7.5 worksheet

- A *complex number* is a number of form $a+bi$, where $a,b \in \mathbb{R}$, and $i = \sqrt{-1}$. The set of all complex numbers is denoted $\mathbb{C}$

  $(i^2 = -1)$

- Addition, multiplication, division all work the same as usual:
  - $(3+(\sqrt{2})i)+(5-i) = 3+5+\sqrt{2}\,i - i = 8 + (\sqrt{2}-1)i$
  - $(2-5i)-(1-5i) = 2-5i-1+5i = 2-1\cancel{-5i}\cancel{+5i} = 1$
  - $(1+3i)(1-3i) = 1\cdot 1 + \cancel{1\cdot(-3i)} + \cancel{(3i)1} + (3i)(-3i)$

    $= 1 + -3\cdot 3\cdot \underbrace{i\cdot i}_{=-1} = 1 + (-9)\cdot(-1) = 1+9 = 10$

- With complex numbers, there's another important operation called *complex conjugation*:

  $$\overline{a+bi} = a - bi \qquad (a,b \in \mathbb{R})$$

For instance:

- $\overline{3-i} = \overline{3+(-1)i} = 3-(-1)i = 3+i$
- $\overline{5} = \overline{5+0i} = 5-0i = 5$

Facts: $\overline{z} = z$ if and only if $z \in \mathbb{R}$

- $\overline{z\cdot w} = \overline{z}\cdot\overline{w}$
- $\overline{z+w} = \overline{z}+\overline{w}$

for all $z,w \in \mathbb{C}$

- One reason conjugation is nice: simplifying fracationsc

  $\frac{1}{1+i} = a+bi$? $\quad \frac{1}{1+i}\cdot\frac{1-i}{1-i} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i$

- Big important fact (fundamental theorem of algebra): if $f(x)$ is any polynomial, then, using complex numbers, we can: completely factor $f(x)$ into linear terms

  $$f(x) = (x-c_1)\cdot(x-c_2)\cdot \ldots \cdot (x-c_n)\cdot a$$

  complex #s

  the most you can factor using $\mathbb{R}$.

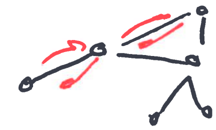
e.g. $f(x) = x^4 - 9x^3 + 30x^2 - 42x + 20 = (x-1)(x-2)(x^2-6x+10) = (x-1)(x-2)(x-(3-i))\cdot(x-(3+i))$

Roots of $f(x)$: $x=1,\ x=2,\ x=3-i,\ x=3+i$

complex conjugates!

7.4 worksheet

Markov chain:

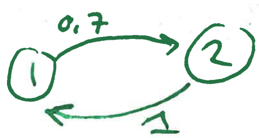


Recall from last time:

- If A is the transition matrix of a Markov chain, then 1 is an eigenvalue of A
- Any eigenvector w/ eigenvalue 1 is an "equilibrium state." $\quad$ gmult$(1) = 1$
- Every state of the system approaches an equilibrium state as time $\to \infty$.

1. Consider a Markov chain (think: block party) where there are two nodes (houses). At each time interval, 70% of the people at node 1 go to node 2, and all the people from node 2 go to node 1. Suppose we start with 120 people at node 1 and 50 people at node 2. How many people will there be at each node after a really really long time?

$\boxed{A^t(c_1v_1 + c_2v_2) \to c_1v_1}$



$A = \begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix}$ $\quad \lambda = 1$ is an eigenval. Find the corresp. eigenvector.

$$\ker(A - 1 \cdot I_2) = \ker\begin{pmatrix} -0.7 & 1 \\ 0.7 & -1 \end{pmatrix} = \operatorname{span}\underbrace{\begin{bmatrix} 1 \\ 0.7 \end{bmatrix}}_{\text{an eigenvector w/ } \lambda = 1} = \operatorname{span}\begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

So answer is $c \cdot \begin{bmatrix} 10 \\ 7 \end{bmatrix}$. $c = ?$ There are 170 ppl total.

Need $c \cdot 10 + c \cdot 7 = 170 \rightsquigarrow c = 10$

answer: $\begin{bmatrix} 100 \\ 70 \end{bmatrix}$

Some closely related questions:

2. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, and let $\vec{x}_0 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Find a "closed formula" for $A^t\vec{x}_0$.

Find eigenvalues: $\lambda_1 = 3, \ \lambda_2 = 2$.

Find eigenvectors:

v_1: $\ker\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

v_2: $\ker\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x}_0 = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to A^t\vec{x}_0 = A^t\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$= A^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2A^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 3^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot 2^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Find a closed formula for A^t. (Hint: diagonalize A)

Set $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\rightsquigarrow A = S\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot S^{-1}$

$$A^t = \left(S\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}S^{-1}\right)^t = \underbrace{\left(S\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\cancel{S^{-1}}\right) \cdot \left(\cancel{S}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\cancel{S^{-1}}\right) \cdots \left(\cancel{S}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}S^{-1}\right)}_{t \text{ times}} = S\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^t S^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

- Another important fact: if $f(x)$ is a polynomial with real coefficients and z is a root of $f(x)$, then

$\bar{z}$ is also a root of $f(x)$

- E.g. find all Eigenvalues (real and complex!) of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix} \quad \det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -4 \\ 0 & 4 & 3-\lambda \end{pmatrix} = (1-\lambda)\underbrace{\left((3-\lambda)^2+16\right)}=0$$

$\lambda = 1$

$(3-\lambda)^2 = -16$

$(3-\lambda) = \sqrt{-16} = \pm 4i$

$\rightarrow 3 \pm 4i = \lambda$

$\lambda = 1, \quad \lambda = 3+4i, \quad \lambda = 3-4i$

conjugate pair

- What are the eigenvectors?

- **Fact**: The eigenvalues and eigenvectors of a matrix with real entries come in ______________ pairs

- Exercise: find the eigenvalues and eigenvectors of $\begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$.