Mostly from: Fall 2015 midterm 1

(4 points) Can you find a matrix A with 3 rows and 5 columns such that $A\vec{x} = \vec{0}$ has a unique solution? If so, write down such a matrix. If not, explain why not.

$\leadsto [\quad]^{3}_{5}$ columns must be lin dependent $\Rightarrow \ker A \neq \{0\}$.

$\begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{bmatrix}$

Alt: unique sol to $A\vec{x} = \vec{0}$ requires A to have no free variables

(4 points) Can you find a matrix A with 4 rows and 3 columns that satisfies $rank(A) = 2$ and $nullity(A) = 2$? If so, write down such a matrix. If not, explain why not.

rank = dim(im(A)) = # pivots in RREF(A) = max # lin indep cols of A

nullity = dim(ker(A)) = # free variables

rank(A) + nullity(A) = # columns $= 3$

Answer: not possible.

e.g. $A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ 5 × 5 rank = 0, nullity = 5: $A\vec{x} = 0$ for all $\vec{x} \in \mathbb{R}^5$

$\ker(A) = \mathbb{R}^5$

(4 points) Can you find a 2×2 matrix A satisfying $image(A) = ker(A)$? If so, write down such a matrix. If not, explain why not.

$\leadsto$ Yes. e.g. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has ker = im = span $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

e.g. $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ has ker = im = span $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(4 points) Can you find a matrix A with 3 rows and 4 columns such that $ker(A) = span \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. If so, write down such a matrix. If not, explain why not.

$\hookrightarrow$ Yes. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(4 points) Write down the matrix of any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that satisfies $T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$.

(4 points) Is the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix}$ invertible? Justify your answer with a computation

Here is a matrix M and its row reduction:

$$M = \begin{bmatrix} 2 & 2 & 8 & 16 & 28 \\ 2 & 3 & 9 & 18 & 37 \\ 3 & 3 & 12 & 24 & 42 \end{bmatrix} \qquad \operatorname{rref}(M) = \begin{bmatrix} 1 & 0 & 3 & 6 & 5 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

~~(a) (6 points) Compute a basis for the kernel of M.~~ ↑ 3rd col = 3 (1st col) + 1 · second col

Which of these are bases of im(M)?

a) $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ Yes! these corresp. to pivot cols in RREF(M).

b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ No! $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \in \operatorname{im} M$, but not in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

c) $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 9 \\ 12 \end{bmatrix}$ Yes! From RREF(M): $\begin{bmatrix} 8 \\ 9 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

So $\operatorname{span}\left(\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 9 \\ 12 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}\right) = \operatorname{im}(A)$

And these 2 vect. are lin indep.

d) $\begin{bmatrix} 8 \\ 9 \\ 12 \end{bmatrix}, \begin{bmatrix} 16 \\ 18 \\ 24 \end{bmatrix}$

Question 3 (24 points)

A stock fund is a bundle of stocks which can be purchased together at a fixed ratio. Purchasing 1 dollar of the Gauss stock fund will buy you 0.70 dollars of stock in software companies and 0.30 dollars of stock in automotive companies. Purchasing 1 dollar of the Cayley stock fund will buy you 0.40 dollars of software stocks and 0.60 dollars of automotive stocks.

(a) (5 points) Write down a 2×2 matrix A that maps the vector $\begin{pmatrix} \text{dollars of Gauss stock} \\ \text{dollars of Cayley stock} \end{pmatrix}$ to the vector $\begin{pmatrix} \text{dollars of software stock} \\ \text{dollars of automotive stock} \end{pmatrix}$.

(b) (6 points) If you desire 3 dollars of software stock and 3 dollars of automotive stock, but you can only buy the Gauss and Cayley funds, how much of each should you purchase?

a) $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ $\quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$ $\rightarrow$ $A = \begin{array}{cc} & \begin{array}{cc} G & C \end{array} \\ & \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{array}{c} S \\ A \end{array} \end{array}$

b) $A^{-1} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -4/3 \\ -1 & 7/3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Setup: $A \cdot \begin{bmatrix} G \\ C \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. What is $\begin{bmatrix} G \\ C \end{bmatrix}$?

Alt: $\left[\begin{array}{cc|c} 0.7 & 0.4 & 3 \\ 0.3 & 0.6 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \end{array}\right]$

E-mail: any questions to discuss on Monday!

(c) (5 points) After a year, automotive stock halves its value and software stock doubles its value. Write down the matrix which transforms the value of each of your stocks a year ago into their values now. Call this matrix B.

(d) (4 points) Write a formula in terms of B and A that gives the matrix which converts the vector $\begin{pmatrix} \text{dollars of Gauss stock bought a year ago} \\ \text{dollars of Cayley stock bought a year ago} \end{pmatrix}$ to $\begin{pmatrix} \text{dollars of software stock now} \\ \text{dollars of auto. stock now} \end{pmatrix}$? You need not evaluate this formula.

(e) (4 points) A year ago, you purchased g_1 dollars of Gauss stock and c_1 dollars of Cayley stock. Now your brother decides he wants to start investing and wants to hold the same amount of automotive stock and the same amount of software stock as you; let g_2 and c_2 be the amounts of Gauss and Cayley stock he needs to purchase. Write a formula for $\left(\begin{smallmatrix} g_2 \\ c_2 \end{smallmatrix}\right)$ in terms of $\left(\begin{smallmatrix} g_1 \\ c_1 \end{smallmatrix}\right)$, A and B. Once again, you need not evaluate this formula.

Question 2 (11 points)

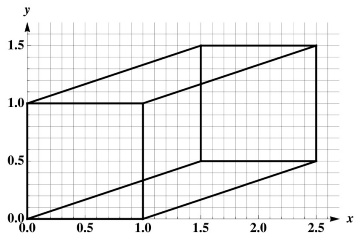
(a) (5 points) Determine if the vectors below are linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

(b) (6 points) Let $\vec{w}$ be the vector below, and let $\vec{v}_1$ and $\vec{v}_3$ be as above. For which value(s) of b are the vectors $\vec{v}_1$, $\vec{w}$, and $\vec{v}_3$ linearly *dependent*?

$$\vec{w} = \begin{bmatrix} 1 \\ -1 \\ b \\ 2 \end{bmatrix}$$

Question 4 (17 points)
We place a wire-frame cube one unit on each side on a table. We create a coordinate system with one corner of the cube at the origin, vertical edge of the cube along the positive z axis and the other two edges along the positive x and y axes. The cube's shadow is shown below.



(a) (5 points) Assuming that the map which sends a point to its shadow is linear, give the matrix of this map.

(b) (6 points) Give a formula parametrizing all points in $\mathbb{R}^3$ whose shadow is at $(4, 7)$.

(c) (6 points) We rotate the cube $45°$ counterclockwise around the z-axis (so that the positive x-axis is turned towards the positive y-axis). Where now is the shadow of the corner which, before the rotation, was at $(1, 1, 1)$?