Math 214 7.3 worksheet

Facts and defs: Let A be an $n \times n$ matrix

- An *eigenvector* of A is a nonzero $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$.
- The function $p_A(x) = \det(A - xI_n)$ is a polynomial in x, called the *characteristic polynomial* of A. A number $\lambda \in \mathbb{R}$ is an eigvenvalue of A if and only if $p_A(\lambda) = 0$. The *algebraic multiplicity* of λ is the biggest power of $(\lambda - x)$ dividing $p_A(x)$.
- The *eigenspace* of λ is $E_\lambda = \ker(A - \lambda I_n)$. This is the set of eigenvectors of A with eigenvalue λ (along with the zero vector, which doesn't count as an eigenvector).
- A is *diagonalizable* if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. If $\mathscr{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is such a basis, and $S = \left[\vec{v}_1, \ldots, \vec{v}_n\right]$, then

  $S^{-1}AS =$

1. Find the eigenvectors of the following matrices. Are they diagonalizable? What are the algebraic multiplicites of the eigenvalues?

   (a) $\begin{bmatrix} 0 & -2 & 2 \\ -3 & -2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

Definition: the *geometric* multiplicty of an eigenvalue λ is ____________________

Fact: a matrix is diagonalizable if and only if ____________________

Fact: the geometric multiplicity of an eigenvalue is always __________ than its algebraic multiplicity.

In other words, each additional (linearly independent) eigenvector for λ gives an additional factor of __________ dividing __________

Let's try to prove this fact!

2. Find the determinant of the following matrix (it's not as bad as it looks!)

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 2 & 0 & 6 & 9 \\ 0 & 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

In general:

3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let S be an invertible matrix. What is $\det(S^{-1}AS)$?

4. Let A be a square matrix and $B = S^{-1}AS$. Show that $p_A(x) = p_B(x)$.

Now let's prove the theorem!

Theorem If A is a square matrix and λ is an eigenvalue of A, then $\operatorname{gmult}(\lambda) \leqslant \operatorname{algmult}(\lambda)$

Proof