## Two lemmas on Matlis Duality

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Matlis duality is an important tool in commutative algebra. It states:

**Theorem 0.1** ([Hoc, Theorem 5.1, Theorem 5.4]). Let  $(R, \mathfrak{m}, k)$  be a local ring, and let  $E = E_R(k)$  be the injective hull of the residue field of R. If R is complete, then the functor,  $(\cdot)^{\vee}$ , defined by

$$M^{\vee} := \operatorname{Hom}_{R}(M, E),$$

induces an equivalence of categories from Noetherian R-modules to Artinian R-modules and vice-versa, and further  $(M^{\vee})^{\vee} = M$  for all Noetherian/Artinian M.

Even if R is not complete, then  $\widehat{R} = \operatorname{Hom}_R(E, E)$ .

If you're not familiar with Matlis duality, then the first five chapters of [Hoc] are a great reference. The following two lemmas are used often in the older literature on test ideals, e.g. [Tak06]. Indeed, Takagi seems to refer to the first lemma itself as "Matlis duality." I couldn't find a reference for these results, so I'm putting them here:

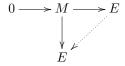
**Lemma 0.2** (c.f. [BH98], exercise 3.2.15c). Let  $(R, \mathfrak{m}, k)$  be a complete local ring, and let  $E = E_R(k)$ . If  $M \subseteq E$  is a submodule, then

$$\operatorname{Ann}_E\operatorname{Ann}_RM=M.$$

Similarly, if  $I \subseteq R$  is an ideal, then

$$\operatorname{Ann}_R\operatorname{Ann}_E I=I$$

*Proof.* I claim that  $(M^{\vee})^{\vee} = \operatorname{Ann}_{E} \operatorname{Ann}_{E} M$ . Then the result follows by Matlis duality. First, note that by the universal property of injective modules, for any map  $M \to E$ , we can fill in the diagram:



In other words,  $\operatorname{Hom}(E, E) \to \operatorname{Hom}(M, E)$ . But, since R is complete, any map  $\operatorname{Hom}(E, E)$  is just given by multiplication by an element of R. Thus  $M^{\vee}$  is a quotient of R. Now we ask: what is the kernel of the quotient? The kernel of this quotient is the set of maps  $E \to E$  that restrict to 0 on M. Since maps  $E \to E$  are given by multiplication by elements of R, we see that the kernel is  $\operatorname{Ann}_R(M)$ . Thus  $M^{\vee} = R/\operatorname{Ann}_R(M)$ 

By the "first isomorphism theorem" (or, if you like, the universal property of quotient modules), an element of  $\operatorname{Hom}_R(R/\operatorname{Ann}_R(M), E)$  is the same as a map  $R \to E$  that restricts to 0 on  $\operatorname{Ann}_R(M)$ . A map  $R \to E$  is completely determined by where we send  $1 \in R$ , so we see that  $(M^{\vee})^{\vee} = \operatorname{Hom}_R(R/\operatorname{Ann}_R(M), E) = \operatorname{Ann}_E \operatorname{Ann}_R M$ .

The second statement is proved in the same way.

**Lemma 0.3.** Notation as in lemma 0.2. Let  $M \subseteq E$  be a submodule and  $I \subseteq R$  be an ideal. Then

$$(0:(M:I)_E)_R = I \cdot (0:M)_R$$

In other words.

$$\operatorname{Ann}_R(M:I)_E = I \cdot \operatorname{Ann}_R M$$

*Proof.* First I'll show that  $(M:I)_E = \operatorname{Ann}_E(I \cdot \operatorname{Ann}_R(M))$ . To do so, we start by showing the left-hand side is smaller than the right-hand side. So let  $x \in (M:I)_E$ . Then  $I \cdot x \subseteq M = \operatorname{Ann}_E \operatorname{Ann}_R M$ . In other words,  $I \operatorname{Ann}_R M \cdot x = 0$ , as desired. To get the opposite inclusion, let  $y \in \operatorname{Ann}_E(I \cdot \operatorname{Ann}_R(M))$ . By definition,  $\operatorname{Ann}_R(M) \cdot (Iy) = 0$ . In other words,  $Iy \subseteq \operatorname{Ann}_E \operatorname{Ann}_R M = M$ , as desired.

From the above, it follows that

$$\operatorname{Ann}_R(M:I)_E = \operatorname{Ann}_R \operatorname{Ann}_E (I \cdot \operatorname{Ann}_R(M)) = I \cdot \operatorname{Ann}_R(M).$$

## References

- [BH98] W. Bruns and H.J. Herzog. *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.
- [Hoc] Mel Hochster. Local cohomology. http://www.math.lsa.umich.edu/~hochster/615W11/loc.pdf. Notes from Math 615, Winter 2011. Accessed 2017-05-12.
- [Tak06] Shunsuke Takagi. Formulas for multiplier ideals on singular varieties. Amer. J. Math., 128(6):1345–1362, 2006.