Midterm 2 review

November 1, 2022

42. Consider an $n \times m$ matrix

$$A = QR,$$

where Q is an $n \times m$ matrix with orthonormal columns and R is an upper triangular $m \times m$ matrix with positive diagonal entries $r_{11}, \ldots, r_{mm}$. Express $\det(A^T A)$ in terms of the scalars r_{ii}. What can you say about the sign of $\det(A^T A)$?

(Hint: recall that $(AB)^T = B^T A^T$)

47. If $A = QR$ is a QR factorization, what is the relationship between $A^T A$ and $R^T R$?

48. Consider an invertible $n \times n$ matrix A. Can you write A as $A = LQ$, where L is a *lower* triangular matrix and Q is orthogonal? *Hint*: Consider the QR factorization of A^T.

37. For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

describe the images and kernels of the matrices A, A^2, and A^3 geometrically.

38. Consider a square matrix A.

a. What is the relationship among $\ker(A)$ and $\ker(A^2)$? Are they necessarily equal? Is one of them necessarily contained in the other? More generally, what can you say about $\ker(A)$, $\ker(A^2)$, $\ker(A^3)$, ...?

b. What can you say about $\operatorname{im}(A)$, $\operatorname{im}(A^2)$, $\operatorname{im}(A^3)$, ...?

Hint: Exercise 37 is helpful.

11. Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose for two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^2$ we have $T(\vec{v}_1) = 3\vec{v}_1$ and $T(\vec{v}_2) = 4\vec{v}_2$. What can you say about $\det A$? Justify your answer carefully.

Let $M = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 4 \end{bmatrix}$. Find an orthonormal basis for the image of M.

Let V be the image of the matrix M above. Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find $\text{proj}_V(\vec{v})$.