Math 1220-003, Summer 2018

Exam 1

Please write your name on the front and back of the exam. Remember to turn off your phone before starting this exam. Show all of your work for full credit. You may not use any notes or calculators during this exam.

Name: Solutions

UID: _____

1. (16 points) Determine whether the each of following statements is true or false. If true, write "True." If false, write "False."

(a) $\arctan\left(\tan\left(\frac{3\pi}{4}\right)\right) = \frac{3\pi}{4}$

False

(b) $\tan\left(\arctan\left(1/\sqrt{2}\right)\right) = 1/\sqrt{2}$

True

(c) $\sin\left(1/\sin\left(\frac{3\pi}{4}\right)\right) = \frac{3\pi}{4}$

False

(d) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

False

(e) $\int \tan x \sec^2 x \, dx = \frac{1}{2}\tan^2 x + C$

True

(f) $\frac{d}{dx}e^{5x} = e^{5x}$

False

(g) $\int xe^x \, dx = \frac{x^2}{2}e^x + C$

False

(h) $\frac{d}{dx}2^x = x2^{x-1}$

False

2. A tank of salt water starts with 3kg of salt dissolved in 9L of water. Salt water flows into the tank at a concentration of 2kg/L, at a rate of 3L/min. At the same time, water is flowing out of the tank at a rate of 3L/min. Let $y(t)$ denote the amount of salt in the tank after t minutes. Answer the following:

(a) (2 points) What is the rate (in kg/min) of **salt** flowing **into** the tank at time t?

$\frac{2kg}{L} \cdot \frac{3L}{min} = 6\ kg/min$

(b) (2 points) What is the rate (in kg/min) of **salt** flowing **out of** the tank at time t?

$\frac{y(t)\ kg}{9L} \cdot \frac{3L}{min} = \frac{y(t)}{3}\ kg/min$

(c) (10 points) How many kilograms of salt there in the tank after 30 minutes? Start by setting up a differential equation and solving it for $y(t)$.

$\frac{dy}{dt} = 6 - \frac{y}{3} \Rightarrow \frac{dy}{dt} + \frac{y}{3} = 6$

Integrating factor: $e^{\int \frac{1}{3} dt} = e^{t/3}$

$\Rightarrow \frac{d}{dt}\left[y \cdot e^{t/3}\right] = 6e^{t/3} \Rightarrow ye^{t/3} = 18e^{t/3} + C$

$y(0) = 3 \Rightarrow 3 \cdot 1 = 18 \cdot 1 + C \Rightarrow C = -15$

So, $ye^{t/3} = 18e^{t/3} - 15$. Multiply both sides by $e^{-t/3}$:

$y = 18 - 15e^{-t/3}$

$\Rightarrow \boxed{y(30) = 18 - 15e^{-10}}$

3. (10 points) A laboratory starts with 20 grams of certain radioactive substance. After 10 years, it is observed that only 18 grams of the substance are left. What is the half-life of this substance?

$A = A_0 e^{kt}$

$18 = 20 e^{k \cdot 10} \Rightarrow 0.9 = e^{k \cdot 10}$

$\Rightarrow \ln(0.9) = k \cdot 10$

$\Rightarrow k = \ln(0.9)/10$

$\frac{1}{2} = e^{k t_{half}}$ ← half-life

$\Rightarrow \ln(1/2) = k \cdot t_{half}$

$\Rightarrow t_{half} = \frac{\ln(1/2)}{k} = \frac{\ln(1/2)}{\ln(0.9)/10} = \boxed{\frac{10 \ln(1/2)}{\ln(0.9)}}$

4. Find each of the following derivatives:

(a) (10 points) $\frac{d}{dx}\log_3\left(x^{2x}\right)$

$$\frac{d}{dx}\frac{\ln(x^{2x})}{\ln(3)} = \frac{d}{dx}\frac{2x\ln(x)}{\ln(3)} = \frac{2}{\ln(3)}\frac{d}{dx}x\ln(x)$$

$$\frac{d}{dx}\left(x\ln(x)\right) = 1\cdot\ln(x) + x\cdot\frac{1}{x} = \ln(x)+1$$

$$\Rightarrow \boxed{\frac{2}{\ln(3)}\left[\ln(x)+1\right]}$$

(b) (10 points) $\frac{d}{dx}\frac{(2x+1)^{3/2}}{(\sqrt{x}+1)^4}$

Logarithmic differentiation:

Set $y = \frac{(2x+1)^{3/2}}{(\sqrt{x}+1)^4} \Rightarrow \ln(y) = \frac{3}{2}\ln(2x+1) - 4\ln(\sqrt{x}+1)$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{2}\cdot\frac{1}{2x+1}\cdot\frac{d}{dx}(2x+1) - 4\cdot\frac{1}{\sqrt{x}+1}\frac{d}{dx}(\sqrt{x}+1)$$

$$= \frac{3}{2x+1} - \frac{4}{\sqrt{x}+1}\cdot\frac{1}{2\sqrt{x}} = \frac{3}{2x+1} - \frac{2}{\sqrt{x}+x}$$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{(2x+1)^{3/2}}{(\sqrt{x}+1)^4}\left(\frac{3}{2x+1} - \frac{2}{\sqrt{x}+x}\right)}$$

5. Find the following integrals:

(a) (10 points) $\int x \sin x \, dx$

Integration by parts:

$u = x \quad dv = \sin x \, dx$

$du = 1 \, dx \quad v = -\cos x$

$\Rightarrow \int x \sin x \, dx = -x\cos x - \int -\cos x \, dx$

$= -x\cos x + \underbrace{\int \cos x \, dx}_{\sin x + C}$

$$\boxed{-x\cos x + \sin x + C}$$

(b) (10 points) $\int \sin^4 x \, dx$

$$\int \sin^2 x \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x \, dx$$

$$\int \cos^2 2x \, dx = \int \frac{1+\cos(4x)}{2} \, dx = \frac{x}{2} + \frac{1}{8}\sin(4x)$$

$$\Rightarrow \boxed{\frac{1}{4}\left(x - \sin(2x) + \frac{x}{2} + \frac{1}{8}\sin(4x)\right) + C}$$

(c) (10 points) $\int x \ln x \, dx$

Integration by parts

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$\Rightarrow \frac{1}{2}\ln(x)\cdot x^2 - \int \frac{1}{2}x^2 \frac{1}{x}\,dx = \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx$$

$$= \boxed{\frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C}$$

(d) (10 points) $\int \frac{1}{x^2 - 4x + 20}\,dx$

$$x^2 - 4x + 20 = x^2 - 4x + 4 - 4 + 20 = (x-2)^2 + 16$$

$$\int \frac{1}{(x-2)^2 + 4^2}\,dx = \boxed{\frac{1}{4}\tan^{-1}\left(\frac{x-2}{4}\right) + C}$$

Name: _____

Page	Points	Score
2	16	
3	14	
4	10	
5	20	
6	20	
7	20	
Total:	100	