Optimization Models for Differentiating Quality of Service Levels in Probabilistic Network Capacity Design Problems

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October 7th, 2013

Introduction I

- Network design problems (NDPs) are essential for the development of modern societies
- Objective: Minimize flow cost and arc capacity modification cost

Figure : Internet cable network[1]

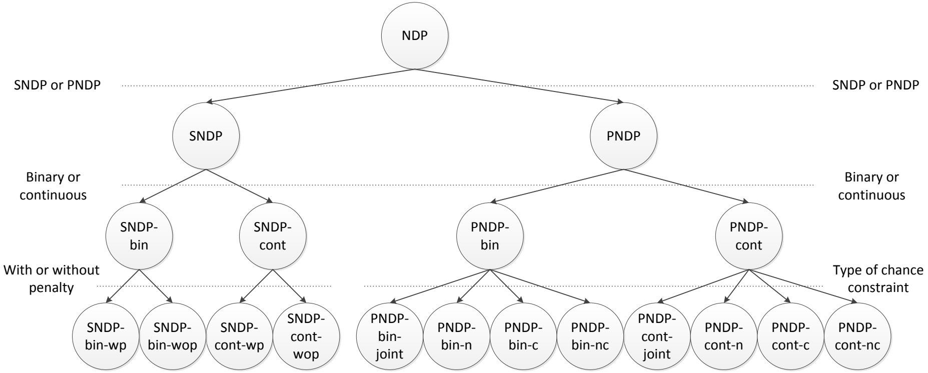
Figure : Road network[2]

[1] Source: http://jamesdudleyonline.com/wp-content/uploads/2011/03/connect-through-the-internet.jpg
[2] Source: http://static.panoramio.com/photos/large/8248689.jpg

Introduction II

- NPDs under demand uncertainty and with multiple commodities
- Capacity design decisions are made before realization of demands
  - Can be continuous or binary
- Probabilistic NDPs (**PNDPs**): Flow decisions made *before* realization of demands
  - Flow decisions are made under probabilistic constraints
  - Probabilistic constraints can be joint, or differentiated by node, commodity, or node and commodity
- Stochastic NDPs (**SNDPs**): Flow decisions made *after* realization of demands
  - *Expected* flow costs
  - Flow decisions may be penalized for unmet demand for greater flexibility in solution

Introduction III



Notation I

Graph: $G(N, A)$

Sets:

- W: Set of commodities
- $O_w \subseteq N$: Set of origins of commodity $w \in W$
- $D_w \subseteq N$: Set of destinations of commodity $w \in W$
- Ω: Set of random scenarios where $\Omega = \{1, \ldots, |\Omega|\}$

Notation II

Parameters:

- c_{ij}: Cost of allocating one unit of capacity at link $(i, j) \in A$
- q_{ij}: Fixed cost of adding link $(i, j) \in A$ when capacity design variables are binary
- a_{ijw}: Unit cost of flowing commodity $w \in W$ on link $(i, j) \in A$
- u_{ij}: Fixed capacity of link $(i, j) \in A$ when capacity design variables are binary
- v_{iw}: Unit penalty cost of unmet demand of commodity w at destination $i \in D_w$
- o_{iw}: Deterministic supply of commodity w at origin $i \in O_w$
- d_{iw}: Random demand of commodity w at destination $i \in D_w$
- ξ^s_{iw}: Realization of random demand d_{iw} in scenario $s \in \Omega$, $\forall w \in W$ and $i \in D_w$
- p^s: Probability of scenario $s \in \Omega$
- $\epsilon, \epsilon_{iw}, \epsilon_i, \epsilon_w$: Risk parameters associated with different forms of chance constraints

PNDP formulations I

PNDP-cont-joint:

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{y}} \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{w\in W}\sum_{(i,j)\in A} a_{ijw}y_{ijw} \\
\text{s.t.} \quad & \sum_{w\in W} y_{ijw} \le x_{ij} && \forall (i,j)\in A && (1)\\
& \sum_{j:(i,j)\in A} y_{ijw} - \sum_{j:(j,i)\in A} y_{jiw} \le o_{iw} && \forall i\in O_w,\ w\in W && (2)\\
& \sum_{j:(i,j)\in A} y_{ijw} - \sum_{j:(j,i)\in A} y_{jiw} = 0 && \forall i\notin O_w\cup D_w,\ w\in W && (3)\\
& \mathbf{x}\ge 0,\ \mathbf{y}\ge 0 && && (4)\\
& \mathbb{P}\left(\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \ge d_{iw},\ \forall i\in D_w, w\in W\right) \ge 1-\epsilon
\end{aligned}
$$

PNDP formulations II

PNDP-cont-n:

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{w\in W}\sum_{(i,j)\in A} a_{ijw}y_{ijw}$$

$$\text{s.t.} \quad (1)\text{–}(4)$$

$$\mathbb{P}\left(\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq d_{iw},\ \forall w \in W\right) \geq 1-\epsilon_i, \quad \forall i \in \bigcup_{w\in W} D_w$$

PNDP-cont-c:

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{w\in W}\sum_{(i,j)\in A} a_{ijw}y_{ijw}$$

$$\text{s.t.} \quad (1)\text{–}(4)$$

$$\mathbb{P}\left(\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq d_{iw},\ \forall i \in D_w\right) \geq 1-\epsilon_w, \quad \forall w \in W$$

PNDP formulations III

PNDP-cont-nc:

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{w\in W}\sum_{(i,j)\in A} a_{ijw}y_{ijw}$$

$$\text{s.t.} \quad (2);(3)$$

$$\sum_{w\in W} y_{ijw} \leq x_{ij} \qquad \forall (i,j)\in A$$

$$\mathbf{x}\geq 0,\ \mathbf{y}\geq 0$$

$$\mathbb{P}\left(\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq d_{iw}\right) \geq 1-\epsilon_{iw}, \quad \forall i\in D_w, w\in W$$

PNDP formulations IV

PNDP-bin-nc:

$$
\begin{aligned}
\min_{\beta,\mathbf{y}} \quad & \sum_{(i,j)\in A} q_{ij}\beta_{ij} + \sum_{w\in W}\sum_{(i,j)\in A} a_{ijw}y_{ijw} \\
\text{s.t.} \quad & (2);(3) \\
& \sum_{w\in W} y_{ijw} \le u_{ij}\beta_{ij} \qquad \forall (i,j)\in A \\
& \beta \in \{0,1\}^{|A|},\ \mathbf{y}\ge 0 \\
& \mathbb{P}\left(\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \ge d_{iw} \right) \ge 1-\epsilon_{iw}, \quad \forall i\in D_w, w\in W
\end{aligned}
$$

SNDP formulations I

SNDP-cont-wop:

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{y}} \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{s\in\Omega} p^s \left(\sum_{w\in W} \sum_{(i,j)\in A} a_{ijw} y^s_{ijw} \right) \\
\text{s.t.} \quad & \sum_{w\in W} y^s_{ijw} \le x_{ij} && \forall (i,j)\in A,\ s\in\Omega && (5) \\
& \sum_{j:(i,j)\in A} y^s_{ijw} - \sum_{j:(j,i)\in A} y^s_{jiw} \le o_{iw} && \forall i\in O_w,\ w\in W,\ s\in\Omega && (6) \\
& \sum_{j:(i,j)\in A} y^s_{ijw} - \sum_{j:(j,i)\in A} y^s_{jiw} = 0 && \forall i\notin O_w\cup D_w,\ w\in W,\ s\in\Omega && (7) \\
& \mathbf{x}\ge 0,\ \mathbf{y}^s \ge 0 && \forall s\in\Omega && (8) \\
& -\sum_{j:(i,j)\in A} y^s_{ijw} + \sum_{j:(j,i)\in A} y^s_{jiw} \ge \xi^s_{iw} && \forall i\in D_w, w\in W,\ s\in\Omega
\end{aligned}
$$

SNDP formulations II

SNDP-cont-wp:

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{y}} \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{s\in\Omega} p^s \left(\sum_{w\in W}\sum_{(i,j)\in A} a_{ijw} y^s_{ijw} + \sum_{w\in W}\sum_{i\in D_w} v_{iw} t^s_{iw} \right) \\
\text{s.t.} \quad & (5)\text{–}(8) \\
& -\sum_{j:(i,j)\in A} y^s_{ijw} + \sum_{j:(j,i)\in A} y^s_{jiw} + t^s_{iw} \geq \xi^s_{iw} \qquad \forall i \in D_w, w\in W,\ s\in\Omega \\
& \mathbf{t}^s \geq 0 \qquad \forall s\in\Omega
\end{aligned}
$$

- SNDPs can be solved as a two-stage problem using Benders' decomposition
- SNDP-cont-wp is typically used to formulate cost-based NDPs
  - A benchmark against which we compare our PNDP-cont reformulations

Big-M reformulation of chance constraints I

Approach:

- Add binary variable z^s that takes value 1 if the chance constraint is violated by demand realization ξ^s and 0 otherwise
- The sum of probabilities of the realizations that violate the chance constraint must not exceed the tolerance level

Big-M reformulation of chance constraints II

Example: **PNDP-cont-nc**

$$\mathbb{P}\left(\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq d_{iw}\right) \geq 1-\epsilon_{iw}, \quad \forall i \in D_w, w \in W$$

Create a new variable z^s_{iw} such that

$$z^s_{iw} = \begin{cases} 1 & \text{if } \sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} < \xi^s_{iw} \\ 0 & \text{if } \sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq \xi^s_{iw} \end{cases}, \quad \forall s \in \Omega, i \in D_w, w \in W$$

Big-M reformulation of chance constraints III

Chance constraint is equivalent to the following set of MIP constraints:

$$-\sum_{j:(i,j)\in A} y_{ijw} + \sum_{j:(j,i)\in A} y_{jiw} - \xi^s_{iw} + \mathbf{M}_{iw} z^s_{iw} \geq 0 \quad \forall s \in \Omega, i \in D_w, w \in W \tag{9}$$

$$\sum_{s\in\Omega} p^s z^s_{iw} \leq \epsilon_{iw} \quad \forall i \in D_w, w \in W \tag{10}$$

$$\mathbf{z}_{iw} \in \{0,1\}^{|\Omega|} \quad \forall i \in D_w, w \in W \tag{11}$$

where $\mathbf{M}$ is an arbitrarily large number.

Polynomial-time algorithm for PNDP-cont-nc I

- An alternative method that does not require the use of binary variables
- Takes advantage of single-line chance constraints
- If

$$\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq \xi_{iw}^{s}$$

  for some realization ξ_{iw}^{s}, then

$$\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq \xi_{iw}^{s'}$$

  for any realization satisfying $\xi_{iw}^{s'} < \xi_{iw}^{s}$.

Polynomial-time algorithm for PNDP-cont-nc II

ALGO1:

for all $w \in W, i \in D_w$

(i) Sort ξ^s_{iw} in ascending order and relabel the scenarios based on this order

(ii) Identify $s' \in \{1, ..., |\Omega_{iw}|\}$ such that

$$\sum_{k=s}^{|\Omega_{iw}|} p^k > \epsilon_{iw} \geq \sum_{k=s'}^{|\Omega_{iw}|} p^k$$

(iii) Replace the $(i, w)^{\text{th}}$ chance constraint with

$$\sum_{j:(j,i)\in A} y_{jiw} - \sum_{j:(i,j)\in A} y_{ijw} \geq \xi^{s'}_{iw} \quad (12)$$

end for

Solve PNDP-cont-nc as

$$\min_{\mathbf{x},\mathbf{y}} \left\{ \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{w\in W} \sum_{(i,j)\in A} a_{ijw}y_{ijw} : \text{ subject to (1)–(4); (12) } \forall w \in W, i \in D_w \right\}$$

Polynomial-time algorithm for PNDP-cont-nc III

- Similar approaches can be used to develop polynomial-time algorithms for special cases of PNDP-cont-n/c
- PNDP-cont-n with each node having demand for no more than 1 type of commodity $\Rightarrow$ single-line chance constraint
- PNDP-cont-c with each commodity having no more than 1 demand node $\Rightarrow$ single-line chance constraint

Results for randomly generated networks I

- Compare computational times and optimal objective values for
  - PNDP-cont-joint
  - PNDP-cont-nc with homogenous (“-ho”) risk parameters
  - PNDP-cont-nc with heterogenous (“-he”) risk parameters

Results for randomly generated networks II

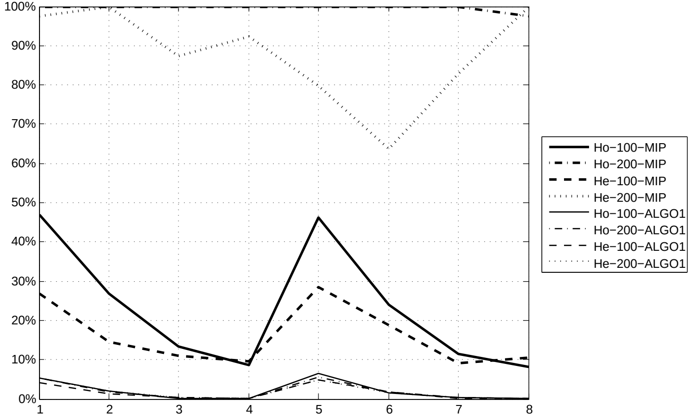


Figure : Percentage comparison of CPU time taken by ALGO1 and the MIP approach for PNDP-cont-nc instances (100% is the largest CPU time)

Results for randomly generated networks III

Summary of observations:

- Optimal objective values decrease as ϵ increases
- PNDP-cont-nc is less sensitive to changes in ϵ than PNDP-cont-joint
- ALGO1 is much more efficient than the MIP approach
- For MIP models, CPU time increases dramatically as ϵ is increased and as $|\Omega|$ is increased
- For ALGO1, CPU time increases is mostly unaffected by changes in ϵ and $|\Omega|$, and the homogeneity of risk parameters

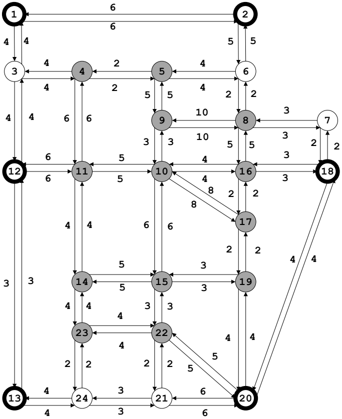
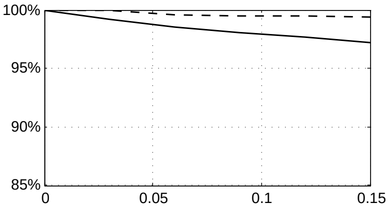


Figure : Sioux Falls road network

Results for real life network II

- High inflow instance
- Higher mean demands for nodes closer to node 10 (center node)
- Compare sensitivity of optimal objective values to ϵ and v
  - PNDP-cont-joint
  - PNDP-cont-nc
  - SNDP-cont-wp

Results for real life network III



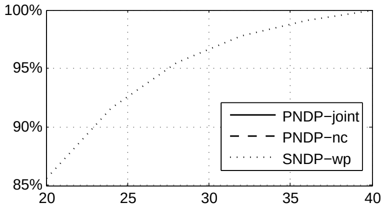


Figure : Percentage comparison of optimal objective values for PNDP-joint, PNDP-nc and SNDP-cont-wp

Results for real life network IV

Summary of observations:

- Optimal values of PNDP-joint and PNDP-nc have a fairly linear relationship with ϵ
- Optimal values of SNDP-cont-wp are concave w.r.t. v (dominant term changes from real cost to penalty cost)
  - Without first experimenting with several values of v, a suitable value for v may not be known
  - PNDP models mitigate ambiguity in solution reliability

Conclusions and future research

Conclusions:

- Developed MIP formulations for PNDP models
- Developed polynomial-time algorithms for PNDP-cont-nc and special cases of PNDP-cont-n/c models that are far more efficient than MIP formulations
- Benchmarked PNDP models against SNDP to find that PNDP models are far less sensitive to small changes in parameters

Future research:

- Risk parameters as variables, to seek and optimal combination of risk versus cost
- Special network topologies that may provide more effective algorithms

Thank you!