<u>Recall</u>: every real $m \times n$ matrix A has a <u>singular value decomposition</u> (SVD):

$$\underbrace{A}_{m\times n} = \underbrace{U}_{\substack{\text{orthogonal} \\ m\times m}} \underbrace{\Sigma}_{\substack{m\times n \\ \text{"almost" diagonal}}} \underbrace{V^T}_{\substack{\text{orthogonal} \\ n\times n}}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix} \quad (n \le m)$$

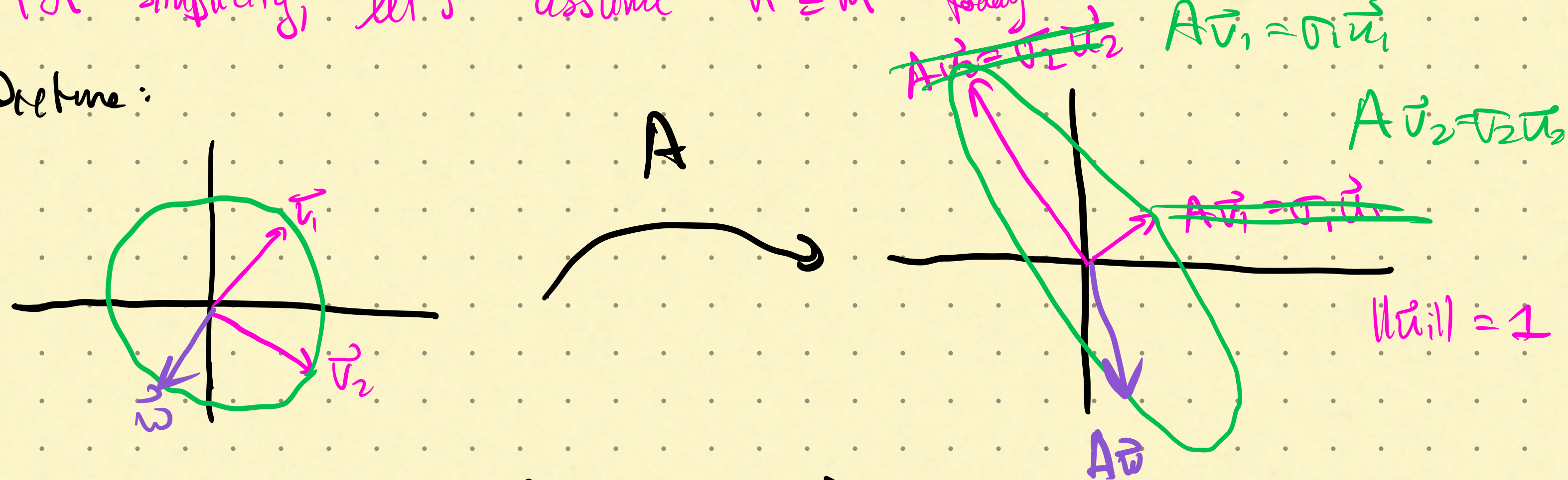
OR $$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 & \cdots & 0 \\ & \ddots & & & & \\ & & \sigma_m & 0 & \cdots & 0 \end{bmatrix} \quad (m \le n)$$

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge 0$$

"singular values"

For simplicity, let's assume $n \le m$ today

Picture:



Let's prove that $A\vec{v}_i = \sigma_i \vec{u}_i$:

$$A = U\Sigma V^T \rightsquigarrow AV = U\Sigma \underbrace{V^T V}_{= I_n}$$

$$\rightsquigarrow A[\vec{v}_1, \ldots, \vec{v}_n] = [\vec{u}_1 \cdots \vec{u}_m] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \end{bmatrix} = [\sigma_1\vec{u}_1, \ldots, \sigma_n\vec{u}_n]$$

$$\Rightarrow A\vec{v}_i = \sigma_i \vec{u}_i \text{ for all } i.$$

<u>Note</u>: from the picture, we can tell how A affects lengths. I.e. if $\|\vec{w}\| = 1$, $\sigma_2 \le \|A\vec{w}\| \le \sigma_1$

In general, if $\vec{w} \in \mathbb{R}^n$ with $\|\vec{w}\| = 1$, then $\sigma_n \le \|A\vec{w}\| \le \sigma_1$

Pf Given SVD $A = U\Sigma V^T$ $U = [\vec{u}_1, \dots, \vec{u}_m]$

$V = [\vec{v}_1, \dots, \vec{v}_n]$

Note: the $\vec{v}_i$ are a basis of $\mathbb{R}^n$ (V is invertible)

$\rightsquigarrow$ we can write $\vec{w} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

The fact that $\|\vec{w}\| = 1$ will put a restriction on the c_i:

$$1 = \|\vec{w}\| = \sqrt{\vec{w}\cdot\vec{w}} = \sqrt{(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)\cdot(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)}$$

$$= \sqrt{\underbrace{c_1\vec{v}_1 \cdot c_1\vec{v}_1}_{c_1^2 \cdot \underbrace{\vec{v}_1\cdot\vec{v}_1}_{1}} + \underbrace{c_1\vec{v}_1\cdot c_2\vec{v}_2}_{c_1c_2\cdot\underbrace{\vec{v}_1\cdot\vec{v}_2}_{0}} + \dots}$$

$\vec{v}_i$ are an ONB

$$= \sqrt{c_1^2 + c_2^2 + \dots + c_n^2}$$

OTOH, $A\vec{w} = c_1 A\vec{v}_1 + \dots + c_n A\vec{v}_n$

$$= c_1\sigma_1\vec{u}_1 + \dots + c_n\sigma_n\vec{u}_n$$

$$\|A\vec{w}\| = \sqrt{(c_1\sigma_1\vec{u}_1 + \dots + c_n\sigma_n\vec{u}_n)\cdot(c_1\sigma_1\vec{u}_1 + \dots + c_n\sigma_n\vec{u}_n)}$$

$\sigma_n \overset{?}{\le}$

$\vec{u}_i$ are an ONB

$$= \sqrt{c_1^2\sigma_1^2 + \dots + c_n^2\sigma_n^2} \qquad \le \sigma_1 ?$$

we know $\sigma_i^2 \le \sigma_1^2$

$$\Rightarrow \|A\vec{w}\| \le \sqrt{c_1^2\sigma_1^2 + c_2^2\sigma_1^2 + \dots + c_n^2\sigma_1^2} = \sigma_1\underbrace{\sqrt{c_1^2 + \dots + c_n^2}}_{1} = \sigma_1$$

Similarly, $\sigma_n \le \|A\vec{w}\|$.

<u>Computing SVD</u>: Given A, how do we actually find U, Σ, V?

Given $A = U \Sigma V^T \rightsquigarrow A^TA = V \Sigma^T \Sigma V^T$

$$= \underbrace{V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^T}_{\text{diagonalization of } A^TA!}$$

$\Rightarrow$ cols of V are eigenvectors of A^TA. Eigenvalues of A^TA are $\sigma_1^2, \dots, \sigma_n^2$

<u>Exc</u> Find SVD of $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ (or at least Σ, V).
(Feel free to use a calc. to find eigenvectors)

<u>Ans</u>: $A^TA = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$.

eigenvectors: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\lambda = 1$; $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\lambda = 16$

σ_i^2

$\sigma_1 = \sqrt{16} = 4$, $\sigma_2 = 1$

need V to be ortho!

$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

To find $\vec{u}_i$: $A\vec{v}_1 = \sigma_1 \vec{u}_1$

$$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4 \cdot \vec{u}_1$$

$$\begin{bmatrix} 8/\sqrt{5} \\ 4/\sqrt{5} \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Similarly, $A\vec{v}_2 = \sigma_2 \vec{u}_2 \Rightarrow \vec{u}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$