Math 214 7.3 worksheet 2

We saw that the algebraic multiplicity of an eigenvalue is always __________ than its geometric multiplicity.

1. For each of the following matrices find: the characteristic polynomial, the algebraic multiplicity of 2, and the geometric multiplicity of 2:

   (a) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ (the (2, 3) entry is the only one that changed)

   (c) $B = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

2. Suppose D is a matrix with characteristic polynomial $p_D(x) = (x-2)^4(x+1)^2(x^2+1)$. What can you say about the geometric multiplicity of $\lambda = 2$ for D?

   While we're at it, what are the dimensions of the matrix D? What is the determinant of D?

Let's prove this fact!

Theorem If A is a square matrix and λ is an eigenvalue of A, then $\text{gmult}(\lambda) \leqslant \text{algmult}(\lambda)$

Proof idea: (1) we can "partially diagonalize" A using the eigenvectors in E_λ, (2) the theorem is clear for this partially diagonal matrix, and (3) it follows that the theorem holds for A.

Proof

- Suppose λ is an eigenvalue of A with $\text{gmult}(\lambda) = k$. We want to show that:

- Because $\text{gmult}(\lambda) = k$, we can find linearly independent eigenvectors

- We can then extend this linearly independent set to a basis of $\mathbb{R}^n$:

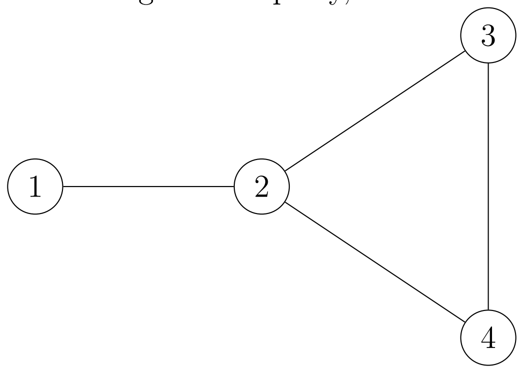
- With respect to this basis, A looks like:
  $S^{-1}AS =$

- Thus the characteristic polynomial of $S^{-1}AS$ is:

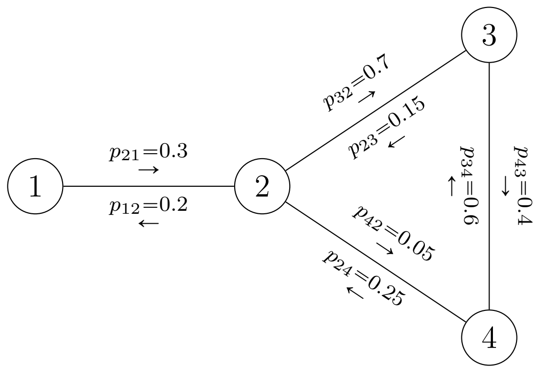
- But by last time, we know:

7.4 worksheet

Suppose there are four houses hosting a block party, connected to each other like so:



Every hour, some percent of people at one house leave and go to another house, as described by the following diagram:



So for instance, the value of $p_{21} = 0.3$ means that house 2 will get 30% of the people in house 1 every hour. This process is an example of a *Markov process*, also called a *Markov chain*.

1. What proportion of people at house 1 will stay at house 1 after an hour? We call this number p_{11}.

2. In general we let p_{ii} be the proportion of people in house i who decide to stay in house i when the hour changes. Find p_{22}, p_{33}, and p_{44} as well.

3. Let $x_i(t)$ be the number of people in house i, t hours after midnight. Let $\vec{x}(t)$ denote the vector

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

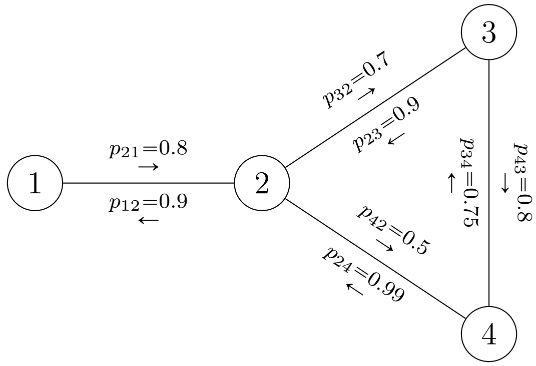
(this is an example of a *vector-valued function*: its input is a number t and its output is a vector $\vec{x}(t)$). Suppose that

$$\vec{x}(0) = \begin{bmatrix} 100 \\ 300 \\ 500 \\ 200 \end{bmatrix}$$

Find $x_2(1)$.

4. Find a matrix A such that $\vec{x}(t+1) = A\vec{x}(t)$. This is called the *transition matrix* of the markov process.

5. Albert takes some measurements of how many people are leaving each house, and gets the following diagram:



Does this diagram make physical sense? Why or why not?

6. Explain the following fact: no matter what numbers p_{ij} we have in our transition matrix A, the columns of A have to add up to 1.

7. Explain why $[1\ 1\ 1\ 1]^T$ is an eigenvector of A^T. What is its eigenvalue?