Agenda
- Review coordinates
- Start 7.1

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$

Recall

"$\mathcal{B}$-coordinates" of $\vec{x}$: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ means $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

exc

Let $\mathscr{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} \right\}$, and let $B = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$.

1. Find the $\mathscr{B}$-coordinates of $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

2. Which of the following formulas is true, for all $\vec{x} \in \mathbb{R}^2$?

$$\underline{B[\vec{x}]_{\mathscr{B}} = \vec{x}}, \quad \text{or} \quad B\vec{x} = [\vec{x}]_{\mathscr{B}} \quad ?$$

(Hint: consider $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. What's $[\vec{x}]_{\mathscr{B}}$? Which of the above formulas holds?)

1) $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ in $\mathcal{B}$ coords: $\begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\vec{v}_1 + 0\vec{v}_2 \rightsquigarrow \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ in $\mathcal{B}$-coords

$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ in $\mathcal{B}$ coords: $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = c_1\vec{v}_1 + c_2 \cdot \vec{v}_2 \rightsquigarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$B \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\mathcal{B}\text{-coords}} = \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\text{std coords}}$

RREF: $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$

2) $B[\vec{x}]_{\mathcal{B}} = \vec{x}$

$\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$B \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathcal{B}\text{ coords}} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{std coords}}$

$B[\vec{x}]_{\mathcal{B}} = \vec{x}, \qquad [\vec{x}]_{\mathcal{B}} = B^{-1}\vec{x}$

$\mathcal{B}$-matrix of a transformation:

let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a lin transform,

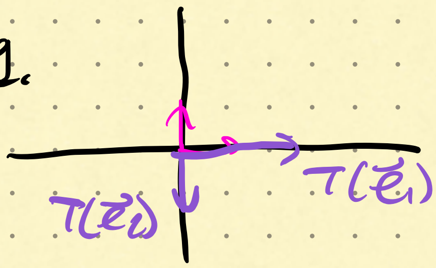
If A is "the matrix of T": $A\vec{x} = T(\vec{x})$

std coord

let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$, $B = [\vec{v}_1 \dots \vec{v}_n]$

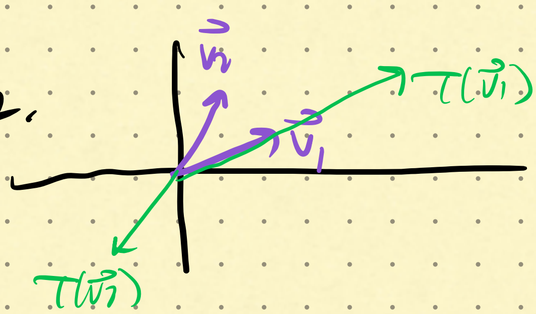
If M is the $\mathcal{B}$-matrix of T, then $M[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}$

eg.



The matrix of T is $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

eg.



The $\{\vec{v}_1, \vec{v}_2\}$-matrix of this transform is $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

eg. let T have the std matrix $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, what is the $\mathcal{B}$-matrix of T?

Compute: $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So the $\mathcal{B}$-matrix of T is $\left[[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \right]$

$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Alternatively: $M = B^{-1}AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

B-matrix of T

$B^{-1} A B [\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}$

convert back into B-coords

std matrix of T

convert to std coords

std matrix of T

Def Let M and N be two $n \times n$ matrices, we say M and N are <u>similar</u> if $M = S^{-1}NS$ for some S

<u>i.e.</u> M and N represent the same lin. transform but with respect to diff bases.

<u>§7.1</u> Eigenvectors

$M = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

$B^{-1}AB = M \iff A = BMB^{-1}$

<u>Theme</u>:
- Diagonal matrices are much nicer to work with than other matrices.
- We often want use a basis which makes our linear transforms into diagonal matrices

<u>eg.</u> $A^{1000000} = (BMB^{-1})^{1000000}$

$= (BM\cancel{B^{-1}})(\cancel{B}MB^{-1}) \cdots (BMB^{-1})$

$= BM^{1000,000}B^{-1} = B\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^{1000000} B^{-1}$

$= B\begin{bmatrix} 2^{1000000} & 0 \\ 0 & (-1)^{1000000} \end{bmatrix} B^{-1}$ (much easier) eg $\det(A) = \det(B^{-1}MB) = \underline{\det(B^{-1})}\ \det(M)\ \underline{\det(B)} = \det\begin{bmatrix}2&0\\0&-1\end{bmatrix}$

$\| $

$(5\cdot -4) - (-6)\cdot 3$ $\qquad$ is $\frac{1}{\det(B)}$ $\qquad = 2(-1)$

more complicated $\qquad$ simpler

Q How can we express a lin transform as a diag matrix? i.e. how can we find the basis in which T is a diag matrix?

Def We say that a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an <u>eigenbasis</u> of T if the $\mathcal{B}$-matrix of T is a diagonal matrix: $\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$

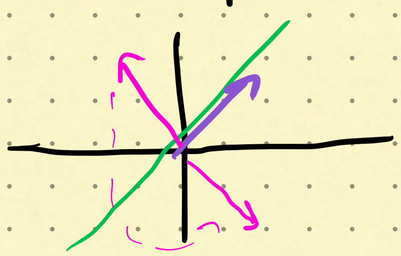
<u>Note</u>: $T(\vec{v}_1) = \lambda_1 \vec{v}_1, \dots, T(\vec{v}_n) = \lambda_n \vec{v}_n$ when this is the case.

Def We say a nonzero vect $\vec{v}$ is an <u>eigenvector</u> of T if $T(\vec{v}) = \lambda \cdot \vec{v}$ for some $\lambda \in \mathbb{R}$

<u>Note</u>: "eigen" means "characteristic" in German

eg. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$, reflection across $y=x$

eigenvectors of T?



- any nonzero vector in span $\begin{bmatrix}1\\1\end{bmatrix}$ works:

$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 1 \cdot \begin{bmatrix}1\\1\end{bmatrix}$

- any nonzero vector in span $\begin{bmatrix}-1\\1\end{bmatrix}$ is an eigenvector:

$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Note: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are a basis of $\mathbb{R}^2$, and eigenvectors of T.

So this is an eigenbasis for T.

Matrix of T w.r.t. this basis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$