1. Let $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ and let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be orthogonal projection onto the line containing $\vec{v}$.

   (a) Find the matrix for T.

   (b) Is the matrix for T invertible?

a) $\dfrac{1}{\vec{v}\cdot\vec{v}} \vec{v}\,\vec{v}^T = \dfrac{1}{4+9+0} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$

$= \dfrac{1}{13} \begin{bmatrix} 4 & 6 & 0 \\ 6 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) No. Many ways to see this:

- T sends several inputs to the same output
- $\ker(T)$ includes $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- RREF of matrix of T only has 2 pivots
- RREF of matrix of T has a row of all zeros.
- $\mathrm{im}(T) \neq \mathbb{R}^3$ (see p. 118 of textbook)

2. Consider the following matrix and its reduced row-echelon form:

$$M = \begin{bmatrix} 3 & 2 & 7 & 1 \\ 4 & 1 & 6 & 5 \\ -1 & 2 & 3 & 3 \end{bmatrix}, \quad RREF(M) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find a basis for $\ker M$

(b) Find three different bases for the image of M.

a) If $\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \in \ker(M)$, then $\begin{matrix} x_1 + x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_4 = 0 \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

Basis: $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ (or: $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, or $2 \cdot \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, etc)

b) $\left\{ \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$,

$\left\{ \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$ (RREF(M) tells us column 1 + 2·column 2 = column 3)

3. Let A be a 3×3 matrix with

$$A \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Find a basis for $\ker A$.

$A \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \Rightarrow A\left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right) = \vec{0}$

so $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \ker A$. Similarly, $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \ker A$,

Q Do $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ span $\ker(A)$?

$\mathrm{Im}(A)$ contains $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, so $\mathrm{im}(A) \neq 0$. Thus $\dim(\mathrm{im}(A)) \geq 1$.

By rank-nullity theorem, $\dim(\ker A) = \#\text{cols} - \dim(\mathrm{im}(A))$

So $\dim(\ker A) \leq 2$. So $\ker(A)$ is spanned by 2 vects or fewer. A Yes!

Question 2 (10 points)
Consider the chemical reaction

$$a\mathrm{NO_2} + b\mathrm{H_2O} \longrightarrow c\mathrm{HNO_2} + d\mathrm{HNO_3},$$

where a, b, c, and d are unknown positive integers. The reaction must be **balanced**; that is, the number of atoms of nitrogen (N), oxygen (O), and hydrogen (H) must be the same before and after the reaction. The term $b\mathrm{H_2O}$ refers to b water molecules, which consists of $2b$ hydrogen and b oxygen atoms. As customary, give the smallest possible positive integer solution.

(a) (4 points) Set up a system in the unknowns.

(b) (2 points) Label each equation with a unit. (What type of thing is being equated to what?)

(c) (4 points) Solve the system to balance the reaction.

System (including "=" and right-hand side)	Units
$a = c + d$	N atoms
$2a + b = 2c + 3d$	O atoms
$2b = c + d$	H atoms

Balanced reaction:

$a =$ 2	$b =$ 1	$c =$ 1	$d =$ 1

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 2 & 1 & -2 & -3 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$a = 2d$, $b = d$, $c = d$

Question 2 (11 points)

(a) (5 points) Determine if the vectors below are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

(b) (6 points) Let $\vec{w}$ be the vector below, and let $\vec{v}_1$ and $\vec{v}_3$ be as above. For which value(s) of b are the vectors $\vec{v}_1$, $\vec{w}$, and $\vec{v}_3$ linearly *dependent*?

$$\vec{w} = \begin{bmatrix} 1 \\ -1 \\ b \\ 2 \end{bmatrix}$$