Math 214-007 Singular Value Decomposition worksheet

1. Let A have the singular value decomposition

$$A = \begin{bmatrix} | & | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \\ | & | & | & | \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} — & \vec{v}_1^\top & — \\ — & \vec{v}_2^\top & — \\ — & \vec{v}_3^\top & — \end{bmatrix}$$

this is the same as $\vec{v}_i$ in ker ΣV^T, (since U has no kernel)

$\vec{v}_1 = [u_1 \cdots u_4] \begin{bmatrix} 5 & & \\ & 3 & \\ & & 0 \\ & & \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix}$

Which vectors $\vec{v}_i$ are in the kernel of A? Which vectors $\vec{u}_i$ are in the image of A?

~~$\vec{v}_i$ in ker $\begin{bmatrix} — \vec{v}_1 — \\ — \vec{v}_2 — \\ — \vec{v}_3 — \end{bmatrix} =$~~

$[u_1 \cdots u_4] \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 5\vec{u}_1$

$\vec{v}_1 \notin$ ker.

$\vec{v}_2 \notin$ ker

$\vec{v}_3 \in \ker A$!

$\begin{bmatrix} — \vec{v}_1^T — \\ — \vec{v}_2^T — \\ — \vec{v}_3^T — \end{bmatrix} \vec{x} = \begin{bmatrix} v_1 \cdot x \\ v_2 \cdot x \\ v_3 \cdot x \end{bmatrix}$

~~the span~~ $\text{im}(A) \ni \vec{u}_1, \vec{u}_2$

2. Let B have the singular value decomposition

$$A = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} — & \vec{v}_1^\top & — \\ — & \vec{v}_2^\top & — \\ — & \vec{v}_3^\top & — \\ — & \vec{v}_4^\top & — \end{bmatrix}$$

Which vectors $\vec{v}_i$ are in the kernel of B? Which vectors $\vec{u}_i$ are in the image of A?

$\ker(A) \ni \vec{v}_3, \vec{v}_4$

$\text{im}(A) \ni \vec{u}_1, \vec{u}_2$

3. Explain how to use the SVD of a matrix to quickly see: its rank, its nullity, an orthonormal basis of its image, and an orthonormal basis of its kernel

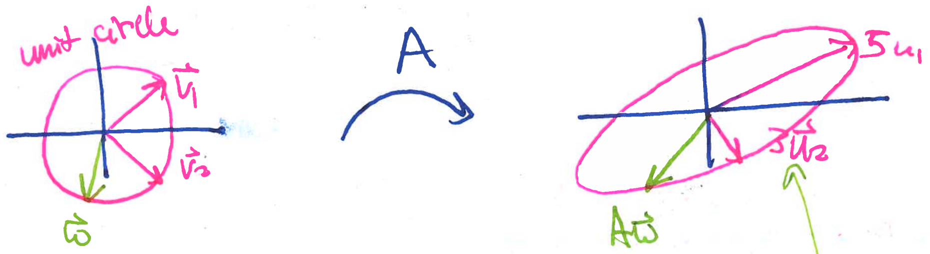
$\text{im}(A) =$ span of $\vec{u}_i$ corresp to nonzero rows in Σ

$\ker(A) =$ span of $\vec{v}_i$ corresp to zero cols in Σ.

4. True or false! (Taken from the textbook)

(a) If A is a 2×2 matrix with singular values 3 and 5, then there is some $\vec{w} \in \mathbb{R}^2$ with $||\vec{w}|| = 1$ and $||A\vec{w}|| = 2$

FALSE: must have $3 \leq ||A\vec{w}|| \leq 5$



(b) If A is a 2×2 matrix with singular values 3 and 5, then there is some $\vec{w} \in \mathbb{R}^2$ with $||\vec{w}|| = 1$ and $||A\vec{w}|| = 4$

TRUE: some point on this ellipse is 4 units away from the origin

(c) The product of the n singular values of an $n \times n$ matrix must be $|\det A|$.

TRUE:
$$A = U \Sigma V^T$$
$$= U \cdot \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} V^T$$

So
$$\det(A) = \underbrace{\det(U)}_{\pm 1 \text{ (U orthogonal)}} \underbrace{\det\left(\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}\right)}_{\sigma_1 \cdots \sigma_n} \underbrace{\det(V^T)}_{\pm 1 \text{ (V orthogonal)}}$$

$= \pm \sigma_1 \cdots \sigma_n$, and $\sigma_1 \cdots \sigma_n \geq 0$. So TRUE.