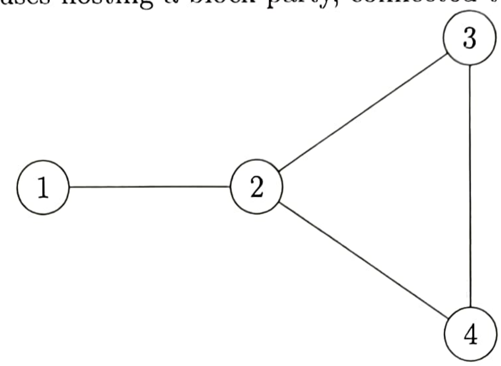
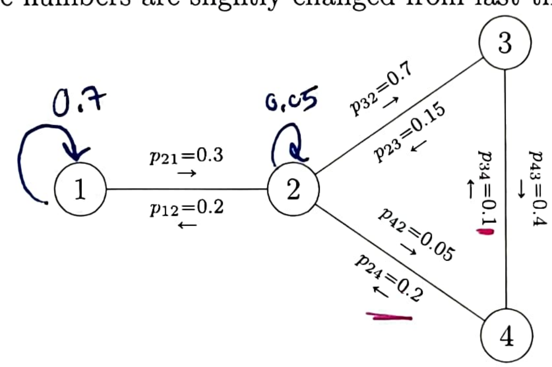
7.4 worksheet

Suppose there are four houses hosting a block party, connected to each other like so:



Every hour, some percent of people at one house leave and go to another house, as described by the following diagram (the numbers are slightly changed from last time)



So for instance, the value of $p_{21} = 0.3$ means that house 2 will get 30% of the people in house 1 every hour. This process is an example of a *Markov process*, also called a *Markov chain.*

1. What proportion of people at house 1 will stay at house 1 after an hour? We call this number p_{11}. everyone in house 1 either stays in house 1, or goes to 2.

   $p_{11} = 0.7$

2. In general we let p_{ii} be the proportion of people in house i who decide to stay in house i when the hour changes. Find p_{22}, p_{33}, and p_{44} as well.

   $p_{22} = 1-(0.2)-(0.05)-(0.7) = 0.05$ $\qquad$ $p_{44} = 0.7$

   $p_{33} = 0.45$

3. Let $x_i(t)$ be the number of people in house i, t hours after midnight. Let $\vec{x}(t)$ denote the vector $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$, and let $\vec{x}(0) = \begin{bmatrix} 100 \\ 300 \\ 500 \\ 200 \end{bmatrix}$. ($\vec{x}(t)$ is an example of a *vector-valued function*: its input is a number t and its output is a vector). Find $x_2(1)$.

   (100 → ppl in house 1 @ t=0; 300 → ppl in house 2 @ t=0; $x_2(1)$ → # ppl in house 2 @ t=1)

   $x_2(1) = 0.3\, x_1(0) + 0.05\, x_2(0) + 0.15\, x_3(0) + 0.2\, x_4(0) =$ ~~140~~ 160

4. Find a matrix A such that $\vec{x}(t+1) = A\vec{x}(t)$. This is called the *transition matrix* of the markov process.

$x_1(t+1) = 0.7\, x_1(t) + 0.2 \cdot x_2(t)$

$x_2(t+1) = 0.3\, x_1(t) + 0.05\, x_2(t) + 0.15\, x_3(t) + 0.2\, x_4(t)$

$x_3(t+1) =$

$x_4(t+1) =$

i^{th} cols

% ppl leaving from house i

(i,j)-entry $= p_{ij}$

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 & 0 & 0 \\ 0.3 & 0.05 & 0.15 & 0.2 \\ 0 & 0.7 & 0.45 & 0.1 \\ 0 & 0.05 & 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$p_{13} = ?$ ppl in house 3 going to house 1

row j tells you about arrows pointing into house j

5. Suppose there are x_1 people at house 1, x_2 people in house 2, etc. This configuration of people $\vec{x}_{eq} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is called an *equillibrium state* if $A\vec{x}_{eq} = \vec{x}_{eq}$. Is there an equilibrium state to block party described above? Give an example of one. It may be helpful to know that the eigenvectors/eigenvalues of A are:

then if $\vec{x}(t) = \vec{x}_{eq}$, then $\vec{x}(t+1) = \vec{x}_{eq}$, $\vec{x}(t+2) = \vec{x}_{eq}$

basis of $\mathbb{R}^4$

$$\vec{v}_1 = \begin{bmatrix} 0.184 \\ 0.276 \\ 0.716 \\ 1 \end{bmatrix}, \lambda_1 = 1 \qquad \vec{v}_2 = \begin{bmatrix} -0.925 \\ -0.222 \\ 0.148 \\ 1 \end{bmatrix}, \lambda_2 = 0.748$$

$\hookrightarrow A\vec{v}_1 = 1 \cdot \vec{v}_1$

$$\vec{v}_3 = \begin{bmatrix} 0.053 \\ -0.102 \\ -0.951 \\ 1 \end{bmatrix}, \lambda_3 = 0.314 \qquad \vec{v}_4 = \begin{bmatrix} -0.416 \\ 1.798 \\ -2.381 \\ 1 \end{bmatrix}, \lambda_4 = -0.163$$

(so A is diagonalizable!)

Any constant times $\vec{v}_1$ gives an equilibrium state of the system. However, to be physically meaningful, we want positive, integer #s of ppl in the houses.

$$1000\,\vec{v}_1 = \begin{bmatrix} 184 \\ 276 \\ 716 \\ 1000 \end{bmatrix}$$ is an equilibrium state.

No matter how many ppl start in each house, the populations of the houses will approach an equilib. state.

6. Show that $\vec{x}(t)$ gets closer and closer to an equilibrium state $\vec{x}_{eq}$ as t gets bigger and bigger (hint: this is a lot like the foxes and hares problem from the homework! We know $x(t) = A^t x(0)$. Write $\vec{x}(0)$ as a linear combination of eigenvectors)

$\rightsquigarrow \vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$

$\vec{x}(t) = A^t\vec{x}(0) = A^t(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + c_3\lambda_3^t\vec{v}_3 + c_4\lambda_4^t\vec{v}_4$

($\lambda_1 = 1$; $\lambda_2^t, \lambda_3^t, \lambda_4^t \to 0$)

$\Rightarrow$ if t is really big, $\lambda_2^t, \lambda_3^t, \lambda_4^t \approx 0$, so $\vec{x}(t) \approx c_1\vec{v}_1$ for some c_1 — equilibrium state

7. **Fact:** This always happens! Every[1] Markov chain has an *equilibrium state* $\vec{x}_{eq}$. Given any initial configuarion $\vec{x}(0)$, the vector $\vec{x}(t)$ will always get closer and closer to (a positive mutliple of) $\vec{x}_{eq}$.

- The reason is the same as before:

If A is the transition matrix, the ~~theorem~~ is that 1 is an eigenvalue ~~eigenvector~~ of A, $|\lambda| < 1$ for all other eigenvalues λ, and the eigenvector w/ eigenvalue 1 will have all positive entries (Perron-Frobenius thm)

Let's solve the following problems using some similar ideas:

For the matrices A and the vectors $\vec{x}_0$ in Exercises 13 through 19, find closed formulas for $A^t\vec{x}_0$, where t is an arbitrary positive integer. Follow the strategy outlined in Theorem 7.1.6 and illustrated in Example 1. In Exercises 16 through 19, feel free to use technology.

13. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Find eigenvectors of A: ~~$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 3, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \lambda_2 = -2$~~

~~Write $\vec{x}_0$ as a combination of $\vec{v}_1, \vec{v}_2$: $\vec{x}_0 = \frac{11}{4}\vec{v}_1 - \frac{1}{4}\vec{v}_2$~~

~~$\rightsquigarrow A^t\vec{x}_0 = A^t\left(\frac{11}{4}\vec{v}_1 - \frac{1}{4}\vec{v}_2\right) = \frac{11}{4}\lambda_1^t\vec{v}_1 - \frac{1}{4}\lambda_2^t\vec{v}_2$~~

For the matrices A and the vectors $\vec{x}_0$ in Exercises 25 through 29, find $\lim_{t\to\infty}(A^t\vec{x}_0)$. Feel free to use Theorem 7.4.1.

25. $A = \begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}$

[1] Technically, we need this to be a *regular* Markov chain, meaning every node of the Markov chain is reachable from every other node

Problem #13 :

$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Find $A^t x_0$

- First, find Eigenvects of A.
  - To do so, find eigenvalues:

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda).$$

$\lambda = 1,\ \lambda = 3$

- $\ker(A - 1 \cdot I_2) = \ker\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = \operatorname{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

  $\ker(A - 3I_2) = \ker\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Write x_0 in terms of eigenvectors:

$$x_0 = c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 1 & 2 \end{array}\right]$$

$$\overset{RREF}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right] \qquad c_1 = -1,\ c_2 = 3$$

$$\Rightarrow x_0 = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- So $A^t x_0 = A^t\left(-\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -A^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3A^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

~~[illegible]~~

$$= -\underbrace{1^t\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is eigenvect w/ } \lambda = 1} + 3 \cdot \underbrace{3^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is eigenvect w/ } \lambda = 3}$$