11/7/22

Continuing 7.1: eigenvectors

Def let $T: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ be a lin. trans, with matrix A.

Then a vector $\vec{0} \neq \vec{v} \in \mathbb{R}^n$ is called an (real) eigenvector of T if

$T(\vec{v}) = \lambda \vec{v}$, for some $\lambda \in \mathbb{R}$.

λ is called the eigenvalue of $\vec{v}$

Def An eigenbasis of T is a basis of $\mathbb{R}^n$ consisting of eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ of T.

If T has an eigenbasis, then let $S = [\vec{v}_1, \dots, \vec{v}_n]$. Then

$$\underbrace{S^{-1}AS}_{\substack{\text{matrix of } T \\ \text{with respect to basis } \{\vec{v}_1, \dots, \vec{v}_n\}}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

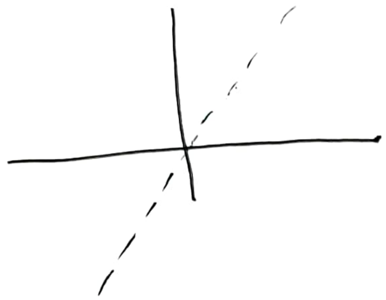
diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal.

We say the transformation T / the matrix A are "diagonalizable" in this case.

Exc let A be an invertible matrix, let $\vec{v}$ be an eigenvect of A w/ eigenval λ.

1. Is $\vec{v}$ an eigenvect of A^3? If so, what is the eigenval?
2. Is $\vec{v}$ an eigenvect of $A + 2I_n$? eigenval?
3. If $\vec{v}$ is an eigen vect of A and of B, is $\vec{v}$ an eigenvect of $A + B$?
4. Show: $\vec{w}$ is an eigenvect of A for all $\vec{0} \neq \vec{w} \in \text{Span}\{\vec{v}\}$.

Particular example for 1: last time: let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across $y=x$,



then $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of T

because $T(\vec{v}) = 1 \cdot \vec{v}$

eigenvalue 1.

$T^2(\vec{v}) = 1 \cdot \vec{v}$ $\quad$ $\vec{v}$ is still an eigen vector of T^2.

$T^3(\vec{v}) = 1 \cdot \vec{v}$ $\quad$ $\vec{v}$ is an eigenvector of T^3

#1, in general: given $A\vec{v} = \lambda \vec{v}$. Question: $A^3 \vec{v} = \underset{\text{const}}{(\ )}\ \vec{v}$

$$A^3\vec{v} = A^2 \cdot (A\vec{v}) = A^2 \lambda \vec{v} = \lambda A^2 \vec{v} = \lambda \cdot A \cdot \underbrace{A\vec{v}} = \lambda A \lambda v = \lambda^2 A \vec{v}$$
$$= \lambda^3 \vec{v}$$

#1: Yes, with eigenval of λ^3.

#2 $\quad (A + 2I_n)\vec{v} = \underbrace{A\vec{v}}_{\lambda\vec{v}} + \underbrace{2I_n \vec{v}}_{2\vec{v}} = \lambda\vec{v} + 2\vec{v} = \underset{\text{const.}}{(\lambda+2)}\vec{v}$

Yes, with eigenval $\lambda + 2$

#3. Set up: $A\vec{v} = \lambda\vec{v}$, $\quad B\vec{v} = \mu\vec{v}$ $\quad$ some const

$$(A+B)\vec{v} = A\vec{v} + B\vec{v}$$
$$= \lambda\vec{v} + \mu\vec{v} = (\lambda+\mu)\vec{v}$$

#4 $\quad \vec{w} = k\vec{v} \qquad A\vec{w} = Ak\vec{v} = kA\vec{v} = \underset{\lambda k \vec{v}}{k\lambda\vec{v}} = \underline{\ \lambda\ }\ \vec{w}$

$\vec{w}$ an eigvector ✓, eigenvalue? λ

Moral: if $\vec{v}$ is an eigenvector of A, w/ eigenval λ,

then everything in span $\{\vec{v}\}$ is also an eigenvect. w/ the same eigenval.

eg. If we say "the eigenvectors of T are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$"

we really mean, span $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and span $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$