$B_1 \psi_L \boldsymbol{w}_M^{L\prime}(w, G') + (B_1 + \frac{1}{\psi_H})\psi_H \boldsymbol{w}_M^{H\prime}(w, G') = B_2,$$

where

$$A_1 = -\frac{(1 - \rho_M)(1 - \rho_Y)w_Y \boldsymbol{\theta}_Y^L(w)}{\boldsymbol{p}(w)},$$

$$A_2 = (1 - \rho_Y)w_Y \left[\frac{(1 - \boldsymbol{\theta}_Y^L(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta}_Y^L(w)(1 + (1 - \rho_M)w_Y)}{\boldsymbol{p}(w)}\right],$$

$$B_1 = -\frac{(1 - \rho_M)(1 - \rho_Y)w_Y \boldsymbol{\theta}_Y^H(w)}{\boldsymbol{p}(w)},$$

$$B_2 = (1 - \rho_Y)w_Y \left[\frac{(1 - \boldsymbol{\theta}_Y^H(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta}_Y^H(w)(1 + (1 - \rho_M)w_Y)}{\boldsymbol{p}(w)}\right].$$

And we have

$$\boldsymbol{w_M^{H'}}(w, G') = \frac{A_2 B_1 - (A_1 + \frac{1}{\psi_L}) B_2}{A_1 B_1 - (A_1 + \frac{1}{\psi_L})(B_1 + \frac{1}{\psi_H})} \frac{1}{\psi_H} \qquad (\text{SM.12})$$

$$\boldsymbol{w_M^{L'}}(w, G') = \frac{A_2(B_1 + \frac{1}{\psi_H}) - A_1 B_2}{(A_1 + \frac{1}{\psi_L})(B_1 + \frac{1}{\psi_H}) - A_1 B_1} \frac{1}{\psi_L}. \qquad (\text{SM.13})$$

The next-period middle-aged wealth w' is given by the optimal portfolio choice of the young and the prices, which, aside from the shock G', are both functions solely of current middle-aged wealth w.

A.2 Existence of Equilibria

This section establishes the existence of the recursive equilibrium for our model. We first state the result and then illustrate the proof.

PROPOSITION. There exists a recursive equilibrium that satisfies equations (SM.1)-(SM.11).

Duffie, Geanokoplos, Mas-Colell, and McLennan (1994) have a result (Proposition 1.2) under quite general conditions that if an equilibrium always exists for any finite-horizon, then an ergodic Markov equilibrium also exists in the infinite horizon. In this section, we will show that their general proof applies to our specific setting. Their proposition is stated as follows.

> **PROPOSITION 1.2**: Let $G : S \rightarrow \mathcal{P}(S)$ be an expectations correspondence and K be a compact subset of S. If, for every $T \in N$, there exists a T-horizon equilibrium $\{S_1, ...S_T\}$ for G such that $S_t \in K$ almost surely for all t, then G has an ergodic Markov equilibrium.

Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) establish the existence of ergodic Markov equilibria for a stochastic overlapping generations model in Theorem 2.1 as an example of their general theory. Their overlapping generations model is an endowment economy with very general assumptions on fundamentals. It allows for l goods, n assets and m types of agents in each cohort with an exogenous Markov shock process. They first establish that an equilibrium exists for any finite horizon of their overlapping generations model, and then invoke Proposition 1.2 to establish that a Markov equilibrium exists in the infinite horizon.

Their example nests our overlapping generations model as a special case with $l = 1$, $n = 2$, $m = 2$, and an i.i.d. shock process, except for two minor differences. One minor difference is that agents live for two periods in their setup instead of three periods in ours. The other difference is that they assume that the utility is von Neumann-Morgenstern, while ours is

Epstein-Zin-Weil. These two minor modifications in our model do not affect the arguments of their proof. The difference between two and three periods does not change the basic argument of the proof. Furthermore, their proof uses only the continuity, monotonicity, and concavity of the utility function, and the Epstein-Zin-Weil preferences have all these three properties. The Epstein-Zin-Weil preferences do not satisfy the *Independence of Irrelevant Alternatives* (IIA) of the von Neumann-Morgenstern utility, but this property is not required for the proof. Thus, the generalization of the preferences does not change the proof either.

This appendix therefore proceeds by (a) giving a formal proof closely following an OLG theorem of Duffie, Geanokoplos, Mas-Colell, and McLennan that an equilibrium exists for any finite horizon, and then (b) formally applying Proposition 1.2 to establish that a Markov equilibrium exists in the infinite horizon.

Below we show that their proof that equilibria exist for any finite horizon $T \in \mathbb{N}$ goes through with our two minor modifications. The original proof was omitted in their paper. We thank Andrew McLennan for providing us a copy of the omitted proof, which is attached with this document for your reference. We go through their proof line by line, taking care of any minor modifications that we have to make for our setup. We adopt their notation in this section of the appendix for an easy comparison.

We now outline a proof, similar to Radner's (1972), that T-horizon equilibria exist for any T. Fix $T \geq 1$. The space of T-period deterministic price paths is P, the set of pairs (p, q) where $p = (p_1, ..., p_T)$ and $q = (q_1, ..., q_T)$ are T-tuples of functions $p_t : G^t \to [0, 1]^2$ and $q_t : G^t \to [0, 1]$ with $(p_t(G_1, G_2, ..., G_t), q_t(G_1, G_2, ..., G_t)) \in \Delta^3$ for all t and $(G_1, G_2, ..., G_t)$. p is asset price and q is goods price. Δ^3 denotes the price simplex. In our model, there are two types of assets, risk-free bonds and risky trees, and one good, implying that p_t has two dimensions and q_t has one dimension.

The truncated consumption set is $\aleph = [0, L]$ where L is a number large enough to include the total endowment in any period. In our setup, $L > \frac{1}{\alpha} > 1$, since $\frac{1}{\alpha}$ is total endowment of goods in every period. The truncated portfolio set is $\Gamma = [-L, L]^2$. For the risky tree, the demand is bounded in $[0, 1]$, and for the risk-free bond, it is bounded by $[-L, L]$. These bounds are introduced to ensure that excess demand is bounded and compact-valued. In equilibrium, these constraints do not bind.

The first difference between our setting and that of Duffie et al. is that we have a three-period rather than two-period generations model. That implies that we need to take care of an extra cohort in each period and one extra period for each generation. As is shown below, this generalization is straightforward.

An excess demand is a tuple $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$, where $\zeta_Y = (\zeta_{Y1}^H, \zeta_{Y1}^L, ..., \zeta_{YT}^H, \zeta_{YT}^L)$, $\zeta_M = (\zeta_{M1}^H, \zeta_{M1}^L, ..., \zeta_{MT}^H, \zeta_{MT}^L)$, $\zeta_O = (\zeta_{O1}^H, \zeta_{O1}^L, ..., \zeta_{OT}^H, \zeta_{OT}^L)$, $\xi_Y = (\xi_{Y1}^H, \xi_{Y1}^L, ..., \xi_{YT}^H, \xi_{YT}^L)$, and $\xi_M = (\xi_{M1}^H, \xi_{M1}^L, ..., \xi_{MT}^H, \xi_{MT}^L)$ are T-tuples of functions $\zeta_{Yt}^i : G^t \to \mathbb{R}$, $\zeta_{Mt}^i : G^t \to \mathbb{R}$, $\zeta_{Ot}^i : G^t \to \mathbb{R}$, $\xi_{Yt}^i : G^t \to \Gamma$, and $\xi_{Mt}^i : G^t \to \Gamma$. Here ζ_{jt}^i denotes the excess demand for goods for cohort $j = \{Y, M, O\}$ and type $i = \{H, L\}$ in period t, and ξ_{jt}^i denotes the excess demand for assets for cohort $j = \{Y, M\}$ and type $i = \{H, L\}$ in period t.

The excess demand $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is *(truncated) individually feasible* if, for all i, t,

and G^t,
$$e^i_{Yt}(G_t) + \zeta^i_{Yt}(G^t) \in \aleph,$$
$$e^i_{Mt}(G_t) + \sum_a \xi^i_{Yat-1}(G_t)d_a(G_t) + \zeta^i_{Mt}(G^t) \in \aleph,$$
$$e^i_{Ot}(G_t) + \sum_a \xi^i_{Mat-1}(G_t)d_a(G_t) + \zeta^i_{Ot}(G^t) \in \aleph,$$

where e^i_{jt} denotes the endowment of agent of type i and age j in period t; ξ^i_{jat} is the excess demand of asset $a = 1, 2$ of type i and age j in period t, and d_a denotes the dividend paid out by asset a in period t. In our setup, only the young have the labor endowment $e^i_{Yt}(G_t)$, which is proportional to the dividend, so $e^i_{Mt}(G_t) = e^i_{Ot}(G_t) = 0$.

The excess demand $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is *budget feasible* for price path (p, q) if, for all i, t, and G^t,
$$p_t(G^t) \cdot \xi^i_{Yt}(G^t) + q_t(G^t) \cdot \zeta^i_{Yt}(G^t) \le 0,$$
$$p_t(G^t) \cdot \xi^i_{Mt}(G^t) + q_t(G^t) \cdot \zeta^i_{Mt}(G^t) \le p_t(G^t) \cdot \xi^i_{Yt-1}(G^{t-1}),$$
$$q_t(G^t) \cdot \zeta^i_{Ot}(G^t) \le p_t(G^t) \cdot \xi^i_{Mt-1}(G^{t-1}).$$

The excess demand $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is *optimal* for price path (p, q) if it is individually feasible, budget feasible for (p, q), and no individual has another individually feasible and budget feasible plan that yields higher utility. Note that since preferences are increasing, continuous, and concave, the set of optimal excess demands for (p, q) is compact and convex.

The second difference between our setting and theirs is that we use the Epstein-Zin-Weil preferences, while they use the von Neumann-Morgenstern utility. Note that the utility function properties used in the proof apply equally to both their and our preferences.

An *aggregate excess demand* is a pair $\Psi = (\psi_\theta, \psi_x)$ where $\psi_\theta = (\psi_{\theta 1}, ..., \psi_{\theta T})$ and $\psi_x = (\psi_{x1}, ..., \psi_{xT})$ are T-tuples of functions $\psi_{\theta t} : G^t \to \mathbb{R}^2$ and $\psi_{xt} : G^t \to \mathbb{R}$. The aggregate excess demand (ψ_θ, ψ_x) is said to be *derived* from $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ if for all t and G^t

$$\psi_{\theta t}(G^t) = \sum_i \left(\xi^i_{Yt}(G^t) + \xi^i_{Mt}(G^t) \right),$$

and

$$\psi_{xt}(G^t) = \sum_i \left(\zeta^i_{Yt}(G^t) + \zeta^i_{Mt}(G^t) + \zeta^i_{Ot}(G^t) \right).$$

Note that if $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is budget feasible for (p, q) and (ψ_θ, ψ_x) is derived from $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$, then

$$p_t(G^t) \cdot \psi_{\theta t}(G^t) + q_t(G^t)\psi_{xt}(G^t) \le 0,$$

for all t and G^t. This is Walras' Law.

For each price path (p, q) let $\Psi(p, q)$ be the set of aggregate excess demands derived from excess demands that are optimal for (p, q). The usual arguments show that Ψ is a compact convex valued upper semicontinuous correspondence. In the proof McLennan

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provided, Duffie, Geanakoplos, Mas-Colell, and McLennan established a generalization of the Debreu-Gale-Kuhn-Nikaido lemma as follows.

Proposition: Let A and B be positive integers, and let $\Delta^A = \{\rho \in \mathbb{R}^A_+ | \sum_a \rho_a = 1\}$. Suppose that $F : (\Delta^A)^B \to (\mathbb{R}^A)^B$ is an upper semicontinuous compact convex valued correspondence whose image is contained in a bounded subset of $(\mathbb{R}^A)^B$, and suppose that $r_b \cdot z_b \leq 0$, $b = 1, ..., B$, whenever $z = (z_1, ..., z_B) \in F(r)$. Then there exists $r^\star \in (\Delta^A)^B$ and $z^\star \in F(r^\star)$ with $z^\star \leq 0$.

This Proposition guarantees the existence of a price path $(p^\star, q^\star)$ and an aggregate excess demand $(\psi^\star_\theta, \psi^\star_x) \in \Psi(p^\star, q^\star)$ with $\psi^\star_\theta(G^t) \leq 0$ and $\psi^\star_x(G^t) \leq 0$ for all t and G^t. Let $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ be an optimal excess demand vector for $(p^\star, q^\star)$ whose derived aggregate excess demand is $(\psi^\star_\theta, \psi^\star_x)$. To show that $(p^\star, q^\star)$ and $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ constitute a T-period equilibrium it now suffices to show that the plans for individuals specified by $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ are optimal, i.e., the constraints $\zeta^{i\star}_{jt} \in \aleph$ and $\xi^{i\star}_{jt} \in \Gamma$ do not rule out utility-improving plans that are budget feasible and individually feasible in the absence of these artificial constraints. But $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ specifies consumption bundles and portfolios that are feasible in the aggregate, so the artificial constraints are satisfied with strict inequality, and the desired result follows from the concavity of all utility functions. Thus, we establish the existence of equilibrium in any finite horizon case for our setting.

Since the amount of each good available in each state and date is bounded uniformly, both above and away from zero, there is a compact set K almost surely containing all states reached in any such T-horizon equilibrium. Therefore, Proposition 1.2 implies the existence of a stationary Markov equilibrium in our setting for the infinite horizon.

To recap the strategy of the proof, we cite Duffie et. al. page 763:

We induce a T-period equilibria for all finite horizons $T \in \mathbb{N}$ as follows. ... For each T these preferences induce a T-period finite-state Radner (1972) style event-tree economy. Then, by an argument similar to Radner's, we establish the existence of an S^T-valued random variable that is a T-horizon equilibrium for G, in the sense of our central results. ... Since the amount of each good available in each state and date is bounded uniformly, both above and away from zero, there is a compact set K almost surely containing all states reached in any such T-horizon equilibrium. Proposition 1.2 now implies the existence of a compact self-justified set J for g, and the desired conclusion follows from Proposition 1.3.

This section shows that their proof applies to our model. Therefore, there exists a stationary Markov equilibrium for our model.

A.3 Equilibrium Solution

Having shown the existence of Markov equilibria, we now describe how to find such an equilibrium. This section details the computation algorithm in our paper. We use backward induction to solve for time-invariant equilibrium functions. We start from a terminal period to serve as the transversality condition and to rule out sunspots. We solve the model backward from the last period. In each period t, we obtain the optimal policy functions $\{c_{Yt}^j(w),\ c_{Mt}^j(w),\ c_{Ot}^j(w),\ \theta_{Yt}^j(w),\ \theta_{Mt}^j(w),\ R_t(w),\ p_t(w),\ Z_t(w, G_{t+1}),\ w_t'(w, G_{t+1})\}$. We move backward until the optimal policy functions in period t and period $t-1$ converge to obtain the time-invariant equilibrium functions $\{c_Y^j(w),\ c_M^j(w),\ c_O^j(w),\ \theta_Y^j(w),\ \theta_M^j(w),$ $R(w),\ p(w),\ Z(w, G'),\ w'(w, G')\}$. Note that we need to index functions by time since the solution iterates backwards until the policy functions converge to the stationary equilibrium. Now we describe this backward induction.

A.3.1 Last Period T

Optimal choices at time T are simple: all agents consume their wealth and the stock price is zero. Since there is no saving, the portfolio share is irrelevant. Thus, for $j = H$ and L, we have

$$\boldsymbol{p_T}(w_M^H, w_M^L) = 0,$$
$$\boldsymbol{c_{YT}^j}(w_M^H, w_M^L) = (1-\alpha)/\alpha,$$
$$\boldsymbol{c_{MT}^j}(w_M^H, w_M^L) = w_M^j,$$
$$\boldsymbol{c_{OT}}(w_M^H, w_M^L) = 1 - \psi_H w_M^H - \psi_L w_M^L.$$

Nonnegative consumption for all agents implies the set of state variables in period T is $\Omega_T^w = \{(w_M^H, w_M^L) \in \mathbb{R}_+^2 | \psi_H w_M^H + \psi_L w_M^L \in [0, 1]\}$.

A.3.2 Period $T-1$

In Period $T-1$, the asset market equilibrium conditions and goods-market equilibrium condition are modified: the marginal propensity to consume and the optimal risky asset share for the young become the same as for the middle aged. The stock price is given by

$$\boldsymbol{p_{T-1}}(w_M^H, w_M^L) = (1-\rho_M)w_Y + (1-\rho_M)\left(\psi_H w_M^H + \psi_L w_M^L\right). \tag{A.1}$$

It is easy to derive the optimal consumption:

$$\boldsymbol{c_{YT-1}^j}(w_M^H, w_M^L) = \rho_M w_Y,$$

$$\boldsymbol{c_{MT-1}^j}(w_M^H, w_M^L) = \rho_M w_M^j,$$

$$\boldsymbol{c_{OT-1}}(w_M^H, w_M^L) = 1 + \boldsymbol{p_{T-1}}(w_M^H, w_M^L) - (\psi_H w_M^H + \psi_L w_M^L)$$
$$= 1 + (1 - \rho_M)w_Y - \rho_M \left(\psi_H w_M^H + \psi_L w_M^L\right). \tag{A.2}$$

We now derive bounds for variables of interest. First, nonnegative consumption implies the set of state variables in period $T - 1$ is

$$\Omega_{T-1}^w = \left\{ (w_M^H, w_M^L) \in \mathbb{R}_+^2 | \psi_H w_M^H + \psi_L w_M^L \in \left[0, \frac{\alpha + (1 - \alpha)(1 - \rho_M)}{\alpha \rho_M}\right]\right\}.$$

This implies $(1 - \rho_M)\frac{1-\alpha}{\alpha} \le \boldsymbol{p_{T-1}}(w) \le (1 - \rho_M)\frac{1}{\alpha \rho_M}$.

Second, $\frac{G_1}{\boldsymbol{p_{T-1}}(w)} \le \boldsymbol{R_{T-1}}(w) \le \frac{G_N}{\boldsymbol{p_{T-1}}(w)}$. If the tree return is larger than R_{T-1} even for the lowest shock realization, i.e., $\frac{G_1}{\boldsymbol{p_{T-1}}(w)} > \boldsymbol{R_{T-1}}(w)$, the households will all borrow in bonds to invest in the tree, and the excess demand of bonds is negative. If the excess return is lower than R_{T-1} even for the highest realization, i.e., $\frac{G_N}{\boldsymbol{p_{T-1}}(w)} < \boldsymbol{R_{T-1}}(w)$, the households will all buy bonds, and the excess demand of bonds is positive. To clear the market, it must be the case that $\frac{G_1}{\boldsymbol{p_{T-1}}(w)} \le \boldsymbol{R_{T-1}}(w) \le \frac{G_N}{\boldsymbol{p_{T-1}}(w)}$.

In period $T - 1$, $\boldsymbol{\theta_{MT-1}^j}(w) = \boldsymbol{\theta_{YT-1}^j}(w)$ since both the young and the middle-aged have only one more period to live. Consider the middle-aged portfolio problem for each w:

$$\boldsymbol{\theta_{MT-1}^j}(w) = \arg\max_{\theta \in \Theta} \left(\sum_{n=1}^N \pi_n \left[1 - \theta + \theta \frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{1-\gamma_j}\right)^{\frac{1}{1-\gamma_i}}.$$

We directly plug in the excess return $\boldsymbol{Z_{T-1}}(w, G_n) = \frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}$ since $\boldsymbol{p_T}(w) = 0$. Nonnegative consumption when old implies that $1 - \theta + \theta \frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)} \ge 0$ for all G_n. This implies $-\frac{1}{\frac{G_N}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}-1} \le \theta \le \frac{1}{1-\frac{G_1}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}}$.

Label the objective function as $\boldsymbol{F_M^j}$:

$$\boldsymbol{F_M^j}(\theta) \equiv \left(\sum_{n=1}^N \pi_n \left[1 - \theta + \theta \frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{1-\gamma_j}\right)^{\frac{1}{1-\gamma_i}}.$$

We solve the optimal θ as the solution to the first order condition

$$(\boldsymbol{F_M^j})'(\theta) = \boldsymbol{F_M^j}(\theta)^{\gamma_j} \sum_{n=1}^N \pi_n \left[1 - \theta + \theta \frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{-\gamma_j} \left[\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)} - 1\right] = 0. \tag{A.3}$$

The solution to the first order condition $\hat{\theta}$ satisfies

$$(\boldsymbol{F_M^j})''(\hat{\theta}) = -\boldsymbol{F_M^j}(\hat{\theta})^{\gamma_j} \sum_{n=1}^N \pi_n \gamma_j \left[1 - \hat{\theta} + \hat{\theta} \frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{-\gamma_j-1} \left[\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)} - 1\right]^2 < 0.$$

The two power terms on the right hand side of the above equation are positive, implying the

second derivative of F_M^j at $\hat{\theta}$ is negative. Thus, the solution $\hat{\theta}$ to the first-order condition is also globally optimal and thus unique. This result can be established straightforwardly by contradiction.

Then we use the market clearing condition for bonds to obtain $R_{T-1}(w)$:

$$\sum_{j=H,L} \psi_j \left[1 - \theta_{MT-1}^j(w; R_{T-1}) \right] \left(w_M^j + w_Y \right) = 0. \tag{A.4}$$

To establish that $\theta_{MT-1}^j(w; R_{T-1})$ is continuous in R_{T-1}, we invoke the implicit function theorem. Since $(F_M^j)'$ is continuous and differentiable in both θ and R, and $(F_M^j)''$ is nonzero, the policy function $\theta_{MT-1}^j(w; R_{T-1})$ given by equation (A.3) is continuous in R_{T-1}. Since $\theta_{MT-1}^j(w; R_{T-1})$ is continuous in R_{T-1}, the excess demand of bonds in the left hand side of equation (A.4) is continuous in R_{T-1}. As R_{T-1} approaches $\frac{G_1}{p_{T-1}}$ from above, the excess returns go up, and θ_{MT-1}^j approaches positive infinity, implying a negative excess demand of bonds. On the other side, as R_{T-1} approaches $\frac{G_N}{p_{T-1}}$ from below, the excess returns go down, and θ_{MT-1}^j approaches positive infinity, implying a positive excess demand of bonds. The continuity of the excess demand function over R_{T-1} implies that there exists at least one R_{T-1} which implies zero excess demand in the bond markets. Therefore, in period $T-1$, we have proven the existence of the interest rate function $R_{T-1}(w)$.

The certainty equivalent total next-period return function for the middle-aged is

$$\phi_{MT-1}^j(w) = R_{T-1}(w) \left(\sum_{n=1}^N \pi_n \left[1 - \theta_{MT-1}^j(w) + \theta_{MT-1}^j(w) \frac{G_n}{p_{T-1}(w) R_{T-1}(w)} \right]^{1-\gamma_j} \right)^{\frac{1}{1-\gamma_j}}.$$

The young's decision on portfolio choice $\theta_{YT-1}^j = \theta_{MT-1}^j$ implies that his next-period wealth $\{ w'_{T-1}(w, G_n) \}$ is given by equations (SM.12) and (SM.13).

A.3.3   Period $T - k$ with $k \geq 2$ (A Generic Period)

Now consider a generic period: $T - k$ with $k \geq 2$. The tree price is given by

$$p_{T-k}(w) = (1 - \rho_Y) w_Y + (1 - \rho_M)(\psi_H w_M^H + \psi_L w_M^L).$$

It is easy to derive the optimal consumption:

$$c_{YT-k}^j(w) = \rho_Y w_Y,$$

$$c_{MT-k}^j(w) = \rho_M w_M^j,$$

$$c_{OT-k}(w) = 1 + p_{T-k}(w) - (\psi_H w_M^H + \psi_L w_M^L).$$

Again, nonnegative consumption implies that the set of state variables in period $T - k$ is

$$\Omega^w_{T-k} = \left\{ (w^H_M, w^L_M) \in \mathbb{R}^2_+ | \psi_H w^H_M + \psi_L w^L_M \in \left[0, \frac{\alpha + (1-\alpha)(1-\rho_Y)}{\alpha \rho_M} \right] \right\}.$$

This implies $\frac{(1-\rho_Y)(1-\alpha)}{\alpha} \leq \boldsymbol{p}_{T-k}(w) \leq \frac{\alpha(1-\rho_M)+(1-\alpha)(1-\rho_Y)}{\alpha \rho_M}$.

Here is some intuition for the bounds on the tree price. The lower bound is given by the minimum possible demand for the tree. Suppose the middle-aged have zero financial wealth, so they have a zero demand for savings. The old always have zero demand for savings. In that case, demand for net savings, which can only be provided by the tree, is equal to young's demand for savings, and the young's demand for savings will determine the tree price. Since the young have to live through the next two periods without labor income, they need savings of $\frac{(1-\rho_Y)(1-\alpha)}{\alpha}$, which gives the lower bound for the tree price. Now consider the upper bound for the tree price. The young still demand $\frac{(1-\rho_Y)(1-\alpha)}{\alpha}$ in savings, but now suppose the middle-aged own all the financial wealth of the economy $1 + p_t$ and so have savings demand of $(1 - \rho_M)(1 + p_t)$. Then the maximum tree price is given by $p_t = \frac{(1-\rho_Y)(1-\alpha)}{\alpha} + (1 - \rho_M)(1 + p_t)$, which can be solved to give the upper bound on the tree price.

Now we need to show the existence of the price function $\boldsymbol{R}_{T-k}(w)$, the law of motion of wealth $\boldsymbol{w}'_{T-k}(w, G_n)$ and the portfolio function $\boldsymbol{\Theta}_{T-k}(w) = \{ \boldsymbol{\theta}^H_{YT-k}(w), \boldsymbol{\theta}^L_{YT-k}(w), \boldsymbol{\theta}^H_{MT-k}(w), \boldsymbol{\theta}^L_{MT-k}(w) \}$ such that (i) given prices and the law of motion for wealth, the portfolio choices $\boldsymbol{\Theta}_{T-k}(w)$ are optimal; (ii) the law of the motion is consistent with the agents' choices; and (iii) the prices clear the markets.

Given the interest rate $\boldsymbol{R}_{T-k}(w)$, and the next-period state variable $\boldsymbol{w}'_{T-k}(w, G_n)$, the agents know the next period tree price $\boldsymbol{p}_{T-k+1}(\boldsymbol{w}'_{T-k}(w, G_n))$, and the total return for the middle-aged next period, $\boldsymbol{\phi}^j_{MT-k+1}(\boldsymbol{w}'_{T-k}(w, G_n))$ by solving the problem of period $T-k+1$. The implied excess returns are

$$\boldsymbol{Z}_{t-k}(w, G_n) = \frac{G_n \left(\boldsymbol{p}_{T-k+1}(\boldsymbol{w}'_{T-k}(w, G_n)) + 1 \right)}{\boldsymbol{p}_{T-k}(w) \boldsymbol{R}_{T-k}(w)}.$$

We have $\boldsymbol{Z}_{t-k}(w, G_1) \leq 1 \leq \boldsymbol{Z}_{t-k}(w, G_N)$ following similar arguments as in period $T - 1$. This implies $\frac{G_1 \left(\boldsymbol{p}_{T-k+1}(\boldsymbol{w}'_{T-k}(w, G_1)) + 1 \right)}{\boldsymbol{p}_{T-k}(w)} \leq \boldsymbol{R}_{T-k}(w) \leq \frac{G_N \left(\boldsymbol{p}_{T-k+1}(\boldsymbol{w}'_{T-k}(w, G_N)) + 1 \right)}{\boldsymbol{p}_{T-k}(w)}$. Thus, using the bounds for the tree prices, we have

$$\Omega^R_{T-k} = \left[\frac{G_1 \rho_M \left(1 - \rho_Y(1-\alpha) \right)}{1 - \alpha \rho_M - (1-\alpha)\rho_Y}, \frac{G_N \left(1 - \rho_Y(1-\alpha) \right)}{\rho_M(1 - \rho_Y)(1-\alpha)} \right].$$

With the information they have, the middle-aged solve the following problem:

$$\boldsymbol{\theta}^j_{MT-k}(w) = \arg\max_{\theta \in \Theta} \left\{ \sum_{n=1}^N \pi_n \left[1 - \theta + \theta \boldsymbol{Z}_{T-k}(w, G_n) \right]^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}. \tag{A.5}$$

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Again as in period $T - 1$, we have a unique global optimizing solution. Nonnegative consumption when old implies that $1 - \theta + \theta Z_{T-k}(w, G_n) \geq 0$ for all G_n. This implies $-\frac{1}{Z_{T-k}(w,G_N)-1} \leq \theta \leq \frac{1}{1-Z_{T-k}(w,G_1)}$. The optimal portfolio choices give $\phi^j_{MT-k}(w)$, which is needed to solve the problem of period $T - k - 1$.

The young's portfolio problem is given by equation (SM.7). Let's rewrite the objective function as follows:

$$
F^j_Y(\theta) = \left\{ \sum_{n=1}^{N} \pi_n \left\{ [1 - \theta + \theta Z_{T-k}(w, G_n)] \phi^j_{MT-k+1} \left(w'_{T-k}(w, G_n) \right)^{1-\rho_M} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}.
$$

Solve for the optimal θ as the solution to the first order condition

$$
(F^j_Y)'(\theta) = F^j_Y(\theta)^{\gamma_j} \sum_{n=1}^{N} \pi_n \phi^j_{MT-k+1} \left(w'_{T-k}(w, G_n) \right)^{(1-\rho_M)(1-\gamma_j)} \frac{Z_{T-k}(w, G_n) - 1}{(1 - \theta + \theta Z_{T-k}(w, G_n))^{\gamma_j}} = 0. \quad \text{(A.6)}
$$

The solution to the first order condition $\hat{\theta}$ also satisfies

$$
(F^j_Y)''(\hat{\theta}) = -F^j_Y(\hat{\theta})^{\gamma_j} \sum_{n=1}^{N} \pi_n \gamma_j \phi^j_{MT-k+1} \left(w'_{T-k}(w, G_n) \right)^{(1-\rho_M)(1-\gamma_j)} \frac{(Z_{T-k}(w, G_n) - 1)^2}{\left(1 - \hat{\theta} + \hat{\theta} Z_{T-k}(w, G_n) \right)^{\gamma_j+1}} < 0.
$$

Thus, there is a unique optimal portfolio share $\theta^j_{YT-k}(w)$. Nonnegative consumption when middle-aged implies that $-\frac{1}{Z_{T-k}(w,G_N)-1} \leq \theta \leq \frac{1}{1-Z_{T-k}(w,G_1)}$.

Thus, given functions $\{R_{T-k}(w), w'_{T-k}(w, G_n)\}$, we can solve for the optimal portfolio $\Theta_{T-k}(w)$ from equations (A.5) and (A.6). For the optimal portfolio choices to be an equilibrium allocation, they must be consistent with the law of motion of wealth given by (SM.12) and (SM.13). Also, they need to satisfy the bond market clearing condition given $R_{T-k}(w)$:

$$
\sum_{j=H,L} \psi_j \left[(1 - \rho_Y) \left[1 - \theta^j_{YT-k}(w) \right] w_Y + (1 - \rho_M) \left[1 - \theta^j_{MT-k}(w) \right] w^j_M \right] = 0.
$$

We cannot establish the uniqueness of the interest rate that clears the bond markets. However, numerically for our parameters, θ^j_{MT-1} decreases with R_{T-1} and the excess demand decreases with R_{T-1}, as illustrated in Figure 9. For a reasonably typical value of w, the top two panels plot the optimal portfolio shares over different values of the risk-free interest rate for the low-risk-tolerant and the high-risk-tolerant households, and the bottom panel plots the excess demand. As the risk-free rate increases and the excess tree return decreases, households decrease their risky asset shares. The excess demand increases with the interest rate. Thus, there exists a unique R_{T-1} that clears the bond market. One stable method for solving $R(w)$ is thus the bisection technique, starting with the lower bound set at a value slightly higher than $\frac{G_1}{P_{T-1}(w)}$ and the upper bound set at a value slightly lower than $\frac{G_N}{P_{T-1}(w)}$.

Figure 9: Numerical Solution Illustration 1
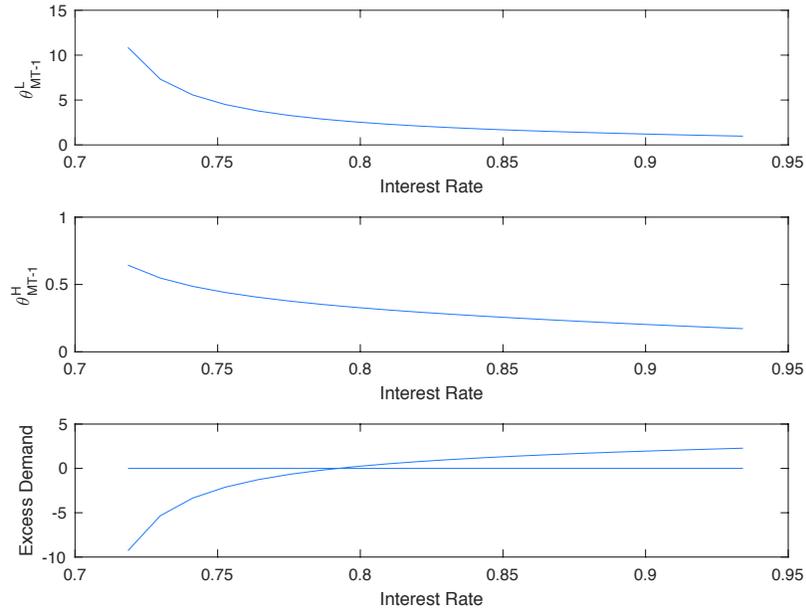
A.3.4 Stationary Equilibrium

We use the above procedure for each period going backward until the policy functions converge. We then have the stationary Markov equilibrium for our model.

Appendix B: Price and Return Dynamics

In this appendix we describe the price and return dynamics implied by the model in more detail.

The dependence of the risky asset share of the young cohort as a whole on the shock right before they begin investing makes it possible to explain the behavior of the price-dividend ratio. Recall that the price-dividend ratio is determined exactly by the total middle-aged wealth in each period. See Equation (36). Because of the Cobb-Douglas production function and the constant average propensity to consume, the invested wealth of the young cohort is a fixed fraction of output, or equivalently, in a fixed ratio to the dividend. Their later wealth in middle age in comparison with the dividend at that time depends on their overall portfolio return compared to the growth rate of the dividend. Let us explain the general level of the price-dividend ratio after each of the four possible histories of the last two shocks.

A good shock leading into the youth of the current middle-aged cohort depressed their risky asset share when young, and they were significantly underlevered relative to the economy as a whole. Then a second good shock will actually cause the growth rate of their wealth to fall behind the growth rate of the dividend (which equals the growth rate of economic production as a whole). Middle-aged wealth that is smaller relative to the expanded size of the economy will then lead to a low price-dividend ratio (in part because dividends are large). In contrast, a bad shock as they go into middle age causes the economy to grow more slowly than their relatively safer portfolio, leading to a high middle-aged wealth to dividend ratio, which in turn implies a high price-dividend ratio.

The bad shock leading into the youth of the current middle-aged cohort led that cohort to have slightly elevated risky asset holdings and thus to be slightly overlevered relative to the economy as a whole. A good shock at middle age then leads them to have a high level of wealth relative to the economy as a whole, leading in turn to a high price-dividend ratio. In contrast, a bad shock at middle age causes them to have a low level of wealth relative to the economy as a whole, leading in turn to a low price-dividend ratio.

Note that the last shock makes a bigger difference for the price-dividend ratio between the (Good, Good) and (Good, Bad) histories than it does between the (Bad, Good) and (Bad, Bad) histories. The reason is that the daring middle-aged agents in the previous period pulled more dramatically ahead of the economy after a good shock leading into their middle age than they fell behind the economy after a bad shock leading into their middle age. This led the current middle-aged cohort to have a significantly underlevered portfolio after a good shock leading into their youth, but only slightly overlevered portfolio after a bad shock leading into their youth.

To summarize the history dependence of prices, we present the key price statistics conditional on the current shock, the past shock, and the two past shocks in Table A1. The kind of reasoning above can be extended to interpret the consequences of a longer history of shocks, but the most recent shocks have the most powerful effect on the model economy. One reason we include the effects of a three-shock history is the identity relating the realized

tree return to the history of the price-dividend ratio:

$$R_t^{\text{Tree}} = \frac{P_t + D_t}{P_{t-1}} = \frac{D_t}{D_{t-1}} \left(\frac{1 + (P_t/D_t)}{P_{t-1}/D_{t-1}} \right) = G_t \left(\frac{1 + p_t}{p_{t-1}} \right).$$

Thus, the realized tree return depends on the values of the price-dividend ratio in two different periods. Among three-shock histories, the realized tree return is highest after (Good, Good, Good) since the two good shocks at $t-2$ and $t-1$ depress p_{t-1}, while the good shock at t gives a high G_t.[11] The realized tree return is lowest after the three-shock history (Good, Bad, Bad) because the combination of a good shock at $t-2$ followed by a bad shock at $t-1$ elevates p_{t-1}, while the bad shock at t gives a low G_t.

Table A1: History Dependence of Key Variables

G_t	P_t/D_t		$\log R_t^{\text{Tree}}$		$\log R_t$		$E(\log R_t^{\text{Tree}})$		$E(\log Z_t)$	
	Bad	Good	Bad	Good	Bad	Good	Bad	Good	Bad	Good
Mean	19.20	18.69	1.61	5.60	1.75	2.10	3.91	4.08	2.16	1.98
G_{t-1}										
Bad	18.99	19.23	1.51	5.60	1.81	1.96	3.96	3.95	2.15	1.98
Good	19.41	19.17	1.71	5.60	1.68	2.23	3.86	4.20	2.18	1.97
G_{t-2}, G_{t-1}										
Bad, Bad	19.07	19.32	1.58	5.67	1.79	1.94	3.94	3.93	2.15	1.99
Good, Bad	18.90	19.15	1.45	5.54	1.84	1.99	3.98	3.97	2.14	1.98
Bad, Good	19.21	17.81	1.53	5.41	1.74	2.33	3.91	4.29	2.17	1.95
Good, Good	19.59	18.52	1.87	5.79	1.62	2.12	3.81	4.11	2.19	1.99

Note: P_t/D_t denotes the price-dividend ratio, $\log R_t$ denotes the log risk free interest rate, $\log R_t^{\text{Tree}}$ denotes the log realized tree return, given by $\log(D_t + P_t) - \log P_{t-1}$, $E(\log R_t^{\text{Tree}})$ denotes the expected log tree return, and $E(\log Z_t)$ denotes the expected log excess return. All prices are annualized. All prices are in percentage, except the price-dividend ratio. G_t, G_{t-1} and G_{t-2} denote the dividend growth rate shock in period t, $t-1$ and $t-2$, respectively.

Table A1 also shows that the risk free rate is higher after a good shock than after a bad shock in period t. This pattern becomes particularly pronounced after a good shock in period $t-1$. The expected tree return is higher after a good shock in period t. However, the pattern does not hold when we consider a two-period history of shocks. After a good shock in period $t-1$, the expected tree return is higher after a good shock. In contrast, after a bad shock in period $t-1$, the expected tree return is higher after a bad shock. The expected excess return is lower after a good shock in period t, given the high aggregate risk tolerance after a good shock in period t. The expected excess return is especially low after a history of the bad and good shocks.

---

[11]The realized tree return is second-highest after the three-shock history (Bad, Bad, Good) not only because two bad shocks at $t-2$ and $t-1$ depress p_{t-1} somewhat, but also because the tail end of a bad shock at $t-1$ followed by a good shock at t puts p_t at a reasonably high level as well as giving a high G_t.

Appendix C: Alternative Parameterizations

We solve the model with different sets of numerical assumptions to determine how robust our primary findings are. First, we recalculate the risk tolerances of cautious and daring agents setting the standard deviation of risk tolerance equal to 0.085, which is half of the standard deviation of the risk tolerances listed in Table 1. We then solve the model with the other assumptions in Table 1 and report results analogous to Table 2 below. Next we we double the standard deviation of risk tolerances. Finally, we both halve and double the standard deviations of economic shocks and report the results below. In each case we find results that are qualitatively similar to those obtained with our main parameterization in the text.

Table A2: Low Dispersion in Tolerance

G_t	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	72	76	69	92	88	70	61
Young	72	76	69	60	61	46	42
Middle	73	76	70	32	27	24	19
Daring	359	388	330	8	12	30	39
Young	360	389	330	5	5	20	18
Middle	359	388	330	2	7	10	22
Total	95	101	90	100.0	100.0	100	100
Young	95	101	90	66	66	66	59
Middle	96	101	90	34	34	34	41

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Table A3: High Dispersion in Tolerance

G_t	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	23	27	18	94	81	25	14
Young	21	27	15	60	62	16	10
Middle	24	28	21	34	19	9	4
Daring	860	1226	495	6	19	75	86
Young	853	1213	492	5	5	63	27
Middle	868	1238	499	1	14	12	59
Total	90	123	56	100.0	100.0	100	100
Young	88	122	54	65	67	79	37
Middle	92	124	59	35	33	22	63

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Table A4: Low Dispersion in Growth

G_t	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	66	69	63	93	89	64	56
Young	66	69	62	60	61	42	38
Middle	66	69	63	32	29	22	18
Daring	454	489	418	7	11	36	44
Young	454	490	418	5	5	26	22
Middle	453	489	418	2	5	10	22
Total	97	103	91	100.0	100.0	100	100
Young	97	103	90	66	66	68	60
Middle	97	103	91	34	34	32	40

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Table A5: High Dispersion in Growth

G_t	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$	
	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	19	22	17	93	75	20	12
Young	19	22	15	59	65	13	10
Middle	20	22	18	34	10	7	2
Daring	746	1091	401	7	25	80	88
Young	750	1100	401	5	6	57	24
Middle	742	1083	401	2	19	23	64
Total	77	108	47	100.0	100.0	100	100
Young	77	108	46	64	70	70	34
Middle	78	107	48	36	30	31	66

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Appendix D: Key Model Variables

We list all of the important model variables with a short description in this table. In the table $j \in \{H, L\}$ and $g \in \{Y, M, O\}$. Note that key model parameters are defined in Table ??.

Table A6: Key Model Variables

Variable	Per D_t	Description
D_t		Total dividends and level of technology
P_t	p_t	Price of one unit of tree
Y_t	y_t	Aggregate output
R_t		Gross risk free rate
Z_t		Excess tree return
C_{gt}^j	c_{gt}^j	Consumption for age g and tolerance j
W_{gt}^j	w_{gt}^j	Wealth for age g and tolerance j
S_{gt}^j	s_{gt}^j	Tree holdings for age g and tolerance j
B_{gt}^j	b_{gt}^j	Bond holdings for age g and tolerance j
θ_{gt}^j		Risky asset share for age g and tolerance j
ϕ_{gt}^j		Certainty equivalent total return for age g and tolerance j