Math 1220-003, Summer 2018

Final Exam Review

1. Find the following limits:

   (a) $\displaystyle\lim_{x\to 0^+} \frac{\cot x}{\sqrt{-\ln x}}$

Solution: This one is tricky. As $\displaystyle\lim_{x\to 0^+} \cot x = \infty$ and $\displaystyle\lim_{x\to 0^+} \sqrt{-\ln x} = \infty$, we see that this limit is indeterminate, of the form $\dfrac{\infty}{\infty}$. Thus we can use L'hopital's rule:

$$\lim_{x\to 0^+} \frac{\cot x}{\sqrt{-\ln x}} = \lim_{x\to 0^+} \frac{-\csc^2 x}{\frac{1}{2}(-\ln x)^{-1/2} \cdot \frac{-1}{x}}$$

Simplifying, we get

$$\frac{-\csc^2 x}{\frac{1}{2}(-\ln x)^{-1/2} \cdot \frac{-1}{x}} = \frac{2x \cdot \sqrt{-\ln x}}{\sin^2 x}$$

Now, the question is: what do we make of the limit $\displaystyle\lim_{x\to 0^+} \frac{2x \cdot \sqrt{-\ln x}}{\sin^2 x}$? The denominator approaches 0. However, plugging in 0 into the numerator, we get $0 \cdot \infty$, which is indeterminate! So before proceeding, we need to figure out the limit $\displaystyle\lim_{x\to 0^+} 2x\sqrt{-\ln x}$. We use L'hopital's rule:

$$\lim_{x\to 0^+} 2x\sqrt{-\ln x} \overset{(1)}{=} \lim_{x\to 0^+} 2\frac{\sqrt{-\ln x}}{\frac{1}{x}} \overset{(2)}{=} \lim_{x\to 0^+} 2\frac{\frac{1}{2}(-\ln x)^{-1/2}\frac{-1}{x}}{\frac{-1}{x^2}} \overset{(3)}{=} \lim_{x\to 0^+} \frac{x}{\sqrt{-\ln x}}$$

Here, (1) follows from simplification, (2) follows from L'Hopital's rule, and (3) follows from simplification. The numerator of this last limit goes to 0 and the denominator goes to ∞ as $x \to 0^+$, so this limit is 0. Thus, we see that both the numerator and denominator of the limit,

$$\lim_{x\to 0^+} \frac{2x\sqrt{-\ln x}}{\sin^2 x}$$

approach 0, so we can use L'Hopital's rule to evaluate this limit:

$$\lim_{x\to 0^+} \frac{2x\sqrt{-\ln x}}{\sin^2 x} \overset{(4)}{=} \lim_{x\to 0^+} \frac{2\sqrt{-\ln x} + 2x \cdot \frac{1}{2}(-\ln x)^{-1/2}\frac{1}{x}}{2\sin x \cos x} \overset{(5)}{=} \lim_{x\to 0^+} \frac{2\sqrt{-\ln x} + (-\ln x)^{-1/2}}{2\sin x \cos x}$$

Here, we're using L'Hopital's rule to get (4) and simplification to get (5). The numerator of the above limit goes to ∞ and the denominator goes to 0 as x approaches 0 from the right, so the limit is ∞.

(b) $\lim_{x\to 0^+} (3x)^{x^2}$

Solution: We want to take natural logs to get rid of the exponent:

$$\lim_{x\to 0^+} \ln\left((3x)^{x^2}\right) \overset{(1)}{=} \lim_{x\to 0^+} x^2 \ln(3x) \overset{(2)}{=} \lim_{x\to 0^+} \frac{\ln(3x)}{\frac{1}{x^2}} \overset{(3)}{=} \lim_{x\to 0^+} \frac{\frac{1}{3x}3}{\frac{-2}{x^3}} \overset{(4)}{=} \lim_{x\to 0^+} \frac{x^2}{-2} = 0$$

Here, (1), (2), and (4) follow from simplification. We get equation (3) using L'Hopital's rule. Finally, we know that

$$\lim_{x\to 0^+} (3x)^{x^2} = \exp\left(\lim_{x\to 0^+} \ln\left((3x)^{x^2}\right)\right) = \exp(0) = 1$$

(c) $\lim_{x\to 0} (\csc^2 x - \cot^2 x)$

Solution: We can solve this problem by making $\csc^2 x - \cot^2 x$ into a fraction and using L'Hopital's rule. However, it's easier just to use the trig identity,

$$1 + \cot^2 x = \csc^2 x$$

It follows that $\csc^2 x - \cot^2 x = 1$, so the limit is 1.

2. Find the following integrals:

(a) $\displaystyle\int \frac{(\ln x)^2}{x}\, dx$

Solution: For this integral, we want to use a u-substitution. Set $u = \ln x$. Then $du = \frac{1}{x}\, dx$. It follows:

$$\int \frac{(\ln x)^2}{x}\, dx = \int u^2\, du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

(b) $\displaystyle\int \frac{x+1}{x(x-1)}\, dx$

Solution: The simplest approach to evaluating this integral is probably to use a partial fraction decomposition. The degree of the numerator is smaller than the degree of the denominator and the denominator is already factored for us, so we don't have to do any additional work before setting up the partial fraction decomposition:

$$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Multiplying both sides by $x(x-1)$, we get

$$x+1 = A(x-1) + Bx$$

Plugging in $x=1$, we get $B=2$, and plugging in $x=0$ gets us $A=-1$. It follows:

$$\int \frac{x+1}{x(x-1)}\,\mathrm{d}x = \int \frac{-1}{x}\,\mathrm{d}x + \int \frac{2}{x-1}\,\mathrm{d}x = -\ln|x| + 2\ln|x-1| + C.$$

(c) $\displaystyle\int \sin^2 x \cos^3 x \,\mathrm{d}x$

Solution: We have sin raised to an even power and cos raised to an odd power. So we want ot save one of the cosines and turn the rest of the expression into sines:

$$\sin^2 x\cos^3 x = \sin^2 x\cos^2 x\cos x = \sin^2 x(1-\sin^2 x)\cos x = (\sin^2 x - \sin^4 x)\cos x$$

Thus, we have

$$\int \sin^2 x\cos^3 x\,\mathrm{d}x = \int (\sin^2 x - \sin^4 x)\cos x\,\mathrm{d}x$$

In this way, we have simplified our original integral into one that we can evaluate using u-substitution: set $u=\sin x$. Then $\mathrm{d}u = \cos x\,\mathrm{d}x$. We get:

$$\int (\sin^2 x - \sin^4 x)\cos x\,\mathrm{d}x = \int u^2 - u^4\,\mathrm{d}u = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

(d) $\displaystyle\int x^2 e^x\,\mathrm{d}x$

Solution: This is a classic example of an integral we can evaluate using integration by parts. We set:

$$\begin{array}{ll} u = x^2 & \mathrm{d}v = e^x\,\mathrm{d}x \\ \mathrm{d}u = 2x\,\mathrm{d}x & v = e^x \end{array}$$

This gets us:

$$\int x^2 e^x\,\mathrm{d}x = x^2e^x - \int 2xe^x\,\mathrm{d}x = x^2e^x - 2\int xe^x\,\mathrm{d}x$$

This last integral is another integral we can solve using integration by parts. We set:

$$\begin{array}{ll} u = x & \mathrm{d}v = e^x\,\mathrm{d}x \\ \mathrm{d}u = \mathrm{d}x & v = e^x \end{array}$$

This gets us:

$$\int xe^x \,\mathrm{d}x = xe^x - \int e^x \,\mathrm{d}x = xe^x - e^x + C$$

Putting it all together, we get:

$$\begin{aligned}\int x^2e^x \,\mathrm{d}x &= x^2e^x - \int 2xe^x \,\mathrm{d}x = x^2e^x - 2(xe^x - e^x + C)\\ &= x^2e^x - 2xe^x + 2e^x + C\end{aligned}$$

(e) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{3-2x^2}}$

Solution: The function $\dfrac{1}{\sqrt{3-2x^2}}$ is pretty similar to something of the form $\dfrac{1}{\sqrt{a^2-x^2}}$, which we know how to integrate (it's on the formula sheet, for instance). The only problem is that we have $2x^2$ instead of x^2. So we simplify:

$$\frac{1}{\sqrt{3-2x^2}} = \frac{1}{\sqrt{2(\frac{3}{2}-x^2)}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3/2-x^2}}$$

Thus,

$$\int \frac{\mathrm{d}x}{\sqrt{3-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}x}{\sqrt{3/2-x^2}} = \frac{1}{\sqrt{2}}\arcsin\left(\frac{x}{\sqrt{3/2}}\right) + C.$$

Note: another approach to this problem would be using the substitution $u = x\sqrt{2}$. Then $2x^2 = u^2$. This is another away to rid ourselves of that meddlesome 2.

3. The half-life of Tritium is 12 years. If you start with 50 grams of Tritium, how much will you have after 100 years?

Solution: Using the formula $A = A_0e^{kt}$, we want to start by solving for k. By definition, the half-life is the time t_0 it takes for $A_0e^{kt_0}$ to equal $\dfrac{1}{2}A_0$. It follows that

$$e^{k\cdot 12} = \frac{1}{2}$$

Solving this equation for k, we get $k = \dfrac{\ln(1/2)}{12}$. Our final answer is then

$$50e^{\frac{\ln(1/2)}{12}\cdot 100}$$

4. Salt water, at a concentration of 2 kg/L, flows into a tank of water at a rate of 5 L/min. Salt water flows out of the tank at a rate of 4 L/min. The tank starts with 10 Liters of water. Find the differential equation describing the amount of salt in the tank after t minutes. (You don't have to solve it).

Solution: Salt is pouring into our tank at a rate of $2\text{kg/L} \cdot 5\text{L/min} = 10\text{kg/min}$. We know that brine (i.e. salt water) is pouring out of our tank at a rate of 4L/min, but we need to know the concentration of this brine at time t in order to know the rate at which *salt* is pouring out of our tank. Let $y(t)$ be the amount of salt in the tank at time t and let $V(t)$ be the volume of water in the tank at time t. Then the concentration of salt water in the tank at time t is $y(t)/V(t)$. Further, we know that $V(t) = 10 + 5t - 4t = 10 + t$. Thus, our differential equation is

$$\frac{dy}{dx} = 10 - \frac{4y}{10+t}$$

5. Solve the differential equation

$$x\frac{dy}{dx} + \ln x = 0$$

given $y(1) = 2$.

Solution: We start by rearranging the equation we're given into the form $\dfrac{dy}{dx} + P(x)y = Q(x)$. This gets us

$$\frac{dy}{dx} = -\frac{\ln x}{x}$$

We could solve this differential equation using an integrating factor, but this turns out not to be necessary; in this case, we can just integrate both sides with respect to x:

$$y = -\int \frac{\ln x}{x}\,\mathrm{d}x = -\frac{(\ln x)^2}{2} + C$$

Here, we used the substitution $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. To solve for C, we use our initial condition $y(1) = 2$:

$$2 = -\frac{(\ln 1)^2}{2} + C = C$$

So our final answer is $y = -\frac{(\ln x)^2}{2} + 2$

6. Find the convergence set of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}(x-1)^n$.

Solution: We use the absolute ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{(n+2)^2|x-1|^{n+1}}{(n+1)!}}{\frac{(n+1)^2|x-1|^n}{n!}} &= \lim_{n\to\infty} \frac{(n+2)^2|x-1|^{n+1}}{(n+1)!} \cdot \frac{n!}{(n+1)^2|x-1|^n} \\
&= \lim_{n\to\infty} \frac{|x-1|(n+2)^2}{(n+1)^3}
\end{aligned}$$

Note that $|x-1|(n+2)^2$ is a degree-2 polynomial in n (remember, x is a constant as far as this limit is concerned), and $(n+1)^3$ is a degree-3 polynomial in n. It follows that the above limit is 0. In particular, the limit is less than 1 no matter what x is. So the convergence set is $(-\infty, \infty)$.

7. Find the first 3 terms of the Taylor series of $\frac{1}{x^3+1}$ at $x = 0$.

Solution: We get the following table:

$$\begin{aligned}
f(x) &= \frac{1}{x^3+1} & f(0) &= 1 \\
f'(x) &= -\frac{3x^2}{(x^3+1)^2} & f'(0) &= 0 \\
f''(x) &= \frac{18x^4}{(x^3+1)^3} - \frac{6x}{(x^3+1)^2} & f''(0) &= 0
\end{aligned}$$

So our Taylor series starts out in a pretty boring way:

$$\frac{1}{0!} + \frac{0}{1!}x + \frac{0}{2!}x^2 = 1$$

8. Find the area of the region enclosed by the curve given in polar coordinates by $r = 2\cos\theta\sqrt{\sin\theta}, 0 \leq \theta \leq \dfrac{\pi}{2}$.

Solution: For this problem, we want to remember the formula, $A = \dfrac{1}{2}\displaystyle\int_{\theta_0}^{\theta_1} f(\theta)^2 d\theta$. Thus the area we're looking for is:

$$A = \frac{1}{2}\int_0^{\pi/2}(2\cos\theta\sqrt{\sin\theta})^2 d\theta = 2\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta.$$

We can evaluate this integral by using the substitution $u = \cos\theta$, $\mathrm{d}u = -\sin\theta$. We get:

$$A = 2\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta = -\frac{2\cos^3\theta}{3}\Big|_0^{\pi/2} = \frac{2}{3}$$