Two lemmas on Matlis Duality

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Matlis duality is an important tool in commutative algebra. It states:

Theorem 0.1 ([Hoc, Theorem 5.1, Theorem 5.4]). *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $E = E_R(k)$ be the injective hull of the residue field of R. If R is complete, then the functor, $(\cdot)^\vee$, defined by*

$$M^\vee := \operatorname{Hom}_R(M, E),$$

induces an equivalence of categories from Noetherian R-modules to Artinian R-modules and vice-versa, and further $(M^\vee)^\vee = M$ for all Noetherian/Artinian M.

Even if R is not complete, then $\widehat{R} = \operatorname{Hom}_R(E, E)$.

If you're not familiar with Matlis duality, then the first five chapters of [Hoc] are a great reference. The following two lemmas are used often in the older literature on test ideals, e.g. [Tak06]. Indeed, Takagi seems to refer to the first lemma itself as "Matlis duality." I couldn't find a reference for these results, so I'm putting them here:

Lemma 0.2 (c.f. [BH98], exercise 3.2.15c). *Let $(R, \mathfrak{m}, k)$ be a complete local ring, and let $E = E_R(k)$. If $M \subseteq E$ is a submodule, then*

$$\operatorname{Ann}_E \operatorname{Ann}_R M = M.$$

Similarly, if $I \subseteq R$ is an ideal, then

$$\operatorname{Ann}_R \operatorname{Ann}_E I = I$$

Proof. I claim that $(M^\vee)^\vee = \operatorname{Ann}_E \operatorname{Ann}_R M$. Then the result follows by Matlis duality. First, note that by the universal property of injective modules, for any map $M \to E$, we can fill in the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & E \\ & & \downarrow & \swarrow & \\ & & E & & \end{array}$$

In other words, $\operatorname{Hom}(E, E) \twoheadrightarrow \operatorname{Hom}(M, E)$. But, since R is complete, any map $\operatorname{Hom}(E, E)$ is just given by multiplication by an element of R. Thus $M^\vee$ is a quotient of R. Now we ask: what is the kernel of the quotient? The kernel of this quotient is the set of maps $E \to E$ that restrict to 0 on M. Since maps $E \to E$ are given by multiplication by elements of R, we see that the kernel is $\operatorname{Ann}_R(M)$. Thus $M^\vee = R/\operatorname{Ann}_R(M)$

By the "first isomorphism thoerem" (or, if you like, the universal property of quotient modules), an element of $\operatorname{Hom}_R(R/\operatorname{Ann}_R(M), E)$ is the same as a map $R \to E$ that restricts to 0 on $\operatorname{Ann}_R(M)$. A map $R \to E$ is completely determined by where we send $1 \in R$, so we see that $(M^\vee)^\vee = \operatorname{Hom}_R(R/\operatorname{Ann}_R(M), E) = \operatorname{Ann}_E \operatorname{Ann}_R M$.

The second statement is proved in the same way. ☐

Lemma 0.3. *Notation as in lemma 0.2. Let $M \subseteq E$ be a submodule and $I \subseteq R$ be an ideal. Then*

$$(0 : (M : I)_E)_R = I \cdot (0 : M)_R$$

In other words,

$$\operatorname{Ann}_R (M : I)_E = I \cdot \operatorname{Ann}_R M$$

Proof. First I'll show that $(M : I)_E = \operatorname{Ann}_E (I \cdot \operatorname{Ann}_R(M))$. To do so, we start by showing the left-hand side is smaller than the right-hand side. So let $x \in (M : I)_E$. Then $I \cdot x \subseteq M = \operatorname{Ann}_E \operatorname{Ann}_R M$. In other words, $I \operatorname{Ann}_R M \cdot x = 0$, as desired. To get the opposite inclusion, let $y \in \operatorname{Ann}_E (I \cdot \operatorname{Ann}_R(M))$. By definition, $\operatorname{Ann}_R(M) \cdot (Iy) = 0$. In other words, $Iy \subseteq \operatorname{Ann}_E \operatorname{Ann}_R M = M$, as desired.

From the above, it follows that

$$\operatorname{Ann}_R(M : I)_E = \operatorname{Ann}_R \operatorname{Ann}_E (I \cdot \operatorname{Ann}_R(M)) = I \cdot \operatorname{Ann}_R(M).$$

□

References

[BH98] W. Bruns and H.J. Herzog. *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[Hoc] Mel Hochster. Local cohomology. `http://www.math.lsa.umich.edu/~hochster/615W11/loc.pdf`. Notes from Math 615, Winter 2011. Accessed 2017-05-12.

[Tak06] Shunsuke Takagi. Formulas for multiplier ideals on singular varieties. *Amer. J. Math.*, 128(6):1345–1362, 2006.