Subadditivity formulas for test ideals

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Symbolic Powers

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The ELS proof uses multiplier ideals $\mathscr{J}(R, \mathfrak{a})$

Hara ’05 runs through the same argument using test ideals $\tau(R, \mathfrak{a})$

Test ideals

Method of proof:

$$\mathfrak{p}^{(nd)} \subseteq \left|\left|\tau(R, \mathfrak{p}^{(nd)})\right|\right| \subseteq \left|\left|\tau(R, \mathfrak{p}^{(d)})\right|\right|^{n} \subseteq \mathfrak{p}^{n}$$

Subadditivity is the second inequality. Only holds for regular rings!

Question: What can we say for more general R?

Char p preliminaries

R Noetherian, characteristic p

$$F^e : R \xrightarrow{F} R \xrightarrow{F} \cdots \xrightarrow{F} R$$

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$r \cdot F^e_* x = F^e_* r^{p^e} x$.

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R-linear maps $\varphi : F^e_* R \to R \Leftrightarrow$ group homomorphisms $\varphi : R \to R$ satisfying $\varphi(r^{p^e} x) = r\varphi(x)$

Char p preliminaries (Test ideals)

Definition (Hochster-Huneke '90, Hara-Takagi '04, Schwede '10):

$\tau(R) =$ unique smallest nonzero ideal satisfying $\varphi(F^e_* J) \subseteq J$ for all e, $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$

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Pairs setting (Hara-Yoshida '03/Hara-Takagi '04): For $\mathfrak{a}_i \subseteq R$, $t_i \in \mathbb{R}$, $\tau(R, \prod_i \mathfrak{a}_i^{t_i})$ is smallest nonzero J satisfying

$$\varphi\left(F^e_* \prod \mathfrak{a}_i^{\lceil t_i p^e \rceil} J\right) \subseteq J$$

($\mathfrak{a}_i^{t_i}$ is just suggestive notation)

Subadditivity

If R is a regular k-algebra, then

$$\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

Proof.

Künneth formula:

$$\tau(R \otimes_k R, \mathfrak{a} \otimes_k R \cdot R \otimes_k \mathfrak{b}) = \tau(R, \mathfrak{a}) \otimes_k \tau(R, \mathfrak{b})$$

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Restriction formula: if $x \in R$ a regular element, then

$$\tau(R/x, (\mathfrak{a}R/x)^s) \subseteq \tau(R, \mathfrak{a}^s)R/x$$

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Proof.

Let $\mu : R \otimes R \to R$ be multiplication, $x \otimes y \mapsto xy$. Let $\Delta = \ker \mu$.

Then Δ is generated by a regular sequence since R is regular

Putting it together:

$$\begin{aligned} \tau(\mathfrak{a}^s\mathfrak{b}^t) &= \tau\left((R \otimes R)/\Delta, \mathfrak{a} \otimes_k R^s \cdot R \otimes_k \mathfrak{b}^t \bmod \Delta\right) \\ &\subseteq \tau(\mathfrak{a} \otimes_k R \cdot R \otimes_k \mathfrak{b}) \bmod \Delta = \tau(\mathfrak{a}^s)\tau(\mathfrak{b}^t) \end{aligned}$$

□

Subadditivity

Restriction formula only works for regular rings! Possible workarounds:

- Abandon old proof (Takagi '07):

$$\mathrm{Jac}(R)\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

- Use cartier algebras

Cartier algebra: $\mathcal{C} = \bigoplus_e \mathcal{C}_e \subseteq \bigoplus_e \mathrm{Hom}_R(F^e_* R, R)$

$\tau(R, \mathfrak{a}^s, \mathcal{C})$ =smallest J st $\varphi\left(F^e_* \mathfrak{a}^{\lceil p^e s \rceil} J\right) \subseteq J$ for all $\varphi \in \mathcal{C}_e$

My cartier algebra

$\varphi \in \mathcal{C}_{\mathrm{compat}}$ if:

$$\begin{array}{ccc} F^e_*(R \otimes_k R) & \dashrightarrow & R \otimes_k R \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \mu} \\ F^e_* R & \xrightarrow{\ \varphi\ } & R \end{array}$$

Then we get this inclusion for free:

$$\tau(R, \mathfrak{a}^s \mathfrak{b}^t, \mathcal{C}_{\mathrm{compat}}) \subseteq \tau(R \otimes R, (\mathfrak{a} \otimes R)^s (R \otimes \mathfrak{b})^t) \bmod \Delta$$

So we have subadditivity:

$$\tau(R, \mathfrak{a}^s \mathfrak{b}^t, \mathcal{C}_{\mathrm{compat}}) \subseteq \tau(R, \mathfrak{a}^s) \tau(R, \mathfrak{b}^t)$$

Questions

For this to be a useful formula, we must answer two questions:

- Is it stronger than Takagi's formula? (Work in progress)

$$\mathrm{Jac}(R)\tau(R, \mathfrak{a}^s\mathfrak{b}^t) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

$$\text{vs.}$$

$$\tau(R, \mathfrak{a}^s\mathfrak{b}^t, \mathcal{C}_{\mathrm{compat}}) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

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$$\tau(R, \mathfrak{a}^s\mathfrak{b}^t, \mathcal{C}_{\mathrm{compat}}) \subseteq \tau(R, \mathfrak{a}^s)\tau(R, \mathfrak{b}^t)$$

- How do we compute $\tau(R, \mathfrak{a}^s\mathfrak{b}^t, \mathcal{C}_{\mathrm{compat}})$? I.e. how do we compute $\mathcal{C}_{\mathrm{compat}}$?

Normal subalgebras of $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ generated by monomials

Correspondence: $k[\mathbf{x}^{a_1}, \ldots, \mathbf{x}^{a_m}]$, $a_i \in \mathbb{Z}^n \Leftrightarrow$ cone in $\mathbb{R}^n$ generated by $a_1, \ldots, a_m$

Monomials in the ring $\Leftrightarrow$ lattice points in the cone

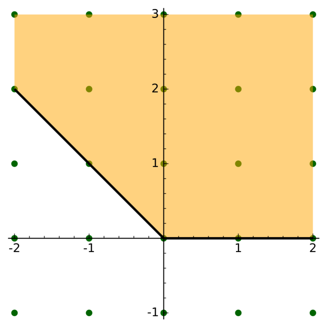

Figure 1: $R = k[x^{-1}y, x]$

Toric examples

Translate cone C to get cone P. Points in $\frac{1}{p^e}\mathbb{Z} \cap P$ correspond to maps $F^e_* R \to R$. These maps generate $\mathrm{Hom}(F^e_* R, R)$ as k-vector space.

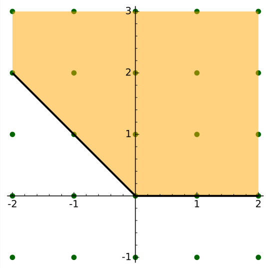

Figure 2:
$R = \mathbb{Z}/3[x^{-1}y, x]$

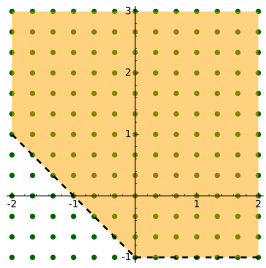

Figure 3:
$\mathrm{Hom}(F^1_* R, R)$

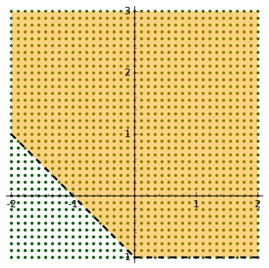

Figure 4:
$\mathrm{Hom}(F^2_* R, R)$

Computing $\mathcal{C}_{\text{compat}}$

Lemma

Let $d \in P$. The map corresponding to d is a generator of $\mathcal{C}_{compat}$ if and only if $P \cap (d - P)$ contains a representative of each equivalence class of $\frac{1}{p^e}\mathbb{Z}/\mathbb{Z}$

E.g.: $R = \mathbb{Z}/5[x, y, z]/(xy - z^2) = \mathbb{Z}/5[y, xy, x^2y]$

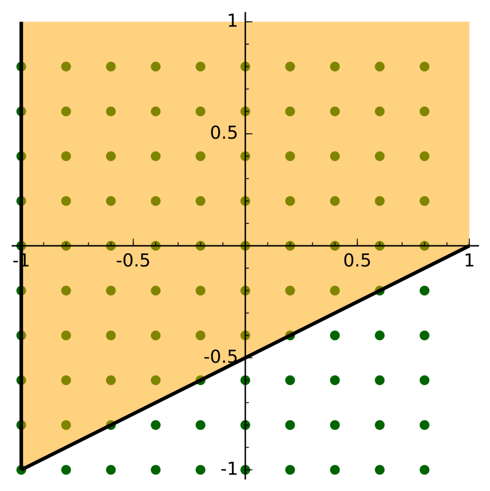

Figure 5: The cone P. Lattice points $\Leftrightarrow$ maps $F^1_* R \to R$

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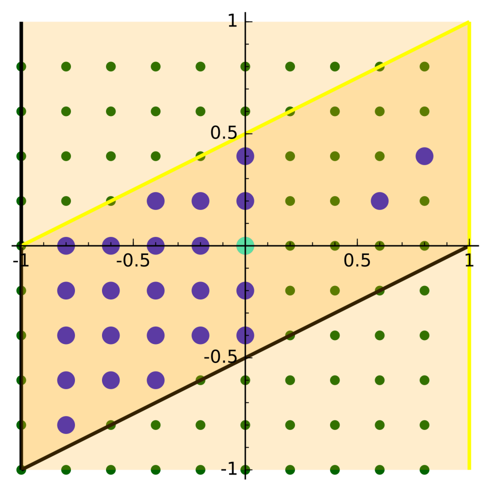

Figure 6: $P \cap ((0, 0) - P)$ contains a representative of each equivalence class of $\frac{1}{5}\mathbb{Z}/\mathbb{Z}$

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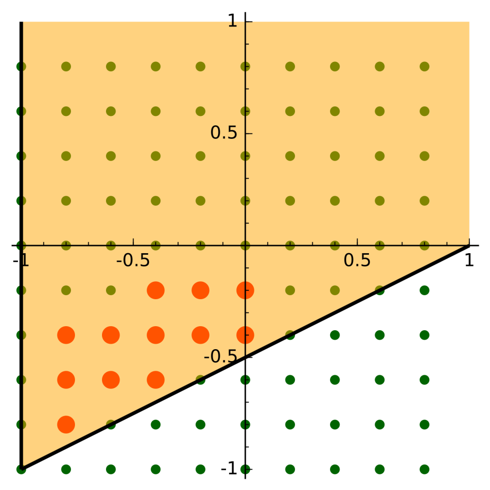

Figure 7: Lattice points corresponding to maps that don't lift

Test ideal comparison

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- E.g. $\tau(R, (1)^1) = \tau(R, (1)^1, \mathcal{C}_{\mathrm{compat}}) = R$
- Thus $\mathrm{Jac}(R)\tau(R, (1)^1) = (x, y, z) \subsetneq \tau(R, (1)^1, \mathcal{C}_{\mathrm{compat}})$