Math 1220-003, Summer 2018

Final Exam

Please write your name on the front and back of the exam. Remember to turn off your phone before starting this exam. Show all of your work for full credit. You may not use any notes or calculators during this exam.

Name: Solutions

UID: ______________________

1. (10 points) True or false:

(a) $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{2\pi}{3}$

False

(b) If $\sum_{i=0}^{\infty} |a_i|$ converges, then $\sum_{i=0}^{\infty} a_i$ converges

True

(c) If $\sum_{i=0}^{\infty} a_i$ converges, then $\lim_{i\to\infty} a_i = 0$

True

(d) If $\lim_{i\to\infty} a_i = 0$, then $\sum_{i=0}^{\infty} a_i$ converges.

False

(e) If the power series $\sum_{i=0}^{\infty} a_i(x-1)^i$ converges at $x=-1$, it converges at $x=2$.

True

must converge on $(-1,3)$

2. Find the following derivatives:

(a) (6 points) $\frac{d}{dx}x^{\sin x}$

$$y = x^{\sin x} \Rightarrow \ln(y) = \sin x \cdot \ln(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \cdot \ln(x) + \frac{\sin x}{x}$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x} \cdot \left[\cos x \cdot \ln x + \frac{\sin x}{x}\right]$$

(b) (6 points) $\frac{d}{dx}\tan\left(e^{3x}\right)$

Chain rule:

$$\sec^2(e^{3x}) \cdot \frac{d}{dx}e^{3x} = \boxed{\sec^2(e^{3x}) \cdot e^{3x} \cdot 3}$$

3. Find the following limits:

(a) (4 points) $\lim_{x\to 0^+} \frac{\sin^{-1} x}{x}$

L'Hopital: $\lim_{x\to 0^+} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

(b) (5 points) $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{x^2}$

$= \lim_{x\to 0^+} \frac{x-1}{x^2} = -\infty$

goes to -1

goes to 0, positive

4. Evaluate the following integrals:

(a) (6 points) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$

$u = 1 - x^2, \quad du = -2x\,dx$

$\Rightarrow -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + C = \boxed{-\sqrt{1-x^2} + C}$

Note: $-\cos(\sin^{-1}(x)) = -\sqrt{1-x^2}$

is also correct.

(b) (6 points) $\displaystyle\int \frac{1}{x^2+2x+5}\,\mathrm{d}x$

$x^2 + 2x + 5 = (x+1)^2 - 1 + 5 = (x+1)^2 + 4$

$\displaystyle\int \frac{1}{(x+1)^2+4}\,dx = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C}$

(c) (6 points) $\int x \sin x \, dx$

Integration by parts:

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$-x\cos x + \int \cos x \, dx = \boxed{-x\cos x + \sin x + C}$

(d) (6 points) $\int \sin^3(x) \cos^4(x) \, dx$

$\int \sin(x) \cdot \sin^2(x) \cos^4(x) \, dx$

$= \int \sin(x) \cdot \left(1 - \cos^2 x\right) \cos^4 x \, dx$

$u = \cos x, \quad du = -\sin x \, dx$

$= -\int (1-u^2) u^4 \, du = \int u^6 - u^4 \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$

$= \boxed{\frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5} + C}$

5. (15 points) For each of the following series, write "CONVERGE" if the series converges and "DIVERGE" if the series diverges. You do not have to show your work for this problem.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^2}}$ Diverges (p-series)

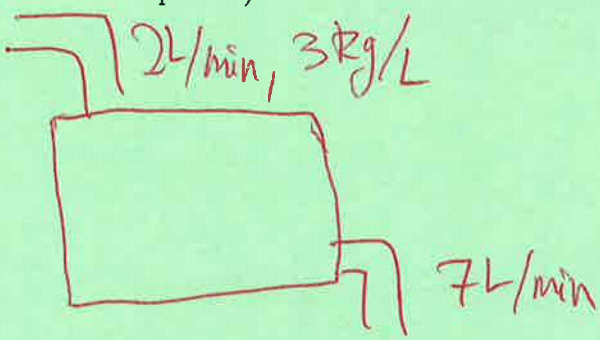
(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+n+1}{n^5+n^4+n^3}$ Converge (Limit comparison)

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(n+1)^2}$ Diverge (Ratio test)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$ Diverge (n^{th} term test)

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}$ Converge (Geometric series)

6. (6 points) Salt water, at a concentration of 3 kg/L, flows into a tank of water at a rate of 2 L/min. Salt water flows out of the tank at a rate of 7 L/min. The tank starts with 10 Liters of water and 1 kilogram of salt. Find the differential equation describing the amount of salt in the tank after t minutes. (You don't have to solve this differential equation).



$$\frac{dy}{dt} = 6 - 7 \cdot \frac{y(t)}{\text{Volume}(t)}$$

$$\text{Volume} = 10 - 7t + 2t = 10 - 5t$$

$$\Rightarrow \quad \frac{dy}{dt} = 6 - \frac{7y(t)}{10-5t}$$

7. (6 points) Find the first three terms of the Taylor series of $f(x) = \dfrac{1}{\sin x}$ at $x = \dfrac{\pi}{2}$.

$f(\pi/2) = 1$

$f'(x) = -(\sin x)^{-2} \cdot \cos x, \quad f'(\pi/2) = 0$

$f''(x) = 2(\sin x)^{-3} \cdot \cos^2 x + -(\sin x)^{-2} \cdot (-\sin x), \quad f''(\pi/2) = 1$

$\underline{1} + \frac{0}{1!}(x - \pi/2) + \frac{1}{2!}(x - \pi/2)^2$

$= 1 + \frac{1}{2}(x - \pi/2)^2$

8. (6 points) Find the convergence set of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n2^n}$.

$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{|x|^n} = \lim_{n\to\infty} \frac{|x| \cdot n}{2(n+1)} = \frac{|x|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x|}{2}$

$\frac{|x|}{2} < 1 \rightsquigarrow -1 < \frac{x}{2} < 1 \rightsquigarrow -2 < x < 2.$

At $x = -2$: $\sum \frac{(-2)^n}{n2^n} = \sum \frac{(-1)^n}{n}$ converges.

At $x = 2$: $\sum \frac{2^n}{n2^n} = \sum \frac{1}{n}$ diverges

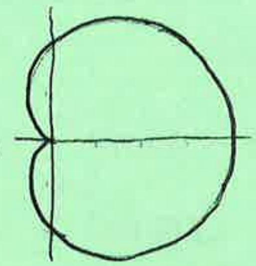
$-2 \le x < 2$

9. (6 points) Match the equation in polar coordinates to the graph.

(a) $r = \sin(3\theta)$

4

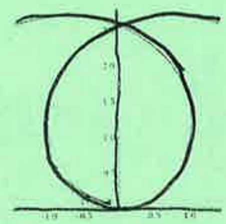
1.



(b) $r = \sin(2\theta)$

6

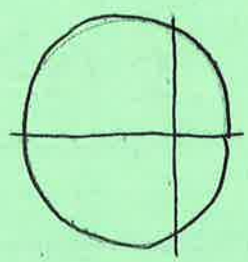
2.



(c) $r = 2 - \cos\theta$

3

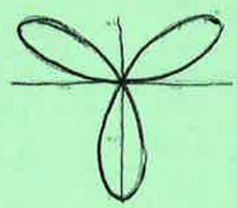
3.



(d) $r = 2 + 2\cos\theta$

1

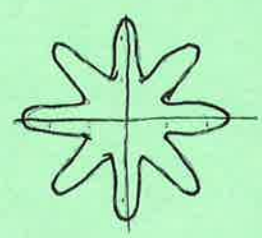
4.



(e) $r = 2 + 3\cos^2(4\theta)$

5

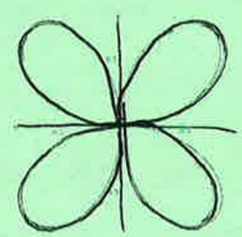
5.



(f) $r = \theta + \sin(\theta)$

2

6.



10. (6 points) Find the area bounded by the polar curve $r = \cos(3\theta)$, $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$.

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} [\cos(3\theta)]^2 \, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos(6\theta)}{2} \, d\theta$$

$$= \frac{1}{4}\int_{-\pi/6}^{\pi/6} 1 + \cos(6\theta) \, d\theta = \frac{1}{4}\left[\theta + \frac{1}{6}\sin(6\theta)\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4}\left[\pi/6 + \cancel{\frac{1}{6}\sin(\pi)} - \left(\frac{-\pi}{6} + \cancel{\frac{1}{6}\sin(-\pi)}\right)\right]$$

$$= \frac{1}{4} \cdot \frac{\pi}{3} = \frac{\pi}{12}$$

Math 1220-003

Form: Hilbert

Name: ______________________

Page	Points	Score
2	10	
3	12	
4	9	
5	12	
6	12	
7	15	
8	6	
9	12	
10	6	
11	6	
Total:	100	