Final Exam Review

Differential equations

1. Salt water at a concentration of 2 kg/L flows into a tank at a rate of 6 L/min. Salt water flows out of this tank at a rate of 4 L/min. Assuming the tank starts with 10 Liters of salt water, write a differential equation describing the amount of salt in the tank after t minutes.

Solution: $\dfrac{dy}{dx} = 12 - \dfrac{4y}{10 + 2t}$

2. Salt water at a concentration of 4 kg/L flows into a tank at a rate of 2 L/min. Salt water flows out of this tank at a rate of 2 L/min. Assuming the tank starts with 20 Liters of salt water, write a differential equation describing the amount of salt in the tank after t minutes.

Solution: $\dfrac{dy}{dx} = 8 - \dfrac{y}{10}$

3. Solve the differential equation: $\dfrac{dy}{dx} + xy = x$, $y(1) = 2$.

Solution: Multiply by the integrating factor $e^{\int x\,dx}$ to solve. $y = 1 + e^{(1-x^2)/2}$

4. Solve the differential equation: $\dfrac{dy}{dx} = 2y$, $y(0) = 3$

Solution: $y = 3e^{2x}$

5. Solve the differential equation: $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2$, $y(1) = 4$.

Solution: $y = \dfrac{x^3}{2} + \dfrac{3}{2}x$

Integrals

6. $\displaystyle\int e^{\cos x} \sin x\,dx$

Solution: Use the substitution $u = \cos x$, $du = -\sin x\,dx$.

7. $\displaystyle\int x \cot^2 x\,dx$

Solution: Use the trig identity $\cot^2 x = 1 - \csc^2 x$. To find the integral of $\int x \csc^2 x \, dx$, use integration by parts with $u = x$, $dv = \csc^2 x \, dx$.

8. $\displaystyle\int \frac{dx}{\sqrt{16 + 4x - 2x^2}}$

Solution: Start by completing the square. The answer is $\dfrac{1}{\sqrt{2}} \arcsin\left(\dfrac{x-1}{3}\right)$.

9. $\displaystyle\int \sin^3(3x)\, dx$

Solution: $\sin^3(3x) = \sin^2(3x)\sin(3x) = (1 - \cos^2(3x))\sin(3x)$. From there, use the substitution $u = \cos(3x)$, $du = -3\sin(3x)$.

10. $\displaystyle\int \frac{\tan x}{\ln|\cos x|}\, dx$

Solution: Use the substitution $u = \ln|\cos x|$, $du = -\tan x$.

11. $\displaystyle\int \frac{x}{\sqrt{x+5}}\, dx$

Solution: Use the substitution $u = \sqrt{x+5}$, so that $u^2 - 5 = x$ and $2\,du = dx$.

12. $\displaystyle\int \frac{x^3}{1-x^2}\, dx$

Solution: Use polynomial long division to write $\dfrac{x^3}{1-x^2} = -x + \dfrac{x}{1-x^2}$. Use the substitution $u = 1 - x^2$ to find $\displaystyle\int \frac{x}{1-x^2}\, dx$. Or you can use a partial fraction decomposition.

Infinite Series

Determine whether the following series converge or diverge. Try and see if you can guess what the answer is going to be before using a convergence test to confirm your answer.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

Solution: Diverges. Use the ratio test

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{n^6+1}$

Solution: Diverges. Use the limit comparison test, and compare to $1/n$

15. $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^n$

Solution: Converges. This is a geometric series.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[6]{n^7}}$

Solution: Converges. This is a p-series with $p = 7/6$.

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n+3}$

Solution: Diverges. This is because of the nth term test.

18. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} e^{-n}$

Solution: Converges. This is a geometric series: $e^{-n} = \left(\frac{1}{e}\right)^n$.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n+1}{n^3+\ln n}$

Solution: Diverges. Use the limit comparison test. Compare to $1/n$.

20. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+5}$

Solution: Converges. Use the limit comparison test. Compare to $1/n^{3/2}$.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 3^n}{(n+1)!}$

Solution: Converges. Use the ratio test.

Power series

Find the convergence sets of the following series

22. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^3+1}$

Solution: $-1 \leqslant x \leqslant 1$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+1} x^n}{2n+3}$

Solution: $-\dfrac{1}{2} < x \leqslant \dfrac{1}{2}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{n+1}$

Solution: $3 < x \leqslant 5$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^{3n}}{(3n)!}$

Solution: $-\infty < x < \infty$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{n!(x+1)^n}{3^n}$

Solution: $x = -1$

Taylor Series

Find the first four terms of the following Taylor series:

27. Taylor series of $\sin^2 x$ centered at $x = 0$

Solution: x^2 (the other terms are 0).

28. Taylor series of e^x centered at $x = 2$

Solution: $e^2 + e^2(x-2) + \dfrac{e^2(x-2)^2}{2} + \dfrac{e^2(x-2)^3}{6}$

29. Taylor series of $\sin x + \cos x$ centered at $x = \dfrac{\pi}{2}$

Solution: $1 - (x - \dfrac{\pi}{2}) - \dfrac{(x-\frac{\pi}{2})^2}{2} + \dfrac{(x-\frac{\pi}{2})^3}{6}$

30. Taylor series of $e^{-x} - 1 + x$ centered at $x = 0$.

Solution: $\dfrac{x^2}{2} - \dfrac{x^3}{6}$

31. Taylor series of $\dfrac{1}{1-x^3}$ centered at $x = 0$.

Solution: $1 + x^3$