

Alternative Methods for Solving Heterogeneous Firm Models

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Abstract

I implement and compare five solution methods for a benchmark heterogeneous firms model with lumpy capital adjustment and aggregate uncertainty. The Krusell Smith algorithm performs best within a group of methods using projection in the aggregate states. Another technique, Parameterization plus Perturbation, is much faster and performs best within a group of methods using perturbation in aggregates. However, projection and perturbation have nonoverlapping strengths and weaknesses. I highlight the resulting tradeoffs with several model extensions. I recommend that researchers apply projection methods to cases with large shocks or nonlinear dynamics, while cases with explicitly distributional channels at work favor perturbation.

Keywords: Heterogeneous Agents, Computational Methods, Lumpy Investment

JEL: C63, E22, E32

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1 Introduction

Heterogeneous agent business cycle models offer the attractive possibility of combining a fully fledged business cycle structure with rich, testable implications for the cross-section of consumer or firm behavior. However, such frameworks, with seminal examples given by the incomplete markets model of Krusell and Smith (1998) and the heterogeneous firms model of Khan and Thomas (2008), pose several practical challenges.¹ First, their solution and simulation are computationally intensive. Second, traditional solution techniques, such as the Krusell Smith (KS) algorithm, rely on approximations to the aggregate state space and must be evaluated ex-post for the internal consistency of these approximations.

Many existing papers help guide applied researchers around these issues in the practical solution of the incomplete markets model, providing alternative solution techniques and computational strategies.² These advances have profitably improved the speed and accuracy of solutions of the incomplete markets model, but the literature lacks a comprehensive analysis of their applicability to the heterogeneous firms context, a fundamentally different economic and computational environment. In particular, the heterogeneous firms model requires a discrete investment choice by agents and typically requires use of Bellman equations rather than Euler equations to characterize optimal policies. At the micro level, the incomplete markets model can also be solved quickly and efficiently using the Endogenous Grid Points method of Carroll (2006), inapplicable to the heterogeneous firms model. Finally, the heterogeneous firms model depends upon prices which are not closed-form functions of moments of the model's cross-sectional distribution, adding computational complexity relative to the standard incomplete markets model.

This paper provides an intentionally practical and applied comparison of solution techniques specifically targeted towards the solution of the heterogeneous firms model. The Khan and Thomas (2008) model is a natural framework on which to base such a comparison because of the large number of papers using a similar underlying structure.³ The heterogeneous firms framework here combines aggregate uncertainty in the form of aggregate productivity shocks together with lumpy capital adjustment costs and a rich cross-sectional distribution of idiosyncratic productivity shocks and capital holdings. I study five algorithms, implementing each solution technique and comparing them along multiple dimensions: their simulated business cycle moments, cross-sectional investment rate moments, impulse response functions, internal accuracy, as well as the computational burden posed by each algorithm. For each solution method I consider here, I provide readily available code

¹These models pose theoretical challenges too. Miao (2006) emphasizes that standard existence proofs for recursive equilibria may not hold in the context of the incomplete markets model with aggregate uncertainty.

²See, among others, Algan et al. (2010b), Algan et al. (2008, 2010a), Den Haan and Rendahl (2010), Den Haan (2010b), Maliar et al. (2010), Reiter (2010c), and Young (2010).

³See, among others, Gourio and Kashyap (2007), Bloom et al. (2016), Khan and Thomas (2013), Bachmann and Bayer (2013), Bachmann and Ma (2015), Bachmann et al. (2013), with earlier work in Khan and Thomas (2003) and Thomas (2002). A complementary literature in heterogeneous agent state-dependent pricing models, typically dependent on the KS algorithm for solutions, includes papers by Vavra (2014), Klenow and Willis (2007), Knotek (2010), Knotek and Terry (2008), and many others.

online. I consider the following five techniques:

1. the traditional KS approach as adapted by Khan and Thomas (2008),
2. the Parameterized Distributions (PARAM) algorithm due to Algan et al. (2008, 2010a),
3. the Explicit Aggregation (XPA) method of Den Haan and Rendahl (2010) as adapted by Sunakawa (2012),
4. the Projection plus Perturbation (REITER) solution technique of Reiter (2009),
5. and the Parameterization plus Perturbation (WINBERRY) method of Winberry (2015).

The KS approach is a natural and important choice because of its wide use in the heterogeneous firms literature to date. The PARAM algorithm is attractive both because it has been studied comprehensively in the context of the incomplete markets model but also because it bears conceptual similarity to another approach, the Backward Induction algorithm of Reiter (2010c). The XPA approach has been studied previously as a solution method for the heterogeneous firms model in Sunakawa (2012), and for comparability I rely on that paper’s adaptation of the original Den Haan and Rendahl (2010) technique. These first three methods each rely upon projection within an approximate aggregate state space. In my context, the term projection simply implies that macro variables enter explicitly into firm decision rules which are computed over a grid in the simplified aggregate state space.

The final two techniques, the REITER and WINBERRY approaches, are conceptually distinct and based on linear perturbation with respect to aggregate shocks. More explicitly, the REITER algorithm characterizes firm decisions with a system of Bellman equations holding at the steady-state of the model and linearizes both these Bellman equations and the dynamics of the micro-level distribution, stored as a histogram, with respect to aggregate productivity. The WINBERRY method, applied by Winberry (2015) to an extended version of the heterogeneous firms model considered here, linearizes the dynamics of the economy around steady-state as in the REITER algorithm. However, the WINBERRY method stores information about the cross-sectional distributions more parsimoniously following a parametric approach.

All of the methods deliver broadly similar overall business cycle dynamics. Furthermore, the micro investment rate moments, a crucial target for calibration in this class of models, are virtually identical across techniques.

Two methods stand out based on their performance in the baseline model. Within the class of projection-based solutions, the KS routine offers superior internal accuracy, although this comes at the cost of high computational intensity. This result should be interpreted as a favorable robustness check to the large number of papers relying on the KS algorithm in the heterogeneous firms context. Within the set of perturbation-based algorithms, the WINBERRY method is attractive because it combines speed and scalability common to all the perturbation techniques with an efficient storage

convention for the cross-sectional distribution. The two broad approaches these methods represent, projection versus perturbation, are not strictly ranked and possess non-overlapping strengths and weaknesses.

In order to provide a more explicit guide for a choice between projection- and perturbation-based solutions, I also extend the model and baseline analysis in three directions designed ex-ante to showcase the distinct strengths of each approach. I narrow my focus to the KS and REITER methods, which my initial analysis suggests are representative for projection and perturbation, respectively.

First, I vary the size of aggregate shocks. For small shocks, KS and REITER macro simulations differ little, consistent with high accuracy for the local REITER solution very near the steady-state of the model. As the shock size grows to high levels, however, the KS and REITER simulations display different volatilities. I therefore recommend projection-based methods, such as KS, for cases with large shocks.

Second, I extend the model to consider a system of size-dependent cyclically varying labor taxes and subsidies. The size-dependent distortions are designed ex-ante to deliver reallocation of labor across firms in a manner which amplifies output, an explicitly distributional channel. The KS and REITER methods deliver similar implications of increased output volatility as the distortions become more severe. However, KS prediction rules based on an approximate aggregate state space become less accurate. By almost fully storing the distribution, the REITER solution captures the effects of size-dependent taxes but also offers dramatic speed gains. In contexts centered on explicitly distributional channels, I therefore recommend perturbation-based solutions such as REITER.

First-order perturbation solutions, like the versions of the REITER technique considered here, also inherently rely upon an assumption of linear dynamics with respect to aggregate shocks. In a third and final extension, I provide an example of a commonly studied case in which this linearity assumption breaks down. I allow the volatility of micro shocks to fluctuate countercyclically, following the conventions of a recently growing literature on uncertainty shocks. Since fluctuations in volatility occur only at the micro level, in principle these dynamics can be captured with a first-order perturbation in aggregates. However, the KS solution delivers higher output volatility as micro-level uncertainty fluctuates more strongly, while the REITER approach delivers reduced output volatility as uncertainty varies more. As Bloom (2009) emphasizes, this class of uncertainty shock models with lumpy input adjustment contain forces pushing output in opposite directions when volatility changes. Linearized solutions in aggregates capture the full quantitative strength of only one of these forces, a dampening effect. In contexts which involve potentially nonlinear dynamics in aggregates, I therefore recommend projection-based methods such as KS.⁴

Section 2 lays out the model and calibration, a direct simplification of Khan and Thomas (2008).

⁴There is no reason why higher-order perturbations capturing nonlinearity can't be implemented. My conclusions apply to the context of linearized solutions. Interested readers can see Reiter (2010b) or Winberry (Forthcoming) for examples of higher-order generalization of the REITER and WINBERRY solution methods.

Section 3 provides a brief overview of each of the solution techniques. Section 4 compares the resulting simulations, impulse responses, accuracy, and time requirements of each solution method. Section 5 compares projection- and perturbation-based approaches in more detail. Section 6 concludes. A set of online appendixes provides details. Appendix A contains detailed explanations of the solution algorithms, Appendix B contains some practical details on the numerical implementation, Appendix C discusses the simulation used to generate nonlinear impulse responses, and Appendix D details my comparison between projection and perturbation techniques.

2 Model and Calibration

The model is a simplified version of the structure in Khan and Thomas (2008). The simplification involves only the removal of maintenance investment and trend investment growth, but crucially maintains aggregate uncertainty, the discrete nature of the investment decision, and idiosyncratic productivity shocks at the microeconomic level. Interested readers can find much more detail on the assumptions underlying this economic structure in Khan and Thomas (2008).

2.1 Households

A unit mass of identical households trade a complete set of state-contingent claims, own a unit mass distribution of firms, and have flow utility given by $U(C, 1 - N) = \log(C) + \phi(1 - N)$, $\phi > 0$. C represents aggregate consumption, and N represents aggregate labor supply. For my purposes, there are two implications of the household problem of importance for the solution of the model. First, firm value maximization is equivalent to maximization of dividends weighted by a marginal utility price p . Second, household labor supply optimality and linear disutility of labor imply a trivial relationship between the wage and price p :

$$p = \frac{1}{C(A, \mu)}, \quad w(A, \mu) = \frac{\phi}{p(A, \mu)}.$$

Above, prices and wages are written in terms of an aggregate state (A, μ) including aggregate productivity A and a cross-sectional distribution μ of capital and productivity, both of which are discussed in more detail below.

2.2 Firms

In each period there is a distribution of firms $\mu(z, k)$ over idiosyncratic productivity and capital levels z and k .⁵ Individual firms are subject to both idiosyncratic and aggregate productivity shocks, which are exogenous and are assumed to follow independent AR(1) processes in logs:

$$\log(A') = \rho_A \log(A) + \sigma_A \varepsilon_A, \quad \log(z') = \rho_z \log(z) + \sigma_z \varepsilon_z$$

⁵Note that although I use the term “firm” throughout the paper for simplicity, such models are typically disciplined by the use of *establishment* data at the microeconomic level, treating individual establishments as separately operating business units. However, for a recent treatment of a heterogeneous firms structure centering on the distinction between establishments and firms, see Kehrig and Vincent (2013).

where innovations to both processes are iid $N(0, 1)$. The state vector for an individual firm is given by $(z, k; A, \mu)$, which contains both the idiosyncratic states for that firm as well as an aggregate state including productivity and all distributional information. Firms also receive a random draw of fixed capital adjustment costs in each period, discussed below. Conditional upon idiosyncratic productivity and capital (z, k) , a firm that chooses labor input n produces output given by the decreasing returns to scale technology $y(z, k, n, A) = zAk^\alpha n^\nu$, where $\alpha + \nu < 1$.

In a rational expectations equilibrium there is a known transition mapping Γ_μ tracking the evolution of the cross-sectional distribution, as well as a mapping Γ_p from the aggregate state to the marginal utility of the representative household-owner p :

$$\mu' = \Gamma_\mu(\mu, A), \quad p = \Gamma_p(\mu, A).$$

Recall that the wage is a simple function of the household marginal utility given linear labor disutility, so these two aggregate mappings fully characterize the aggregate structure of the economy from the perspective of an individual firm. Then, in each period, a firm receives a stochastic draw of a fixed capital adjustment cost ξ , given in units of labor. The firm value function V , adjusted by the marginal utility of the representative households, is therefore given by

$$V(z, k; A, \mu) = \mathbb{E}_\xi \tilde{V}(z, k, \xi; A, \mu).$$

Once a firm receives a draw of a stochastic adjustment cost $\xi \sim G(\xi)$, the firm faces a choice between paying the capital adjustment cost or not adjusting the capital stock

$$\tilde{V}(z, k, \xi; A, \mu) = \max \left\{ -\xi p(A, \mu) w(A, \mu) + V^A(z, k; A, \mu), V^{NA}(z, k; A, \mu) \right\},$$

where the value upon adjustment V^A is given by optimization over investment and labor

$$V^A(z, k; A, \mu) = \max_{k', n} \left\{ p(A, \mu) (zAk^\alpha n^\nu - k' + (1 - \delta)k - w(A, \mu)n) + \beta \mathbb{E}_{\Gamma_\mu, z', A'} V(z', k'; A', \mu') \right\}.$$

If a firm chooses not to adjust its capital stock, then it must face a dynamic payoff V^{NA} which involves optimization of only the labor input n holding future capital levels fixed at the depreciated level from the current period:

$$V^{NA}(z, k; A, \mu) = \max_n \left\{ p(A, \mu) (zAk^\alpha n^\nu - w(A, \mu)n) + \beta \mathbb{E}_{\Gamma_\mu, z', A'} V(z', (1 - \delta)k; A', \mu') \right\}.$$

The nature of the discrete choice problem leads to a cutoff rule for capital investment such that firms adjust their capital stock if and only if the adjustment cost draw ξ is less than

$$\xi^*(z, k; A, \mu) = \frac{V^A(z, k; A, \mu) - V^{NA}(z, k; A, \mu)}{\phi},$$

where the numerator reflects the gains from capital adjustment relative to inaction and the denominator's adjustment by labor disutility ϕ is required to convert from marginal-utility to labor units. Further the distribution of lumpy capital adjustment costs is assumed to be given by $G(\xi) = U(0, \bar{\xi})$, where $\bar{\xi} > 0$ indexes the level of the adjustment friction in the economy.

2.3 Equilibrium

An equilibrium represents a set of firm value functions $\tilde{V}, V, V^A, V^{NA}$, firm policies and adjustment thresholds k', n, ξ^* , prices $p(A, \mu), w(A, \mu)$, and mappings Γ_μ, Γ_p such that

- Firm capital adjustment choices and policies conditional upon adjustment satisfy the Bellman equations defining V, V^A, V^{NA} above, and therefore firm capital transitions are given by

$$k'(z, k, \xi; A, \mu) = \begin{cases} k'(z, k; A, \mu), & \xi < \xi^*(z, k; A, \mu) \\ (1 - \delta)k, & \xi \geq \xi^*(z, k; A, \mu) \end{cases}.$$

- The distributional transition rule used in the calculation of expectations above by firms is consistent with the aggregate evolution of the distributional state

$$\Gamma_\mu(z', k') = \int \int \int I_{\mathbb{A}}(z, k) d\mu(z, k) dG(\xi) d\Phi(\varepsilon_z)$$

$$\mathbb{A}(z', k', \xi, \varepsilon_z; A, \mu) = \{(z, k) | k'(z, k, \xi; A, \mu) = k', z' = \rho_z z + \sigma_z \varepsilon_z\}, \Phi(x) = \mathbb{P}(\varepsilon_z \leq x)$$

- Aggregate output, investment, and labor are consistent with the current distribution μ and firm policies:

$$Y(A, \mu) = \int \int z A k^\alpha n(z, k, \xi; A, \mu)^\nu d\mu(z, k) dG(\xi)$$

$$I(A, \mu) = \int \int (k'(z, k, \xi; A, \mu) - (1 - \delta)k) d\mu(z, k) dG(\xi)$$

$$N(A, \mu) = \int \int n(z, k, \xi; A, \mu) d\mu(z, k) dG(\xi) + \int \int_0^{\xi^*(z, k; A, \mu)} dG(\xi) d\mu(z, k)$$

- Aggregate consumption satisfies the resource constraint

$$C(A, \mu) = Y(A, \mu) - I(A, \mu).$$

- The households are on their optimality schedules for savings and labor supply decisions, i.e. the first-order conditions defining marginal utility and wages hold, and the price mapping is consistent

$$p(A, \mu) = \Gamma_p(A, \mu) = \frac{1}{C(A, \mu)}, \quad w(A, \mu) = \frac{\phi}{p(A, \mu)}.$$

- Aggregate productivity follows the assumed AR(1) process in logs.

2.4 Calibration

The parameter choices used in the solution method comparison below are those chosen by Khan and Thomas (2008). The parameter choices reflect an annual frequency and positive levels of capital adjustment costs at the firm level, as summarized in Table 1. Given that this paper is concerned with the comparison of numerical solution techniques, and that the model is a simplified version of the original structure, these parameter choices should be taken as purely illustrative.

Table 1: Model Calibration

Parameter	Role	Value	Parameter	Role	Value
α	Capital elasticity	0.256	ν	Labor elasticity	0.640
β	Discount rate	0.977	ϕ	Labor disutility	2.4
δ	Capital depreciation	0.069	ρ_A	Aggregate persistence	0.859
σ_A	Aggregate volatility	0.014	ρ_z	Idiosyncratic persistence	0.859
σ_z	Idiosyncratic volatility	0.022	$\bar{\xi}$	Capital adjustment costs	0.0083

Note: The calibration above is based on Khan and Thomas (2008), Table I, reflecting an annual calibration of the heterogeneous firms model with lumpy capital adjustment costs.

3 Solution Methods Overview

3.1 Krusell Smith Algorithm: KS

Khan and Thomas (2008) in the original exposition of the heterogeneous firms model use the first algorithm considered here, the KS approach. Their algorithm extends the one proposed in Krusell and Smith (1998) for use in the incomplete markets model and bases the general equilibrium components of the solution on an approximate aggregation approach.

When solving their dynamic problem, firms approximate the intractable distribution $\mu(z, k)$ over idiosyncratic productivity and capital with some moments m . In practice, m is chosen to simply be the mean aggregate level of capital K . Given this approximation, two sets of forecast rules provide expectations for firms of both the aggregate level of consumption and the evolution of aggregate capital itself. Therefore, the intractable state vector $(z, k; A, \mu)$ for the firm problem discussed above is replaced by $(z, k; A, m)$, and the transition and price mappings are replaced by forecast rules $\hat{m}' = \hat{\Gamma}_m$ and $\hat{p} = \hat{\Gamma}_p$. In practice, the forecast rules are assumed to take a loglinear form conditional upon aggregate productivity, although the algorithm is more flexible in principle.

Solution of the model involves repeated simulation to obtain a fixed point on the forecast mappings for firms. First, a particular set of forecast rules is assumed, allowing for the creation of value functions for the idiosyncratic firm problems using the simplified state space $(z, k; A, m)$. Then, given the idiosyncratic firm value functions, the model is simulated. Throughout this paper unless otherwise noted, aggregate and productivity shocks in the KS method, as well as the PARAM and XPA techniques, are discretized using the Markov chain approximation process of Tauchen (1986). Also, unless otherwise noted, simulation of the cross-sectional distribution of productivity and idiosyncratic capital makes use of the nonstochastic or histogram-based approach in Young (2010) rather than relying on simulation of individual firms. This histogram-based simulation technique avoids the sampling error associated with individual firm simulation and in practice is less computationally burdensome. In each period, market-clearing consumption must be found by repeated reoptimization of firm policies given a guessed price level, the currently simulated histogram of firm states, and continuation values and expectations as dictated by the current rules $\hat{\Gamma}_m$ and $\hat{\Gamma}_p$. This within-period clearing process must be completed each period during simulation

of the model because the moments m do not imply closed-form expression for the prices in this economy, the intertemporal price p and the wage w .⁶ Finally, after simulation is complete, forecast rules are updated on the simulated aggregates. The entire process repeats until a forecast rule fixed point is achieved.

By alternating between solutions for firm value functions given prediction rules for moments and prices and simulation of the economy endogenously solving for prices in each period, this version of the KS algorithm is an example of the “two-step procedure” proposed by Ríos-Rull (1999) for the solution of heterogeneous agents models with unknown sufficient statistics for prices. Further details on the KS solution algorithm, as well the practical choices surrounding the numerical solution of the model can be found in Appendixes A and B.

3.2 Parameterized Distributions Algorithm: PARAM

The PARAM algorithm is based on the work of Algan et al. (2008, 2010a), which was done in the context of the incomplete markets model, and the solution technique bears heavy resemblance to the Backward Induction algorithm of Reiter (2010c). To my knowledge, this paper represents the first application of the PARAM algorithm to a version of the Khan and Thomas (2008) model. The PARAM approach, like the KS method, relies upon an approximation to the aggregate state space, the replacement of the cross-sectional distribution $\mu(z, k)$ with a set of moments m in the dynamic problem of an individual firm.

However, and contrasting with the KS assumption of forecast rules $\hat{\Gamma}_m$ and $\hat{\Gamma}_p$ for the aggregate moments and prices, the PARAM approach instead relies upon a set of “reference moments” m^{ref} , equal to the higher-order centered moments of the cross-sectional distribution of firm capital, conditional upon idiosyncratic productivity. The moments m included in the approximate state for firm dynamic problems are either a subset of or implied by the reference moments m^{ref} , and they can be drawn from a steady-state solution of the model with no aggregate uncertainty if solution of the model without simulation is desired.

Solution of the model involves value function iteration with the simplified state space of $(z, k; A, m)$. Given a guess for the firm value function which can be used in construction of the continuation value in the firm Bellman equations, optimization and calculation of the next iteration of the value function requires calculation of two objects: market-clearing price $p(A, m)$ for construction of current-period returns, and next-period moments m' for input into continuation values. Both p and m' can be computed within the value function iteration step quite naturally by using fixed point iteration. After guessing values for (p, m') , firm policies are computable, and implied aggregates can be obtained by integrating over the cross-sectional distribution of firm-level productivity and capital (z, k) . Such integration is the key step within the PARAM algorithm and is performed numerically using flexible exponential functional forms for the density of capital which exactly match

⁶The comment in Takahashi (2014) on the analysis in Chang and Kim (2007) emphasizes that omission of within-period market clearing in models without closed-form expressions for prices can lead to distorted inferences about the business cycle.

the aggregate moments m together with the higher-order reference moments in the cross-section. Iteration on prices and next-period moments continues until a fixed point is achieved, at which point the next value function iteration step is taken. Once the value function converges, the model is solved.

Note that crucially the PARAM approach does not require simulation and therefore leads to large time savings relative to the KS algorithm’s solution. However, if desired, new values for reference moments can be computed from simulation and updated until an outside fixed point is achieved, similar to the KS technique. In either case, however, simulation in each period requires a fixed-point iteration routine over market-clearing prices and next-period moments, similar to the process within the model solution step and involving integration over parameterized cross-sectional densities. See Appendix A for further details on the PARAM algorithm, as well as the functional forms used for the assumed cross-sectional densities.⁷

3.3 Explicit Aggregation Algorithm: XPA

The XPA solution method relies upon the techniques suggested by Den Haan and Rendahl (2010), as first adapted and applied to the heterogeneous firms model by Sunakawa (2012). The algorithm is similar to the KS method, also making use of an approximation assumption replacing the aggregate state space (A, μ) with the smaller state space (A, m) by relying on a set of moments m . XPA also relies on forecasting rules $\hat{\Gamma}_m$ and $\hat{\Gamma}_p$ for prediction of aggregate moments and prices by firms.

However, there is one main difference between the two techniques. XPA replaces the simulation step of the KS routine with an aggregation across a fixed cross-sectional distribution which is made feasible through the substitution of aggregate states into idiosyncratic policies. In other words, once value functions and policies are obtained based on a simplified state space of $(z, k; A, m)$ and the posited forecast rules, market-clearing prices are obtained by integrating policies over the constant exogenous ergodic distribution of z and ignoring heterogeneity in idiosyncratic capital k . Afterwards, the forecast rules can be updated from the moments and prices generated in this manner until a fixed point is achieved.

As the original work by Den Haan and Rendahl (2010) noted, substitution of aggregate states into idiosyncratic policies creates a Jensen’s inequality-type bias in the forecast system which can be ameliorated in a straightforward way by use of information from the steady-state solution. In particular, the constant terms of the loglinear mappings $\hat{\Gamma}_m$ and $\hat{\Gamma}_p$ are simply shifted after the solution of the model by exactly the amount required to achieve a forecast fixed point at the steady-state model’s capital and consumption levels.⁸ Overall, avoiding aggregation across a full cross-sectional distribution within the solution step allows for large time savings, as emphasized and put to practical use for structural estimation of technology shocks by Sunakawa (2012).

⁷The algorithm laid out in Appendix A, as well as the code posted online, allows for use of fixed steady-state reference moments and alternatively for updating of these moments through simulation. I use only the former in this paper, because doing so by itself already yields economic implications similar to the KS and XPA techniques.

⁸See the full details of this bias correction procedure in Appendix A.4.

3.4 Projection plus Perturbation Algorithm: REITER

The REITER algorithm is based on the work of Reiter (2009) in the context of the incomplete markets model, and it is conceptually linked to a broader set of work on perturbation of models with micro-level heterogeneity (Dotsey et al., 1999; King and Thomas, 2006; Campbell, 1998). The REITER approach has gained traction recently in the analysis of models with nonconvex costs and firm heterogeneity, being used in Costain and Nakov (2011) and Reiter et al. (2013). To my knowledge, this paper is the first application of the approach to a version of the Khan and Thomas (2008) model by itself. However, Reiter et al. (2013) analyzes a similar sticky-capital environment with the addition of a New Keynesian sticky-price structure, and Meier (2016) provides an interesting application of the REITER method to a heterogeneous firms model with time-varying time to build horizons in capital accumulation.

The REITER solution method departs in two important ways from the KS, PARAM, and XPA approaches. First, the algorithm tracks a discretized approximation to the full cross-sectional distribution rather than relying upon an approximate aggregation assumption to reduce the state space. Second, the REITER method relies upon linear perturbation of the model around the steady-state of a model with no aggregate uncertainty, although it still preserves idiosyncratic nonlinearity through a discretization of the firm-level problem. By contrast, the methods considered so far have relied upon projection-based solution techniques. The use of a perturbation approach leads to drastically reduced computational requirements and scalability.

The REITER approach relies upon three steps. The first step imposes almost trivial computational cost: the solution of a steady-state model with no aggregate uncertainty but maintaining micro-level nonlinearity, using a discretization or histogram for idiosyncratic states (z, k) . Then, the second step writes the full, discretized rational expectations equilibrium as the solution to a system of nonlinear equations F . The system is a function of current and lagged values of a large endogenous vector X_t , as well as some exogenous aggregate shocks ϵ_t . In the application to the heterogeneous firms model, the endogenous vector includes aggregate productivity, the cross-sectional histogram weights on each idiosyncratic point, firm values at a set of discrete points, optimal capital adjustment policies, as well as some implied model aggregates including consumption, output, investment, and labor. Therefore, the system F must take into account Bellman equations, distributional transitions, and aggregate equilibrium conditions. The third step involves the application of standard techniques for the solution of dynamic linear rational expectations systems, such as the method of Sims (2002), to the solution of the heterogeneous firms model. Through numerical differentiation, the system F can be written as a linear approximation around the steady-state solution of the model, and then the standard methods for the solution of linear models may be applied. Further discussions of the details of the REITER solution method can be found in Appendix A.

3.5 Parameterization plus Perturbation Algorithm: WINBERRY

The WINBERRY algorithm is based on the work of Winberry (2015), which considers an extended version of the heterogeneous firms model with lumpy capital adjustment allowing for habit formation in household preferences. At its core the WINBERRY technique is similar to the REITER approach, linearly perturbing around the steady-state equilibrium of the economy with respect to aggregate productivity fluctuations.

The main substantive difference between the REITER and WINBERRY techniques lies in their approach to tracking the endogenous evolution of the cross-sectional distribution of capital μ . The REITER approach tracks distributional dynamics using a non-parametric histogram or discretized representation of the distribution. By contrast, the WINBERRY approach parameterizes the distribution μ using the flexible functional forms proposed by Algan et al. (2008, 2010b) as implemented in the PARAM technique. This combination of components drawn from the PARAM and REITER solution techniques motivates my “Parameterization plus Perturbation” label for this method. As in the PARAM method, the parameterized cross-sectional distributions are fully pinned down by a set of higher-order centered moments of capital conditional upon idiosyncratic productivity. The implication of this simplification is that the endogenous vector X_t characterizing the economy contains only the reduced number of cross-sectional capital moments rather than the full histogram or set of bin weights tracked by the REITER solution.

With a smaller set of endogenous variables, the WINBERRY method offers further gains in terms of time savings and computational complexity relative to the REITER approach. In particular, adding micro-level state variables to the baseline structure with WINBERRY in principle only requires storage of a few additional moments to characterize a parameterized joint distribution, while the curse of dimensionality applies to the histogram-based storage of the distribution in a REITER extension at the micro level. Further discussion of the details of the WINBERRY method can be found in Appendix A.

4 Comparing Solutions

This section compares the five alternative solutions to the heterogeneous firms model along multiple dimensions. First, I simulate the model unconditionally, comparing business cycle aggregates, cross-sectional distributions, and micro investment rate moments. Then, I compute simulation-based analogues to impulse response functions to an aggregate productivity shock. For the methods relying upon projection in aggregates and state-space reduction, I compare internal accuracy statistics. Finally, I evaluate the computational time requirements of each method.

Throughout the quantitative comparisons, I hold the details of the numerical implementation constant across methods to the extent possible, i.e. the projection grid ranges and densities do not vary across methods and similar interpolation and optimization techniques are used when solving Bellman equations. Appendix B provides additional details about the numerical choices made in

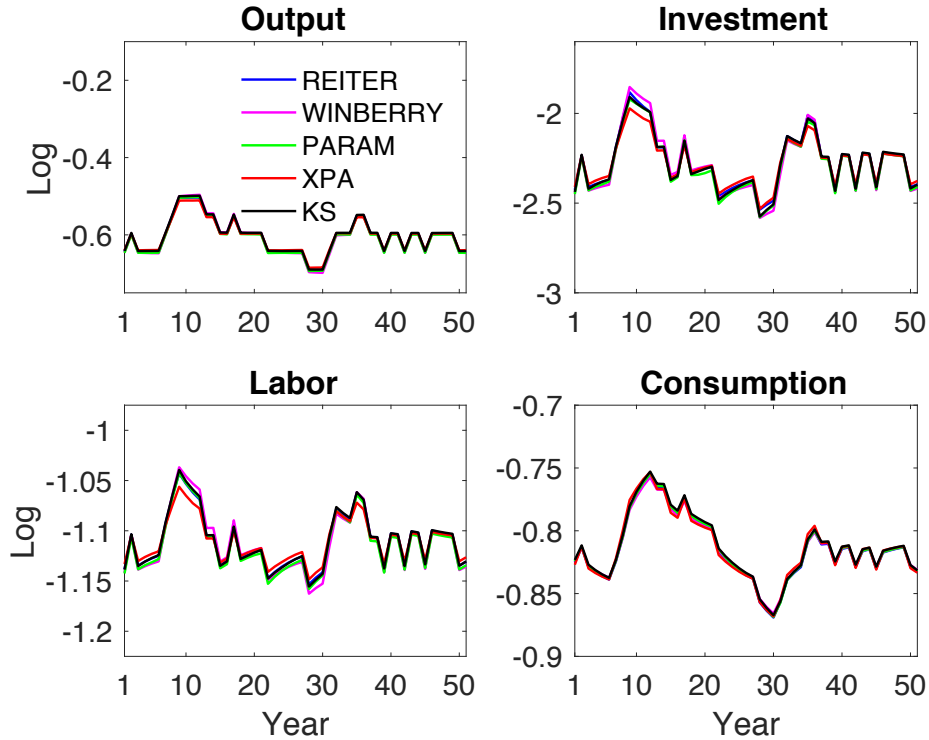


Figure 1: Unconditional Business Cycle Simulation

Note: The figure plots a representative 50-year portion of the unconditional simulation of the model in the KS, XPA, PARAM, REITER, and WINBERRY solutions. The KS solution is in black, XPA in red, PARAM in green, REITER in blue, and WINBERRY in magenta. The exogenous discretized aggregate productivity process over this period is plotted in the left panel of Appendix Figure C.3 and is held constant across methods.

the implementation of each method.

4.1 Unconditional Business Cycle Simulation

To begin the comparison, Figure 1 plots a representative 50-year portion of a larger 2000-year unconditional simulation for each technique, displaying log aggregate output, investment, labor, and consumption. Recall that I discretize the aggregate productivity series in my implementation of the KS, PARAM, and XPA methods, while the linearized REITER and WINBERRY solutions admit continuous local shocks to aggregate productivity.⁹ In order to generate comparable simulations, I compute a set of continuous productivity shocks duplicating the discretized aggregate productivity process and input these shocks into the REITER and WINBERRY solutions to produce Figure 1. The left panel of Appendix Figure C.3 plots the common exogenous productivity series for this range of the simulation.

The simulated fluctuations in Figure 1 are in general quite similar across solution methods, but a

⁹Appendix B provides a business cycle plot in Figure B.2 for the KS and REITER methods allowing for continuously varying aggregate productivity as a robustness check to the main text's assumption of discretized aggregate productivity. The qualitative results of this section are unchanged in the continuous-shock environment.

Table 2: HP-Filtered Business Cycle Statistics

Method	Output	Investment	Labor	Productivity	Consumption
Volatility					
KS	(2.5018)	3.9402	0.6841	0.5789	0.3834
XPA	(2.3002)	3.6069	0.6036	0.6297	0.4513
PARAM	(2.5153)	3.9573	0.6888	0.5758	0.3797
REITER	(2.4616)	3.8101	0.6687	0.5884	0.3951
WINBERRY	(2.5831)	3.9573	0.7022	0.5607	0.3533
Output Correlation					
KS	1.0000	0.9780	0.9661	0.9999	0.8883
XPA	1.0000	0.9764	0.9606	0.9996	0.9287
PARAM	1.0000	0.9783	0.9661	0.9998	0.8846
REITER	1.0000	0.9813	0.9656	1.0000	0.8994
WINBERRY	1.0000	0.9859	0.9748	0.9997	0.8974

Note: The top panel of the table reports the percentage standard deviation of output, investment, labor, exogenous productivity, and consumption for the KS, XPA, PARAM, REITER, and WINBERRY solutions. Each series is first HP-filtered in logs with a smoothing parameter of 100. The first column, in parentheses, reports the raw standard deviation of output, and columns 2-5 report the standard deviation of the indicated aggregate relative to the standard deviation of output. The bottom panel reports the correlation of each indicated business cycle aggregate with aggregate output. All statistics are computed from a 2000-year unconditional simulation of the model, after first discarding an initial 500 years. The exogenous aggregate productivity series is held constant across methods.

few patterns stand out to the naked eye. First, labor and investment fluctuations are somewhat less volatile for the XPA solution than the other projection based solutions KS or PARAM. Second, the WINBERRY solution exhibits marginally more volatility in investment and labor than the REITER solution. In Table 2, I report a standard set of HP-filtered business cycle moments, with volatilities in the top panel and output correlations in the bottom. Just as in Figure 1, the business cycle moments are in general quite similar across methods. In fact, the first column of the top panel reveals that each method implies a standard deviation of output of around 2.5%. Consistent with the patterns in Figure 1, the filtered investment and labor series are relatively more volatile, and slightly more correlated with output, for the REITER and WINBERRY solutions than for the remaining techniques, although these differences are not large enough to be economically significant. Table 2 does also reveal one outlier among the projection-based methods: output, investment, and labor are less volatile in the XPA solution than for the KS or PARAM methods. Taken as a whole, Figure 1 and Table 2 suggest that all of the solution methods yield qualitatively similar implications for aggregate business cycle series and moments.

Figure 2 plots the cross-sectional distribution of capital for each solution method for a representative period in the unconditional simulation of the model. The KS, XPA, and REITER methods each store distributional information non-parametrically as weights on a dense discretization of the capital grid. Discrete idiosyncratic productivity realizations across firms create spikes in the capital distribution for these methods, and the resulting distributions are virtually indistinguishable. By contrast, parameterization of the capital densities in the PARAM and WINBERRY methods yields

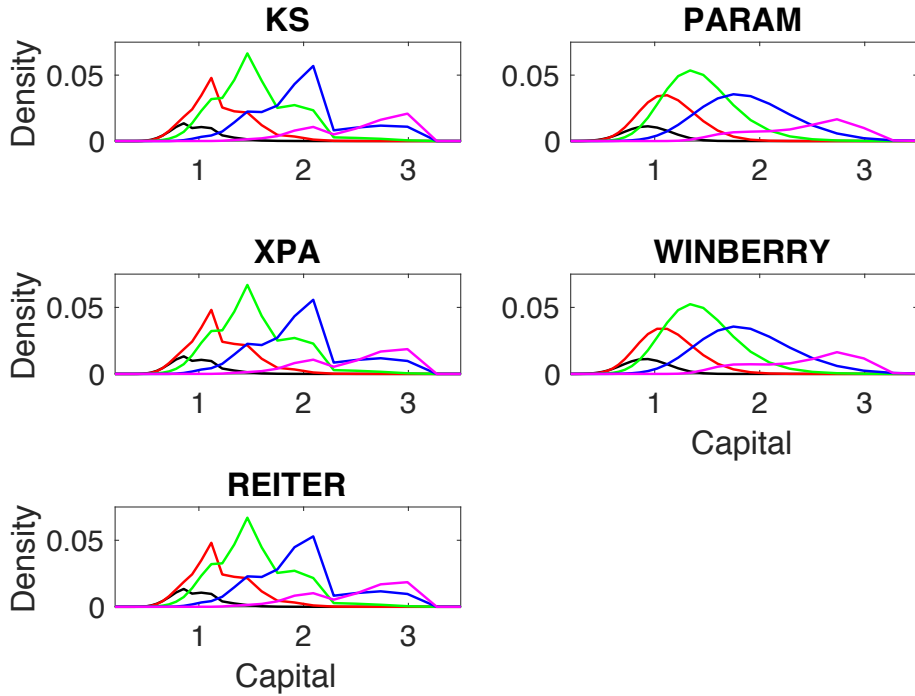


Figure 2: Cross-Sectional Distributions of Capital

Note: Each panel in the figure plots the cross-sectional distribution of capital in a single representative year drawn from the unconditional simulation of the model with a single solution method. At the micro level for all methods, productivity z takes one of 5 values. Each line within the panel plots the cross-sectional distribution of capital for a micro-level productivity level, with the lowest z_1 in black, the next highest z_2 in red, the next highest z_3 in green, the next highest z_4 in blue, and the highest z_5 in magenta. The three methods in the left panels - KS, XPA, and REITER - use a non-parametric histogram-based storage convention for the cross-sectional distribution, while the methods in the right two panels - PARAM and WINBERRY - use a parametric family of densities to store the distribution. The exogenous aggregate productivity series is held constant across methods.

smooth cross-sectional distributions which are nonetheless comparably shaped and positioned.

Table 3 reports a range of micro investment moments computed from the cross-sectional distributions of each solution method. The moments analyzed here include the mean and standard deviation of the investment rate, together with the probability of investment inaction, positive and negative investment spikes, and positive and negative investment overall. Comfortingly, since these moments typically serve as crucial calibration or estimation targets (Khan and Thomas, 2008; Bachmann and Bayer, 2014), each solution method delivers broadly similar implications for the cross-section of investment. Around three-quarters of firms are inactive in each period, with around one-fifth of firms exhibiting both positive investment spikes or positive investment overall. Fewer periods see negative investment rates or spikes.¹⁰

¹⁰By contrast with Khan and Thomas (2008), which allows for costless maintenance investment, my simplified structure features higher levels of inaction and lower levels of negative investment. The model here therefore delivers moments broadly similar to those of the “Traditional Model” of Table II in Khan and Thomas (2008). In Appendix B, I extend the model to allow for maintenance investment in the KS solution. Appendix Figure B.1 reveals that simulated business cycle aggregates differ little from those in the baseline KS solution without maintenance investment. Table

Table 3: Microeconomic Investment-Rate Moments

	KS	XPA	PARAM	REITER	WINBERRY	CENSUS	IRS
$\frac{i}{k}$	0.0947	0.0951	0.0931	0.0900	0.0947	0.122	0.119
$\sigma\left(\frac{i}{k}\right)$	0.2597	0.2628	0.2578	0.2526	0.2627	-	0.20
$\mathbb{P}\left(\frac{i}{k} = 0\right)$	0.7693	0.7726	0.7771	0.7798	0.7757	0.081	0.302
$\mathbb{P}\left(\frac{i}{k} \geq 0.2\right)$	0.1724	0.1709	0.1730	0.1634	0.1716	0.186	0.174
$\mathbb{P}\left(\frac{i}{k} \leq -0.2\right)$	0.0280	0.0276	0.0263	0.0253	0.0257	0.018	-
$\mathbb{P}\left(\frac{i}{k} > 0\right)$	0.1890	0.1865	0.1852	0.1803	0.1852	0.815	-
$\mathbb{P}\left(\frac{i}{k} < 0\right)$	0.0417	0.0409	0.0378	0.0399	0.0391	0.104	-

Note: The rows of the left panel of the table above report the mean value, across periods, of the indicated microeconomic moment of the cross-sectional distribution of investment rates $\frac{i}{k}$ in an unconditional simulation of the KS, PARAM, XPA, REITER, and WINBERRY methods. The first row reports the level of investment rates, the second row the cross-sectional standard deviation of investment rates, the third column the probability of investment inaction, the fourth (fifth) columns the probability of positive (negative) investment spikes larger in magnitude than 20%, and the sixth (seventh) columns the probability of strictly positive (negative) investment rates. All statistics are computed from a 2000-year unconditional simulation of the model, after first discarding an initial 500 years. The exogenous aggregate productivity series is held constant across methods. The rows of the right panel report the same statistics, when available, computed from data. The column labelled ‘‘CENSUS’’ is drawn from Cooper and Haltiwanger (2006) and reports investment statistics from a balanced panel of US manufacturing establishments from 1972-1988 based on US Census Bureau micro data (see that paper’s Table 1). The column labelled ‘‘IRS’’ is drawn from Zwick and Mahon (Forthcoming) and reports investment statistics from an unbalanced panel covering a wider range of US firms from 1998-2010 based on IRS micro data (see that paper’s Table B.1).

To provide a rough empirical benchmark for comparison, Table 3’s CENSUS column reports investment statistics based on Census micro data on US manufacturing establishments in the period 1972-1988 and taken from Cooper and Haltiwanger (2006). Table 3’s IRS column reports investment statistics drawn from IRS micro data on a broad range of US firms from 1998-2010, as reported by Zwick and Mahon (Forthcoming). The mean, standard deviation, and spike rates in the investment rate distribution are quite similar to their data counterparts for each solution method considered in this paper. The inaction rates in the data, by contrast, are smaller than in the simulated data for each method, a difference that is due to my choice to remove costless maintenance investment from the model. As I show in an extension of the KS solution in Appendix B, matching the inaction rates in the simulated data is possible with virtually no change to aggregate dynamics - but substantial complication of the model’s notation - if firms are allowed to become active over investment in some small range without adjustment cost payment.

4.2 Impulse Response Functions

Now I turn to conditional or impulse response analysis. Some concrete decisions must inevitably be made about the manner in which to simulate the underlying object of interest, i.e. the average change in the forecast of a given series in response to a shock to aggregate productivity of a certain size. Two considerations will always face a researcher working with nonlinear discretized models like those considered here. First, given the nonlinear structure of the KS, PARAM, and XPA solutions,

B2 reports that the probability of investment inaction falls.

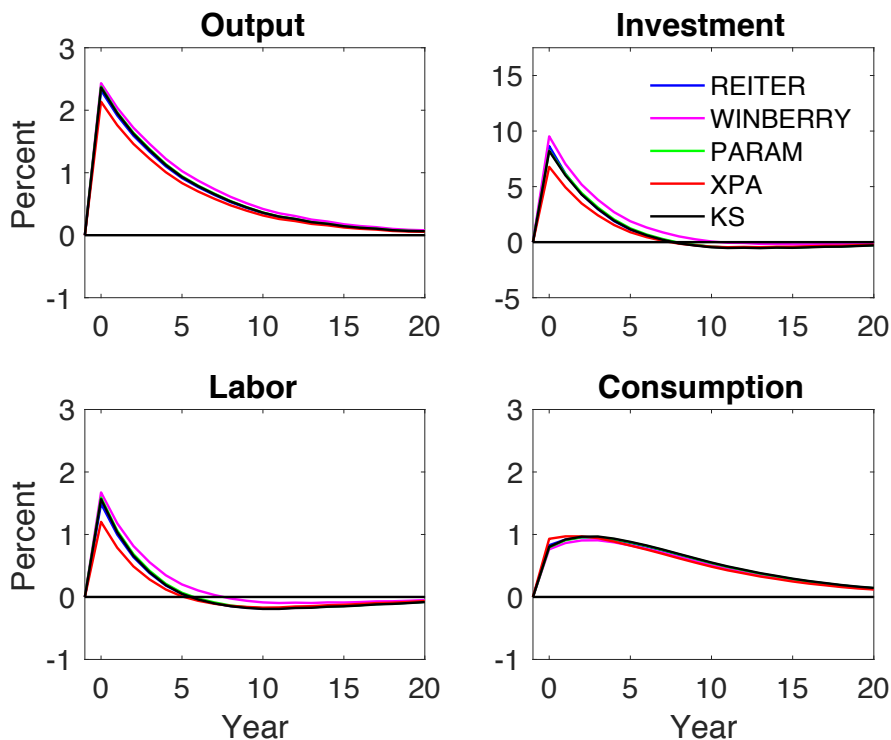


Figure 3: Impulse Response to a Positive Aggregate Productivity Shock

Note: The figure plots simulated impulse responses to a positive one standard deviation (1.4%) aggregate productivity shock for the KS, XPA, PARAM, REITER, and WINBERRY solutions. The KS solution is in black, XPA in red, PARAM in green, REITER in blue, and WINBERRY in magenta. Each line represents a simulated “generalized impulse response” as defined by Koop et al. (1996). This simulation-based impulse response calculation for nonlinear models involves the comparison of 2000 independent simulations of 50-year length, with and without exogenous positive shocks to aggregate productivity. The right panel of Appendix Figure C.3 plots the underlying exogenous shock to aggregate productivity.

the average conditional response to a shock will depend both upon initial conditions and upon the size of the shock. Second, I may wish to consider a shock scaled to a certain average size, such as the calibrated standard deviation of the underlying true aggregate productivity process, but a discrete Markov chain only admits discrete innovations in the aggregate productivity series. Neither challenge is present with the linearized solutions from the REITER and WINBERRY methods, since in those cases a classical impulse response is computable directly from the coefficients defining the model solution. In this case, linearity guarantees that the impulse response scales directly with shock size and doesn’t vary with initial conditions.

To create an approximation to the average conditional response in my context, I simulate to compute the “generalized nonlinear impulse responses” of Koop et al. (1996), although for simplicity I refer to these simply as “impulse responses.” The approach relies upon a large number of pairs of simulations, with one “shock” simulation and one “no shock simulation.” Within each pair the two simulations are run under identical exogenous shock processes with one difference. At a designated period I impose a positive shock to aggregate productivity in the shock simulation, allowing the

aggregates to evolve as normal afterwards. The average percentage difference, across simulation pairs, between the shocked and no shock simulations provides a measure of the average innovation to a given series in response to a productivity shock. To generate a flexibly-sized aggregate shock using discretized productivity, I simply convexify the shock arrival within each simulation pair described above, imposing a shock only with a probability calculated to generate any desired average change in aggregate productivity. Appendix C provides the details of this Koop et al. (1996) approach.

Figure 3 plots the impulse response to a one standard deviation (1.4%) positive aggregate productivity shock for output, investment, labor, and consumption. The responses are qualitatively identical across all methods: an increase in aggregate productivity leads immediately to a jump in output, labor, investment, and consumption.

4.3 Accuracy Statistics

Firms investing in the KS, PARAM, and XPA solutions rely upon the reduced aggregate state space (A, K) to form expectations both about market-clearing prices today p as well as the aggregate capital level in the next period K' . In the KS and XPA solutions, firms use explicit loglinear forecast rules. The PARAM method does not rely on an explicit forecast rule, but PARAM does endogenously generate a mapping over a projection grid on (A, K) to clearing levels of (p, K') . By linearly interpolating this mapping I can generate a forecast system for price and aggregate capital from the PARAM solution.

Using the embedded KS, XPA, and PARAM prediction rules, I produce two different sets of forecasts for aggregate capital and prices: “static” and “dynamic.” A quick overview of these time series concepts is in order. Using actual simulated data as inputs to the prediction rules produces static forecasts. Given this model’s timing, static forecasts are for the current year (prices p) or for one year ahead (capital K'). Dynamic forecasts are produced recursively by forward iteration of the prediction rules. In other words, dynamic forecasts at a two year horizon use one year ahead forecasts as inputs, dynamic forecasts at a three year horizon use two year ahead forecasts as inputs, and so on. Much of the early literature on heterogeneous agents business cycle models presented the R^2 of the forecasting regressions, a function of the static forecasting errors, as a gauge of internal accuracy. However, as emphasized by Den Haan (2010a), accuracy statistics based on dynamic forecasts offer a more stringent accuracy criterion since errors in the prediction rules can accumulate as the horizon increases.

For the KS, XPA, and PARAM solutions, Figure 4 plots capital and price series from the unconditional simulation, together with the associated static and dynamic forecasts.¹¹ For ease of reference, I trivially transform price p to units of consumption C via $\log(C) = -\log(p)$. For KS and PARAM, the realized values of consumption and capital are visually indistinguishable from the static and dynamic forecasts. However, for the XPA simulation, the actual data differs

¹¹Den Haan (2010a) refers to the comparison of dynamic forecasts and simulated data included in Figure 4 as a “fundamental accuracy plot.”

Table 4: Internal Accuracy of Forecast Systems

	C , KS	C , XPA	C , PARAM	K' , KS	K' , XPA,	K' , PARAM
Den Haan Statistics						
Maximum	0.1127	1.2102	0.2201	0.3760	3.4568	0.4857
Mean	0.0500	0.3393	0.0879	0.2312	1.1751	0.2135
Root Mean Squared Error						
$A = A_1$	0.0562	0.7733	0.0157	0.0554	0.6426	0.0697
$A = A_2$	0.0526	0.3115	0.0233	0.0549	0.3632	0.0686
$A = A_3$	0.0466	0.0205	0.0139	0.0517	0.1904	0.0552
$A = A_4$	0.0422	0.3341	0.0165	0.0458	0.3466	0.0365
$A = A_5$	0.0396	0.7355	0.0150	0.0452	0.5705	0.0273
Forecast Regression R^2 's						
$A = A_1$	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000
$A = A_2$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$A = A_3$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$A = A_4$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$A = A_5$	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000

Note: Each projection-based solution method in this paper - KS, XPA, and PARAM - contains embedded forecast rules for the aggregate price p (equivalently, consumption C) and next period's aggregate capital level K' based on an approximate aggregate state space (A, K) . These forecast rules imply static and dynamic forecasts of each series, as discussed in the main text. The top panel of this table reports statistics proposed by Den Haan (2010a): the maximum and mean percentage differences between realized data and dynamic forecasts for the indicated rule and method. The second panel reports the root mean squared error, in percentages, based on static forecasts and conditioned upon the level of aggregate productivity. The bottom panel reports the R^2 of the forecast rules for each solution, a function of the static forecast errors also conditional upon the level of aggregate productivity. The exogenous aggregate productivity series underlying these statistics is held constant across methods and represents a separate 2000-year draw of productivity realizations than the aggregate productivity series used in the KS solution.

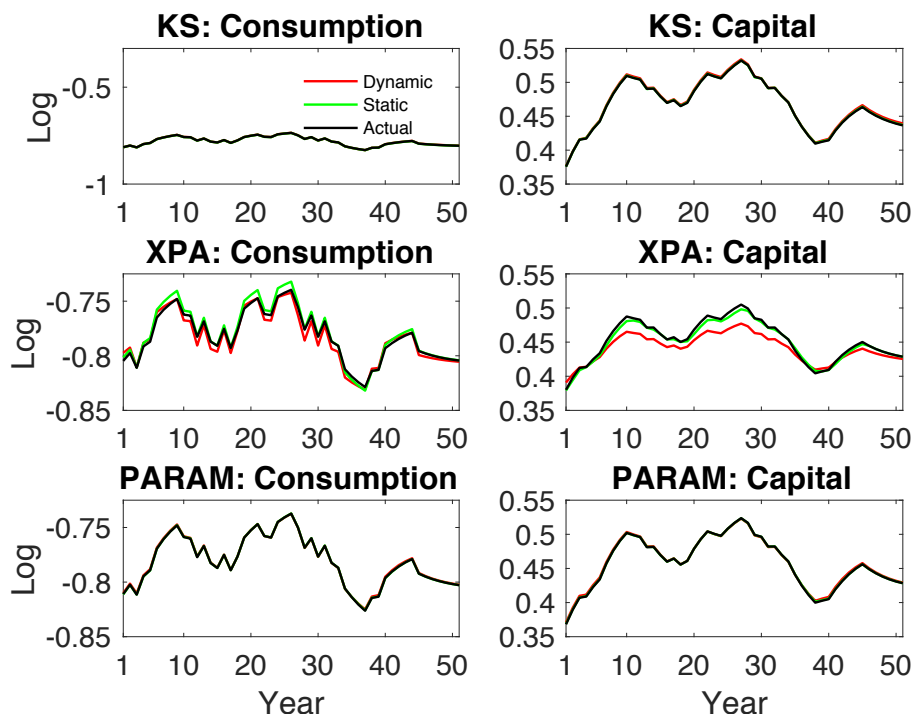


Figure 4: Static Forecasts, Dynamic Forecasts, and Actual Data

Note: Each projection-based solution method in this paper - KS, XPA, and PARAM - contains embedded forecast rules for the aggregate price p (equivalently, consumption C) and next period's aggregate capital level K' based on an approximate aggregate state space (A, K) . These forecast rules imply static and dynamic forecasts of each series, as discussed in the main text. Each panel in the figure plots the realized level of consumption or capital (in black) together with the static (in green) and dynamic (in red) forecasts for the indicated method. The 50-year period is a representative portion drawn from an unconditional simulation of the model under each solution. The exogenous aggregate productivity series underlying these plots is held constant across methods and represents a separate 2000-year draw of productivity realizations than the aggregate productivity series used in the KS solution.

perceptibly from the forecasts. Table 4 reports three sets of accuracy statistics for each solution: Den Haan (DH) statistics following Den Haan (2010a), root mean squared errors (RMSEs), and regression R^2 s. The DH statistics are the mean and maximum percentage difference between the realized series and the dynamic forecasts. The RMSE statistics and regressions R^2 s are standard diagnostics computed directly from the static forecasts. The RMSE and R^2 results are conditioned on aggregate productivity, mirroring the underlying forecast system. All of the statistics come from a 2000-year simulation of the model, using exogenous shocks distinct from those used in the estimation of the forecast rules themselves.

The KS solution delivers the most accurate consumption and capital forecasts, with the maximum DH statistic, the highest percentage difference between realized data and dynamic forecasts well below 0.4% over the entire 2000-year horizon of the simulation. However, the PARAM solution delivers accuracy only slightly worse than the KS method, with maximum forecast errors for consumption and capital only about 0.1% larger. The XPA method results in substantially less

Table 5: Computational Time

Time (in seconds)	Solution	Unconditional Simulation	IRF Simulation
KS	2562	413	34912
XPA	55	646	54688
PARAM	236	189	14691
REITER	12	16	18
WINBERRY	11	10	18

Note: The quantities refer to the runtime on an iMac with a four-core eight-thread 4.4GHz Intel Core i7 processor with 32 GB of RAM. The model solution and simulation code is written in parallelized Fortran for all methods. All of the code used to produced these results can be found on Stephen Terry’s website. “Solution” refers to the time required for calculation of a forecast rule fixed point (KS and XPA), completion of value function iteration (PARAM), or calculation of the linearized model solution representation (REITER and WINBERRY). Unconditional simulation refers to the time required for unconditional simulation of a 2000-year economy with identical exogenous aggregate shocks across solution methods and an initial simulation range of 500 years discarded. IRF simulation refers to the time required for simulation of a conditional impulse response function to an aggregate productivity shock using the method of Koop et al. (1996), with a simulation length of 50 years, 2000 replications, and shocks held constant across solution methods. All models are solved using comparable idiosyncratic and aggregate grids, identical Bellman equation or policy iteration tolerances, and identical forecast rule initial conditions. See Appendix B for details.

accurate forecasts, with dynamic forecasts of capital and consumption differing from their realized levels by more than 3% and 1%, respectively, during some periods. The maximum DH statistics are the most stringent of the diagnostics reported in Table 4, but the ranking of accuracy across these methods - KS the most accurate, followed by PARAM and then XPA - is also supported by the mean DH statistics and the RMSE diagnostics for most series and productivity states. The forecast regression R^2 's are above 0.9999 for each method at all times, highlighting the weakness of this metric as a relative gauge of internal accuracy.

4.4 Computational Time

High computational time for a solution method hinders its practical usefulness. This subsection breaks down the time requirements of each method. Although runtime comparisons inevitably depend upon the efficiency of coding choices, the programming language used, and the details of the numerical approach, a few considerations allay those concerns in my analysis. All of the code is written in Fortran by the same researcher, liberally parallelizing when possible and executing the programs on the same hardware under the same conditions.¹² The single exception is a call to MATLAB within the REITER and WINBERRY solutions to make use of a standard linear rational expectations system solver due to Sims (2002).¹³ Given this uniformity in implementation, which is unusual in solution method comparisons of this type in the heterogeneous agents computational literature, I feel comfortable in relative comparisons across methods. Table 5 reports the runtimes.

For model solution, KS algorithm takes 10 and 45 times as long as PARAM or XPA, respectively,

¹²The results in Aruoba and Fernández-Villaverde (2015), extrapolated to this context, would suggest that the use of Fortran delivers runtimes faster than most alternative languages commonly used by economists.

¹³In particular, the linear solution step for the REITER and WINBERRY techniques uses the `gensys` software from Chris Sims’ website.

due to the necessity of repeated model simulation to find a forecasting system fixed point. By avoiding simulation, each of those two alternative approaches reduces solution time substantially. The steady-state solution with no aggregate uncertainty, an initial input into the REITER and WINBERRY methods, can be solved within a couple of seconds. The numerical differentiation and solution of the resulting linear system takes only a few more seconds for both solution techniques. Overall, for the numerical choices made here, both the REITER and WINBERRY methods produces a solution within 11-12 seconds, faster by around a factor of 5 than the nearest projection-based competitor (XPA).

Simulation speeds fall into two distinct groups as well. The projection-based KS, PARAM, and XPA approaches are costly, because each approach requires iteration on either the market-clearing price (KS, XPA), or the price, next period's capital stock, and the approximating coefficients of a simulated cross-sectional density (PARAM). Although the PARAM technique takes the least time within this group for simulation, costs are roughly comparable. By contrast, once a linear representation of the equilibrium is obtained in the REITER or WINBERRY solution steps, simulation is far less costly and 10-20 times faster than the nearest projection-based competitor (PARAM). The speed advantage scales with the size of the simulation, with the REITER and WINBERRY solution speeds in the longer impulse response function simulation around a factor of 800-900 times lower than PARAM. Summing across solution and simulation steps, the computational time requirements of the WINBERRY technique are lower than for the REITER approach because the dimensionality of the linearized equilibrium is smaller in the WINBERRY solution.

Clearly, the projection-based KS, XPA, and PARAM solutions demand substantially more computational time than the perturbation-based REITER or WINBERRY approaches. However, the XPA and PARAM solutions reduce model solution times significantly by sidestepping the requirement of repeated simulation in the KS method. Overall, WINBERRY is the fastest solution method considered in the paper.

5 Projection vs. Perturbation in Aggregates

The results above suggest that neither the perturbation-based approaches nor the projection-based solution techniques are strictly dominant. On the one hand, perturbation-based solutions like the REITER method offer faster computation. Perturbation in aggregates also offers lower cost storage of the cross-sectional distribution of capital as well as important scalability by sidestepping the curse of dimensionality. On the other hand, the business cycle dynamics of the projection-based solutions like the KS method may be able to capture larger shocks or nonlinear dynamics more accurately. In this section, I explore the tradeoffs between projection- and perturbation-based solution methods in more detail. I narrow my focus to KS and REITER, which my analysis above suggests are representative of projection and perturbation, respectively.

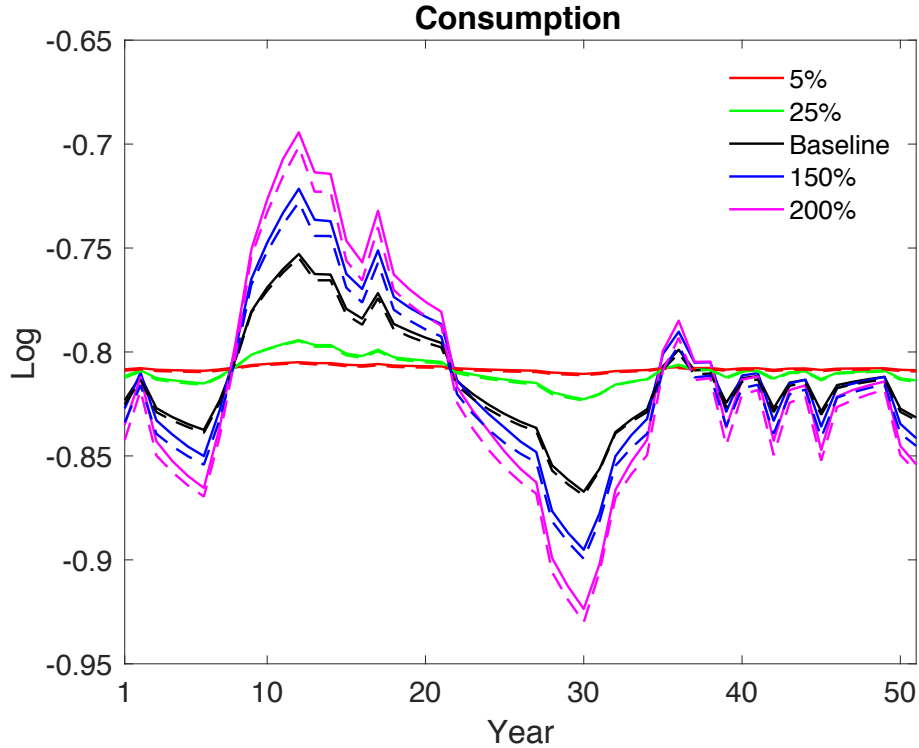


Figure 5: Scaling the Size of Aggregate Shocks

Note: The figure plots a representative 50-year portion of the simulated level of aggregate consumption for both the KS solution (solid lines) and the REITER solution (dashed lines). The standard deviation σ_A of aggregate productivity shocks varies over 5% of its baseline level (red lines), 25% (green), the baseline itself (black), 150% (blue), and 200% (magenta). The underlying innovations to aggregate productivity are held constant as the scaling of these shocks varies across comparisons.

5.1 Size of Shocks

As an initial check, I vary the size of the aggregate shocks in the baseline model. I consider economies with the standard deviation of aggregate productivity shocks σ_A ranging from 5% to 200% of the baseline calibration. Figure 5 plots a representative portion of the simulated consumption path for the KS (solid lines) and REITER (dashed lines) solutions given different aggregate volatility levels. Across each method and aggregate volatility level, the underlying shocks ε_A to the macroeconomy are held constant, and only the scaling or effective size of the shock varies. As the size of the aggregate shocks increases, the deviation between the REITER and KS solutions grows. For volatility at 5% of the baseline level, in red, the endogenous consumption paths in Figure 5 are difficult to distinguish across the KS and REITER solutions, while for shocks twice as large as the baseline, in magenta, the deviations are quite substantial.

More systematically, Appendix Table D1 reports the mean and maximum percentage deviations between the KS and REITER solutions for output, investment, labor, and consumption over the full 2000-year unconditional simulation of the model for each aggregate shock size. For the case

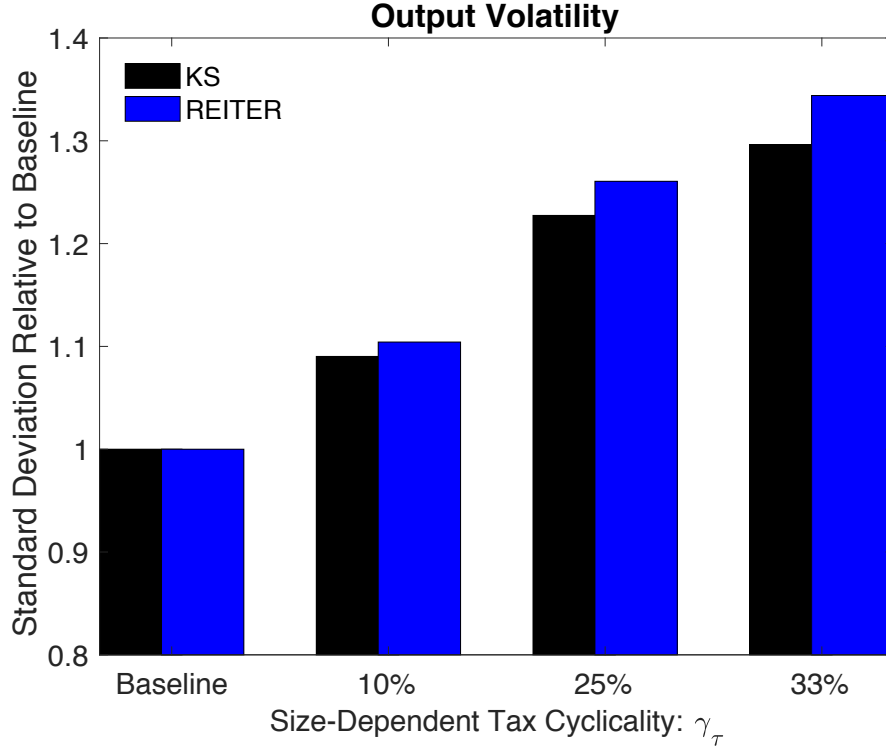


Figure 6: Output Volatility with Different Size-Dependent Tax Fluctuations

Note: The figure plots the volatility of output over a 2000-year unconditional simulation of the model under the KS solution (in black bars) and the REITER solution (blue). Each set of bars represents volatility under a different level of the cyclical elasticity γ_τ of size-dependent taxation which is described in the main text. The aggregate output data is HP-filtered in logs using a smoothing parameter 100, and volatilities are reported as standard deviations relative to the baseline case with $\gamma_\tau = 0$ for the indicated method. The exogenous aggregate productivity series is held constant across methods.

with the lowest aggregate volatility, 5% of baseline, the maximum percentage differences of output, investment, labor, and consumption are all at least an order of magnitude smaller than the baseline case.

Since REITER is based on an assumption of linearity near the steady-state of the model, it is natural and comforting that the KS and REITER methods are consistent for small macro shocks. However, based on the observed divergence between the two methods for larger shock sizes in Table D1 and Figure 5, I recommend projection-based solutions such as KS for cases with large shocks.

5.2 Size-Dependent Taxation

This subsection considers an extension to the baseline model with a cyclically varying size-dependent system of labor taxes and subsidies. The extension nests the baseline model at the steady-state but is designed to deliver amplification of output fluctuations in the presence of aggregate productivity shocks. This effect operates through an explicitly distributional channel.

In particular, I assume that the fiscal authority chooses a micro capital threshold k^* together

with a tax elasticity γ_τ governing fluctuations in a tax rate $\tau(A)$ via

$$\log(1 + \tau(A)) = \gamma_\tau \log(A).$$

The fiscal authority then distorts labor costs at firms relative to the household's marginal rate of substitution between consumption and leisure w . The effective labor cost ω at a firm with capital k given aggregate productivity A satisfies

$$\omega(A, k) = \begin{cases} w(1 - \tau(A)), & k \geq k^* \\ w(1 + \tau(A)), & k < k^* \end{cases},$$

where the implicit labor subsidies and taxes are funded through lump-sum transfers from the households. Appendix D provides details, but the intuition is simple. During booms with $A > 1$, the size-dependent fiscal system increases labor costs at small firms and reduces labor costs at large firms. The reverse occurs during busts. The result of this time-varying system of distortions, which is of course suboptimal, is to push labor towards large firms during booms and towards small firms during busts. Since firms with more capital have on average higher productivity due to persistence in productivity, these distortions amplify output fluctuations. The efficacy of this amplification mechanism depends crucially on the fraction of firms with capital above the threshold k^* in any particular period.

Figure 6 charts HP-filtered output volatilities for the KS and REITER solutions in the baseline economy as well as with $\gamma_\tau = 10\%$, 25% , and 33% .¹⁴ As the distortions vary more strongly, output volatility grows by around a third for both the KS and REITER solutions. However, note that the approximation of the cross-sectional distribution μ by aggregate capital K alone leads to less forecast accuracy in the KS solution as γ_τ grows. The max DH statistic for aggregate consumption, the largest percentage difference between realized consumption and dynamics forecasts of consumption in the KS solution, more than triples from 0.11% to 0.36% as I move from the baseline solution to the $\gamma_\tau = 33\%$ case. For capital, forecast errors by the same metric more than double from 0.37% to 0.81% .

The size-dependent distortion extension suggests that the explicit storage of the full cross-sectional distribution of capital in the perturbation-based REITER method allows for analysis of explicitly distributional mechanisms over the business cycle without a meaningful difference in economic implications relative to the projection-based KS method. However, the approximation to the aggregate state space in the KS solution, at least without a more computationally costly extension to additional moments, begins to yield poorer prediction rules as γ_τ grows. Given its speed advantages, this context favors REITER. I therefore recommend perturbation-based solutions in contexts where the dominant economic forces are explicitly distributional.

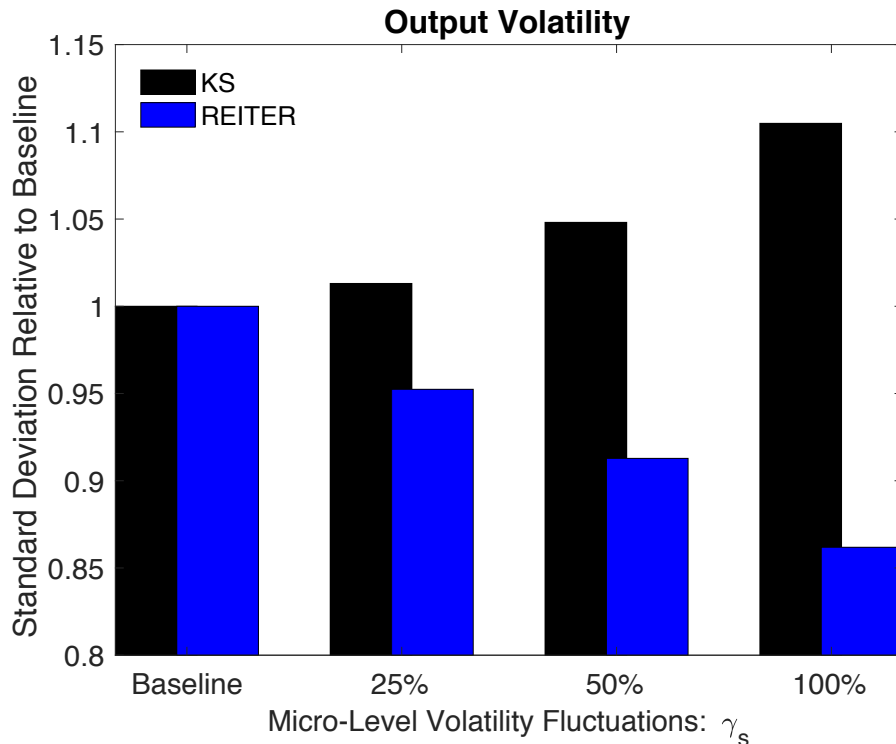


Figure 7: Output Volatility with Micro Uncertainty Fluctuations

Note: The figure plots the volatility of output over a 2000-year unconditional simulation of the model under the KS solution (in black bars) and the REITER solution (blue). Each set of bars represents volatility under a different level of the parameter γ_s described in the main text and governing the countercyclicality of micro volatility. The aggregate output data is HP-filtered in logs using a smoothing parameter 100, and volatilities are reported as standard deviations relative to the baseline case with $\gamma_s = 0$ for the indicated method. The exogenous aggregate productivity series is held constant across methods.

5.3 Micro Uncertainty Fluctuations

A recent literature on uncertainty emphasizes that firm-level shocks appear more volatile during downturns (Bloom, 2009; Bloom et al., 2016) and that this basic fact may affect the efficiency of firm investment, aggregate productivity, financial markets, and output fluctuations over the cycle (Senga, 2015; Bachmann and Bayer, 2013; Arellano et al., 2016). In this subsection, I extend the model with fluctuations in the volatility of micro shocks. I assume that micro productivity follows

$$\log(z') = \rho_z \log(z) + s(A')\sigma_z \varepsilon_z, \quad \varepsilon_z \sim N(0, 1)$$

$$\log(s(A)) = -\gamma_s \log(A).$$

When $\gamma_s = 0$ this nests the baseline, but with $\gamma_s > 0$ micro shock volatility is countercyclical. I tie micro volatility directly to aggregate productivity, avoiding the need to include an additional

¹⁴Interested readers can find a representative plot of simulated output, investment, labor, and consumption for the KS and REITER solutions for each of the size-dependent taxation extensions in Appendix Figure D.4.

micro volatility state variable. Also, by abstracting from fluctuations in the volatility of shocks to aggregate productivity, which would immediately require a more computationally costly higher-order perturbation approach, I allow for micro uncertainty fluctuations to impact the economy within the linearized REITER solution.

Figure 7 charts HP-filtered output volatilities for the KS and REITER solutions in the baseline economy as well as with $\gamma_s = 25, 50,$ and 100 .¹⁵ As micro uncertainty varies more, output volatility in the KS solutions increases by over 10% from baseline to $\gamma_s = 100$. By contrast, in the same range for γ_s , output volatility declines in the REITER solutions by over 10%.

Two distinct economic forces are at work. As emphasized by Bloom (2009) and Bloom et al. (2016), decreasing returns to scale imply convex input demands. Therefore, by Jensen’s inequality increased volatility of micro shocks leads to higher labor use in the short term and higher investment and capital accumulation in the medium term, a force termed the “Oi-Hartman-Abel” effect. Since micro uncertainty increases with negative aggregate productivity shocks, the Oi-Hartman-Abel effect tends to dampen output volatility. By contrast, a second force, the “real options effect” or the “wait-and-see effect,” works through the presence of lumpy capital adjustment costs. In the presence of higher micro volatility the option value of remaining inactive and responding optimally to the more dispersed shocks in future goes up. The result is lower rates of capital adjustment and lower investment. Because an investment freeze tends to misalign micro productivity and capital stocks, misallocation of capital inputs increases and output is pushed further down. Higher volatility is present during a negative aggregate productivity shock, working to amplify aggregate output volatility through this real options channel.

Although in general the relative strength of the Oi-Hartman-Abel effect and the real options effect is ambiguous, work based on lumpy capital adjustment and micro uncertainty shocks emphasizes that the real options effect tends to dominate in the short term for this class of models and standard calibrations (Bloom, 2009; Bloom et al., 2016; Bachmann and Bayer, 2013). While the projection-based KS method, without any reliance upon linearity with respect to aggregate shocks, captures the higher relative strength of the real options effect, linearized REITER does not. The quantitative strength of this channel apparently depends upon nonlinear dynamics with respect to aggregate shocks, present in the projection-based KS solution alone. There is of course no barrier to the implementation of higher-order perturbations in extended versions of the REITER solution. While such extensions might deliver a dominant real options effect and higher output volatility with uncertainty fluctuations, they would be more computationally costly than the linearized REITER solution, eroding its advantage in terms of computational speed and simplicity. Therefore, I recommend the use of projection-based solutions such as the KS method in the presence of potentially nonlinear aggregate dynamics.

¹⁵Interested readers can find a representative plot of simulated output, investment, labor, and consumption for the KS and REITER solutions for each of the micro volatility fluctuation extensions in Appendix Figure D.5.

6 Conclusion

This paper compares the KS, XPA, PARAM, REITER, and WINBERRY solution methods for a benchmark model of firm heterogeneity with lumpy capital adjustment and aggregate productivity shocks. The main qualitative implications of aggregate productivity shocks are the same for all the solution methods, and the moments of investment at the micro level, a crucial target for calibration, are virtually identical across methods. Within the set of projection methods relying on an approximation to the aggregate state space, the KS solution achieves the highest internal prediction accuracy, at the cost of substantial computational time requirements. Within the set of perturbation methods, the WINBERRY solution with distributional parameterization is slightly faster than the REITER method and delivers equivalent business cycle dynamics.

This paper also investigates the tradeoff between projection-based solution techniques and perturbation-based solution techniques. For larger shocks, or for models where nonlinearity with respect to aggregate shocks may matter such as with fluctuations in volatility, projection methods perform best. For contexts in which a researcher desires to scale up the aggregate complexity of the model but sidestep the curse of dimensionality, and for mechanisms which operate through an explicitly distributional channel such as through size-dependent distortions, the perturbation-based approaches are attractive.

In addition to the techniques analyzed in this paper, at least three paths forward for computational analysis of heterogeneous agents business cycle models seem quite promising. First, projection-based methods which sidestep the curse of dimensionality in some fashion might be used to study models which combine a rich aggregate state space with large shocks or highly nonlinear dynamics. Judd et al. (2012) proposes a simulation-based method for reducing the size of a projection grid which McKay (2016) applies to study an incomplete markets model with fluctuations in idiosyncratic risk and seven continuous aggregate state variables. Alternatively, Gordon (2011) uses sparse Smolyak projection grids to study the projection-based solution of heterogeneous agent models with the full cross-sectional distribution as a state variable. Second, perturbation-based methods which sidestep the curse of dimensionality can already be quickly scaled up to consider aggregate features like nominal rigidities together with micro heterogeneity. McKay and Reis (Forthcoming) applies the REITER method to study the impact of automatic fiscal stabilizers on volatility with nominal rigidities and incomplete markets, while Costain and Nakov (2011) and Reiter et al. (2013) apply the REITER approach to heterogeneous firm investment and pricing models with nominal rigidities. Perturbation-based solutions also lend themselves naturally to standard likelihood-based structural estimation exercises, as emphasized by McKay (2013) and Winberry (Forthcoming). Third, recent projects such as Achdou et al. (2015) emphasize continuous time formulations of models with micro heterogeneity. In this formulation, occasionally binding constraints often appear as tractable boundary conditions. Furthermore, transition matrices are typically quite sparse given small exogenous shock sizes within an instant, making solutions based

on perturbation with respect to aggregate shocks a particularly natural approach. The future looks bright for research into business cycle models with rich heterogeneity for firms and households.

References

- Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll (2015), “Heterogeneous Agent Models in Continuous Time.” Working paper.
- Algan, Yann, Olivier Allais, and Wouter J Den Haan (2008), “Solving Heterogeneous-Agent Models with Parameterized Cross-Sectional Distributions.” *Journal of Economic Dynamics and Control*, 32, 875–908.
- Algan, Yann, Olivier Allais, and Wouter J Den Haan (2010a), “Solving the Incomplete Markets Model with Aggregate Uncertainty Using Parameterized Cross-Sectional Distributions.” *Journal of Economic Dynamics and Control*, 34, 59–68.
- Algan, Yann, Olivier Allais, Wouter J Den Haan, and Pontus Rendahl (2010b), “Solving and Simulating Models with Heterogeneous Agents and Aggregate Uncertainty.” *Handbook of Computational Economics*.
- Arellano, Cristina, Yan Bai, and Patrick J. Kehoe (2016), “Financial Frictions and Fluctuations in Volatility.” Working paper.
- Aruoba, Borağan and Jesús Fernández-Villaverde (2015), “A Comparison of Programming Languages in Economics.” *Journal of Economic Dynamics and Control*.
- Bachmann, Rüdiger and Christian Bayer (2013), “Wait-and-See Business Cycles?” *Journal of Monetary Economics*, 60, 704–719.
- Bachmann, Rüdiger and Christian Bayer (2014), “Investment Dispersion and the Business Cycle.” *American Economic Review*, 104, 1392–1416.
- Bachmann, Rüdiger, Ricardo J Caballero, and Eduardo MRA Engel (2013), “Aggregate Implications of Lumpy Investment: New Evidence and a DSGE Model.” *American Economic Journal: Macroeconomics*, 5, 29–67.
- Bachmann, Rüdiger and Lin Ma (2015), “Lumpy Investment, Lumpy Inventories.” *Journal of Money, Credit and Banking*.
- Bloom, Nicholas (2009), “The Impact of Uncertainty Shocks.” *Econometrica*, 77, 623–685.
- Bloom, Nicholas, Max Floetotto, Nir Jaimovich, Itay Saporta-Eksten, and Stephen J Terry (2016), “Really Uncertain Business Cycles.” Working paper.
- Campbell, Jeffrey R (1998), “Entry, Exit, Embodied Technology, and Business Cycles.” *Review of Economic Dynamics*, 1, 371–408.
- Carroll, Christopher D (2006), “The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems.” *Economics Letters*, 91, 312–320.
- Chang, Yongsung and Sun-Bin Kim (2007), “Heterogeneity and Aggregation: Implications for Labor-Market Fluctuations.” *American Economic Review*, 97, 1939–1956.

- Cooper, Russell W and John C Haltiwanger (2006), “On the Nature of Capital Adjustment Costs.” *The Review of Economic Studies*, 73, 611–633.
- Costain, James and Anton Nakov (2011), “Distributional Dynamics under Smoothly State-Dependent Pricing.” *Journal of Monetary Economics*, 58, 646–665.
- Den Haan, Wouter J (2010a), “Assessing the Accuracy of the Aggregate Law of Motion in Models with Heterogeneous Agents.” *Journal of Economic Dynamics and Control*, 34, 79–99.
- Den Haan, Wouter J (2010b), “Comparison of Solutions to the Incomplete Markets Model with Aggregate Uncertainty.” *Journal of Economic Dynamics and Control*, 34, 4–27.
- Den Haan, Wouter J and Pontus Rendahl (2010), “Solving the Incomplete Markets Model with Aggregate Uncertainty using Explicit Aggregation.” *Journal of Economic Dynamics and Control*, 34, 69–78.
- Dotsey, Michael, Robert G King, and Alexander L Wolman (1999), “State-Dependent Pricing and the General Equilibrium Dynamics of Money and Output.” *Quarterly Journal of Economics*, 655–690.
- Gordon, Grey (2011), “Computing Dynamic Heterogeneous-Agent Economies: Tracking the Distribution.” Working paper.
- Gourio, François and Anil K Kashyap (2007), “Investment Spikes: New Facts and a General Equilibrium Exploration.” *Journal of Monetary Economics*, 54, 1–22.
- Judd, Kenneth L (1998), *Numerical Methods in Economics*. MIT press.
- Judd, Kenneth L, Lilia Maliar, and Serguei Maliar (2012), “Merging Simulation and Projection Approaches to Solve High-Dimensional Problems.” NBER working paper 18501.
- Kehrig, Matthias and Nicolas Vincent (2013), “Financial Frictions and Investment Dynamics in Multi-Plant Firms.” *US Census Bureau Center for Economic Studies Paper No. CES-WP-56*.
- Khan, Aubhik and Julia K Thomas (2003), “Nonconvex Factor Adjustments in Equilibrium Business Cycle Models: Do Nonlinearities Matter?” *Journal of Monetary Economics*, 50, 331–360.
- Khan, Aubhik and Julia K Thomas (2008), “Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics.” *Econometrica*, 76, 395–436.
- Khan, Aubhik and Julia K Thomas (2013), “Credit Shocks and Aggregate Fluctuations in an Economy with Production Heterogeneity.” *Journal of Political Economy*, 121, 1055–1107.
- King, Robert G and Julia K Thomas (2006), “Partial Adjustment without Apology.” *International Economic Review*, 47, 779–809.
- Klenow, Peter J and Jonathan L Willis (2007), “Sticky Information and Sticky Prices.” *Journal of Monetary Economics*, 54, 79–99.
- Knotek, Edward (2010), “A Tale of Two Rigidities: Sticky Prices in a Sticky-Information Environment.” *Journal of Money, Credit and Banking*, 42, 1543–1564.
- Knotek, Edward and Stephen J Terry (2008), “Alternative Methods of Solving State-Dependent Pricing Models.” Working paper.

- Koop, Gary, M Hashem Pesaran, and Simon M Potter (1996), “Impulse Response Analysis in Nonlinear Multivariate Models.” *Journal of Econometrics*, 74, 119–147.
- Krusell, Per and Anthony A Smith, Jr (1998), “Income and Wealth Heterogeneity in the Macroeconomy.” *Journal of Political Economy*, 106, 867–896.
- Maliar, Lilia, Serguei Maliar, and Fernando Valli (2010), “Solving the Incomplete Markets Model with Aggregate Uncertainty Using the Krusell–Smith Algorithm.” *Journal of Economic Dynamics and Control*, 34, 42–49.
- McGrattan, Ellen R (1996), “Solving the Stochastic Growth Model with a Finite Element Method.” *Journal of Economic Dynamics and Control*, 20, 19–42.
- McKay, Alisdair (2013), “Idiosyncratic Risk, Insurance, and Aggregate Consumption Dynamics: A Likelihood Perspective.” Working paper.
- McKay, Alisdair (2016), “Time-Varying Idiosyncratic Risk and Aggregate Consumption Dynamics.” Working paper.
- McKay, Alisdair and Ricardo Reis (Forthcoming), “The Role of Automatic Stabilizers in the US Business Cycle.” *Econometrica*.
- Meier, Matthias (2016), “Time to Build and the Business Cycle.” Working Paper.
- Miao, Jianjun (2006), “Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks.” *Journal of Economic Theory*, 128, 274–298.
- Reiter, Michael (2009), “Solving Heterogeneous-Agent Models by Projection and Perturbation.” *Journal of Economic Dynamics and Control*, 33, 649–665.
- Reiter, Michael (2010a), “Approximate and Almost-Exact Aggregation in Dynamic Stochastic Heterogeneous-Agent Models.” Working paper.
- Reiter, Michael (2010b), “Nonlinear Solution of Heterogeneous Agent Models by Approximate Aggregation.” Working paper.
- Reiter, Michael (2010c), “Solving the Incomplete Markets Model with Aggregate Uncertainty by Backward Induction.” *Journal of Economic Dynamics and Control*, 34, 28–35.
- Reiter, Michael, Tommy Sveen, and Lutz Weinke (2013), “Lumpy Investment and the Monetary Transmission Mechanism.” *Journal of Monetary Economics*, 60, 821–834.
- Ríos-Rull, José-Víctor (1999), “Computation of Equilibria in Heterogeneous-Agent Models.” In *Computational Methods for the Study of Dynamic Economies* (Ramon Marimon and Andrew Scott, eds.), 238–264, Oxford University Press.
- Senga, Tatsuro (2015), “A New Look at Uncertainty Shocks: Imperfect Information and Misallocation.” Working paper.
- Sims, Christopher A (2002), “Solving Linear Rational Expectations Models.” *Computational Economics*, 20, 1–20.
- Sunakawa, Takeki (2012), “Applying the Explicit Aggregation Algorithm to Discrete Choice Economies: with an Application to Estimating the Aggregate Technology Shock Process.” Working paper.

- Takahashi, Shuhei (2014), “Heterogeneity and Aggregation: Implications for Labor-Market Fluctuations: Comment.” *American Economic Review*, 104, 1446–1460.
- Tauchen, George (1986), “Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions.” *Economics Letters*, 20, 177–181.
- Thomas, Julia K (2002), “Is Lumpy Investment Relevant for the Business Cycle?” *Journal of Political Economy*, 110, 508–534.
- Vavra, Joseph (2014), “Inflation Dynamics and Time-Varying Volatility: New Evidence and an Ss Interpretation.” *Quarterly Journal of Economics*, 129, 215–258.
- Winberry, Thomas (2015), “Lumpy Investment, Business Cycles, and Stimulus Policy.” Working paper.
- Winberry, Thomas (Forthcoming), “A Toolbox for Solving and Estimating Heterogeneous Agent Macro Models.” *Quantitative Economics*. Work in progress.
- Young, Eric R (2010), “Solving the Incomplete Markets Model with Aggregate Uncertainty using the Krusell–Smith Algorithm and Non-Stochastic Simulations.” *Journal of Economic Dynamics and Control*, 34, 36–41.
- Zwick, Eric and James Mahon (Forthcoming), “Tax Policy and Heterogeneous Investment Behavior.” *American Economic Review*.