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Warm-up: $A = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$.

Q Which of the following are eigenvectors of A?

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix} = (-1)\cdot\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ eigenvalue is -1

$\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

not eigenvector.

$\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = 6\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v}}$

eigenvalue of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

($\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvect)

Q (7.1 #68) find $A^{100}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We saw $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 6\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So $A^2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = A\cdot\left(6\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = 6^2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A^3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 6^3\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

etc.

$\rightsquigarrow A^{100}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 6^{100}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Similarly, $A^{100}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = (-1)^{100}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Q find $A^{100}\begin{bmatrix} 7 \\ 0 \end{bmatrix} = A^{100}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = A^{100}\begin{bmatrix} 5 \\ 2 \end{bmatrix} - 2\cdot A^{100}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= (-1)^{100}\begin{bmatrix} 5 \\ 2 \end{bmatrix} - 2\cdot 6^{100}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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§7.2 Q How can we find the eigenvectors of a given matrix?

Two steps:

- First, find the eigenvalues. (§7.2)
- Use this to find the corresponding eigenvector. (§7.3)

Let A be an $n \times n$ matrix. To find the eigenvals of A,

note:

λ is an eigenval of $A \iff \exists \vec{v}: A\vec{v} = \lambda\vec{v} = \lambda I_n \vec{v}$

$\iff \exists \vec{v}: (A - \lambda I_n)\vec{v} = \vec{0}$

So λ is an eigenval of A $\iff$ $\underbrace{(A - \lambda I_n)}_{\text{not invertible.}}$ has a nonzero kernel.

$\iff \det(A - \lambda I_n) = 0$

eg. What are the eigenvalues of $\begin{bmatrix} 6 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$?

Ans $\det\left(\begin{bmatrix} 6 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 6-\lambda & 1 & 4 \\ 0 & 1-\lambda & 2 \\ 0 & 3 & 2-\lambda \end{bmatrix}\right)$

$= (6-\lambda)\left((1-\lambda)(2-\lambda) - 2 \cdot 3\right) = 0$

$(6-\lambda)(\lambda^2 - 3\lambda - 4) = 0$

$(6-\lambda)(\lambda-4)(\lambda+1) = 0$

Eigenvalues of A are $6, 4, -1$.

Exc Find the eigenvalues of $\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

$\det\left(\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix}\right) = -\lambda(-1-\lambda) - 6 = 0$

$\Rightarrow \lambda^2 + \lambda - 6 = 0$

$\Rightarrow (\lambda+3)(\lambda-2) = 0$

$\lambda = -3, \quad \lambda = 2$

Note If A is an $n \times n$ matrix, then $\det(A - \lambda I_n)$ is always a degree-n polynomial in λ.
This is called the "characteristic polynomial" of A.
Its roots are the eigenvalues of A.

- let $p_A(\lambda)$ be the characteristic polynomial of A.

Then $p_A(0)$ = constant term of the polynomial

$$= \det\left[A - 0 \cdot I_n\right] = \det(A)$$

eg. $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$, $\quad p_A(\lambda) = \lambda^2 \underbrace{+\lambda}_{\text{coeff} = +1 = -\text{trace}(A)} \underbrace{-6}_{= \det(A)}$

trace$(A) = -1$

Def If A is a square matrix, then trace(A) = sum of diagonal entries of A.

Fact If A is a 2×2 matrix, then char poly of A is $p_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$

eg. $A = \begin{bmatrix} 6 & 4 \\ 20 & -7 \end{bmatrix}$. Then $p_A(\lambda) = \lambda^2 - (6-7)\lambda + (6 \cdot -7 - 4 \cdot 20)$

In general, If A is an $n \times n$ matrix, then

$$p_A(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \text{tr}(A) \lambda^{n-1} + \cdots + \det(A)$$

Returning to our Q: how do find the eigenvects of a matrix?

Recall λ is an eigenval of A $\iff \exists \underset{\uparrow}{\underline{\vec{v}}} : (A - \lambda I_n)\vec{v} = 0$

this is the eigenvector.

Once we have our eigenvalue λ, we can find the eigenvector(s) with this eigenvalue by computing $\ker(A - \lambda I_n)$

eg. $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$. We found $\lambda = -3$, $\lambda = 2$

To find eigenvecto corresp to $\lambda = -3$:

find $\ker\left(\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \ker\left(\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \right)$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \text{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

This tells us that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvect. w/ eigenval -3 $= \text{span} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Confirm: $\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.