11/21 The geometry of complex eigenvalues

Q Describe what the following lin. transform does (in words, or in pictures)

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

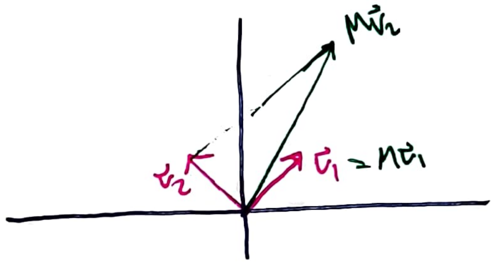
One way:

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$$

the matrix of M, with respect to the basis $\underset{\vec v_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\vec v_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$, is

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ → horiz shear:

So, what M does is

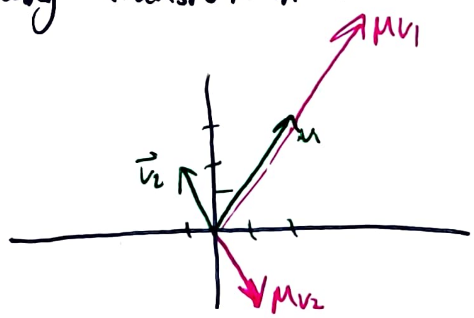


M is a shear in the direction of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

exc Describe the effects of the following transformations:

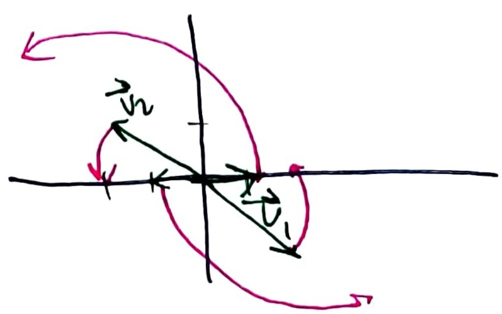
- $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}^{-1}$

  eigenvectors SDS^{-1} eigenvalues.

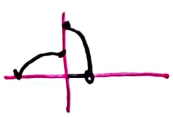


stretching by 2 in $\vec v_1$ direction, and reflecting ~~across~~ ~~line perp to~~ v_2

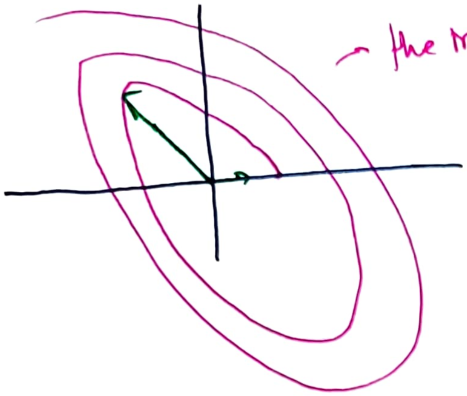
- $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1}$



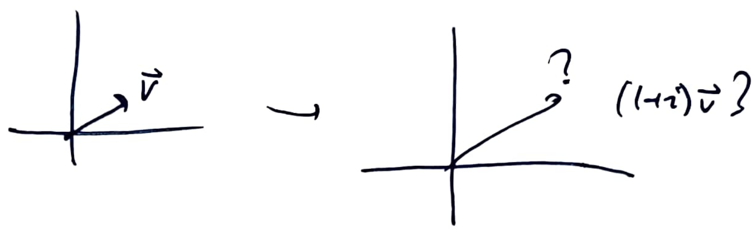
$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$



rotation w/ scaling



Q What do complex eigenvalues mean geometrically?



Answer Rotations!

Thm (7.5.3) Let A be a real 2×2 matrix with eigenvalues $\lambda = a \pm bi$ and eigenvectors $\vec{v} \pm \vec{w} i$

Then
$$A = [\vec{v}\ \vec{w}] \underbrace{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}_{\text{rotation (by } \tan^{-1}(\frac{b}{a})\text{), combined w/ scaling by } \sqrt{a^2+b^2}} [\vec{v}\ \vec{w}]^{-1}$$

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exc $A = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$. Find the eigenvectors, and use them to describe A geometrically.

Eigenvalues: $\det(A - \lambda I_2) = \lambda^2 + 5 = 0 \rightsquigarrow \lambda = \pm i\sqrt{5}$

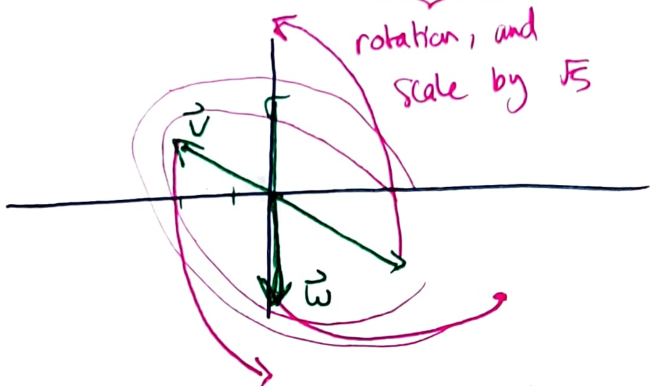
Eigenvector of $\lambda = +i\sqrt{5}$: $\ker \begin{bmatrix} 1-i\sqrt{5} & 2 \\ -3 & -1-i\sqrt{5} \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} -2 \\ 1-i\sqrt{5} \end{bmatrix} \right\}$

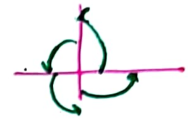
$\Rightarrow \quad \lambda = -i\sqrt{5}$ has eigenvector $\begin{bmatrix} -2 \\ 1+i\sqrt{5} \end{bmatrix}$

so $\quad \lambda = i\sqrt{5}$ has eigenvector $\begin{bmatrix} -2 \\ 1-i\sqrt{5} \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{v}} + i \underbrace{\begin{bmatrix} 0 \\ -\sqrt{5} \end{bmatrix}}_{\vec{w}}$

$= a+bi$ where $a=0$, $b=\sqrt{5}$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & -\sqrt{5} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -\sqrt{5} \\ +\sqrt{5} & 0 \end{bmatrix}}_{\text{rotation, and scale by } \sqrt{5}} \begin{bmatrix} -2 & 0 \\ 1 & -\sqrt{5} \end{bmatrix}^{-1}$$





exc Do the same for $A = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$

eigenvals: $\quad p_A(\lambda) = (3-\lambda)(-3-\lambda) + 10 = 0$

$\lambda^2 + 1 = 0 \quad \leadsto \lambda = \pm i \leadsto a=0,\ b=1$

eigenvector for $\lambda = i$: $\ker \begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 5 \\ 3-i \end{bmatrix} \right\}$

$\underbrace{\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{\vec{v}} + i \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{\vec{w}}$

$\leadsto A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}^{-1}$