Midterm 2 review

November 1, 2022

42. Consider an $n \times m$ matrix

$$A = QR,$$

where Q is an $n \times m$ matrix with orthonormal columns and R is an upper triangular $m \times m$ matrix with positive diagonal entries $r_{11}, \ldots, r_{mm}$. Express $\det(A^T A)$ in terms of the scalars r_{ii}. What can you say about the sign of $\det(A^T A)$?

(Hint: recall that $(AB)^T = B^T A^T$)

n, m

$\det(A^T A)$

"lower-triangular" "upper triang."

$\det$ of $\begin{bmatrix} a_{11} & & a_{mm} \\ & 0 & \end{bmatrix} =$

$\det \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ & & a_{mm} \end{bmatrix} = a_{11} \cdots a_{mm}$

$\det \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = (-1)^{3+3} \cdot 6 \cdot \det \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

$= 6 \cdot 3 \cdot 1$

$A^T = R^T Q^T$

$A^T A = R^T \boxed{Q^T Q} R = R^T R$

$\underset{\shortparallel}{} \; I_m$

$\det(A^T A) = \det(R^T R) = \det\left(\begin{bmatrix} r_{11} & 0 & \cdots & 0 \\ r_{12} & \ddots & & 0 \\ \vdots & & & r_{mm} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots \\ 0 & r_{22} & \cdots \\ 0 & \cdots & r_{mm} \end{bmatrix} \right)$

$= \det\left(\begin{bmatrix} r_{11} & & 0 \\ & \ddots & \\ & & r_{mm} \end{bmatrix} \right) \det\left(\begin{bmatrix} r_{11} & & \\ & \ddots & \\ 0 & & r_{mm} \end{bmatrix} \right)$

$= (r_{11} \cdots r_{mm})^2$

47. If $A = QR$ is a QR factorization, what is the relationship between $A^T A$ and $R^T R$?

48. Consider an invertible $n \times n$ matrix A. Can you write A as $A = LQ$, where L is a *lower* triangular matrix and Q is orthogonal? *Hint*: Consider the QR factorization of A^T.

47. $A^T A = R^T \underbrace{Q^T Q}_{\shortparallel} R = R^T R$

48. $A^T = QR$? Yes!

Want $A = L \cdot Q$

Transpose both sides:

$(A^T)^T = R^T Q^T$

$A = \underbrace{R^T}_{\text{lower triangular}} \underbrace{Q^T}_{\text{orthogonal}}$

When does $A = QR$?

orthogonal

If A orthogonal: $A = Q \cdot I_m$

when cols of A are lin. indep.

M orthogonal $\Leftrightarrow$ M^T is orthogonal.

37. For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$\ker(A) = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

$\text{im}(A) = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

describe the images and kernels of the matrices A, A^2, and A^3 geometrically.

$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\ker = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$\text{im} = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\ker(A^3) = \mathbb{R}^3$

$\text{im}(A^3) = \{\vec{0}\}.$

38. Consider a square matrix A.

a. What is the relationship among $\ker(A)$ and $\ker(A^2)$? Are they necessarily equal? Is one of them necessarily contained in the other? More generally, what can you say about $\ker(A)$, $\ker(A^2)$, $\ker(A^3)$, …?

b. What can you say about $\text{im}(A)$, $\text{im}(A^2)$, $\text{im}(A^3)$, …?

Hint: Exercise 37 is helpful.

$\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \cdots$

$\text{Im}(A) \supseteq \text{Im}(A^2) \supseteq \text{Im}(A^3) \supseteq \cdots$

Proof:

if $\vec{x} \in \ker(A^n)$, then $A^{n+1}\vec{x} = A \cdot A^n\vec{x} = A \cdot \vec{0} = \vec{0}$. So $\vec{x} \in \ker(A^{n+1})$.

Thus $\ker(A^n) \subseteq \ker(A^{n+1})$

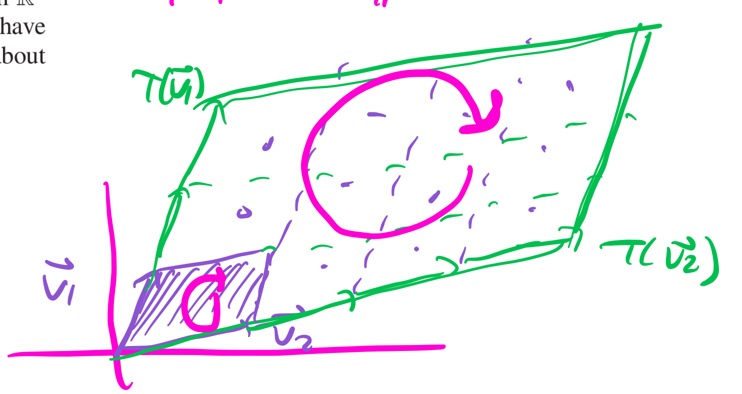
if $\vec{x} \in \text{im}(A^{n+1})$, then $\vec{x} = A^{n+1}\vec{y}$ for some $\vec{y}$. So $\vec{x} = A^n(A\vec{y})$. So $\vec{x} \in \text{im}(A^n)$

Thus $\text{im}(A^{n+1}) \subseteq \text{im}(A^n)$

(assume $\vec{v}_1, \vec{v}_2 \neq 0$)

11. Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose for two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^2$ we have $T(\vec{v}_1) = 3\vec{v}_1$ and $T(\vec{v}_2) = 4\vec{v}_2$. What can you say about $\det A$? Justify your answer carefully.

$T: \mathbb{R}^2 \to \mathbb{R}^2$

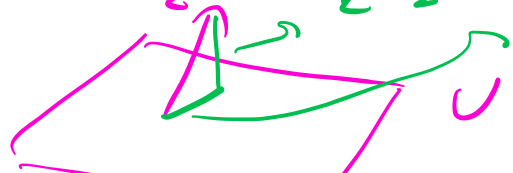


$\det(A) = ?$

area of $\underbrace{\text{big parallelogram}}_{T(\text{parallelogram formed by } \vec{v}_1, \vec{v}_2)}$ $= 12 \cdot$ area of original parallelogram

$\Rightarrow |\det(A)| = 12$

$\det(A) > 0 \Rightarrow$ ans: 12.



$\vec{z} - \vec{z}^{\parallel} = \vec{z}^{\perp}$

$\text{proj}_V \vec{z} = \vec{z}^{\parallel}$

Let $M = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 4 \end{bmatrix}$. Find an orthonormal basis for the image of M.

Let V be the image of the matrix M above. Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find $\text{proj}_V(\vec{v})$.

$\text{proj}_V(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2$

im M