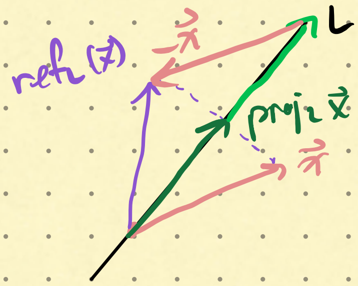
- A couple Qs about exam material,
- Start §5.1

Projection/ Reflection:

- projection onto the line containing $\vec{w}$ is given by the matrix $\frac{1}{\vec{w}\cdot\vec{w}} \vec{w}\,\vec{w}^T$. Works in $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n, \ldots$

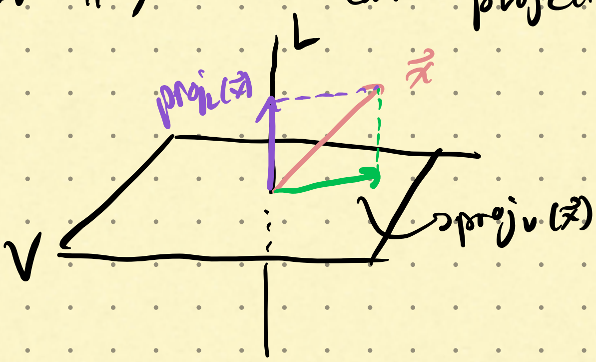
- Reflection across this line:



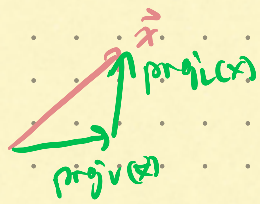
$2\,\text{proj}_L \vec{x} - \vec{x} = \text{ref}_L \vec{x}$

works in $\mathbb{R}^2, \mathbb{R}^3, \ldots$

In $\mathbb{R}^3$, we can project onto planes and reflect across planes.



$\text{proj}_V \vec{x} = ??$



$\vec{x} = \text{proj}_V \vec{x} + \text{proj}_L \vec{x}$

$\Longrightarrow \text{proj}_V \vec{x} = \vec{x} - \text{proj}_L \vec{x}$

Suppose we're given that V is the plane perp to $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

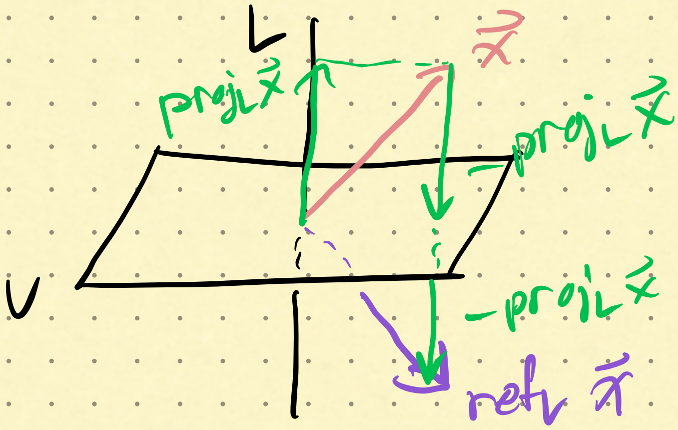
Then matrix for proj_V: if P is the matrix for proj_L,

then $\text{proj}_V(\vec{x}) = I_3 \vec{x} - P\vec{x} = (I_3 - P)\vec{x}$

Matrix for proj_V: $I_3 - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{1+4+1} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

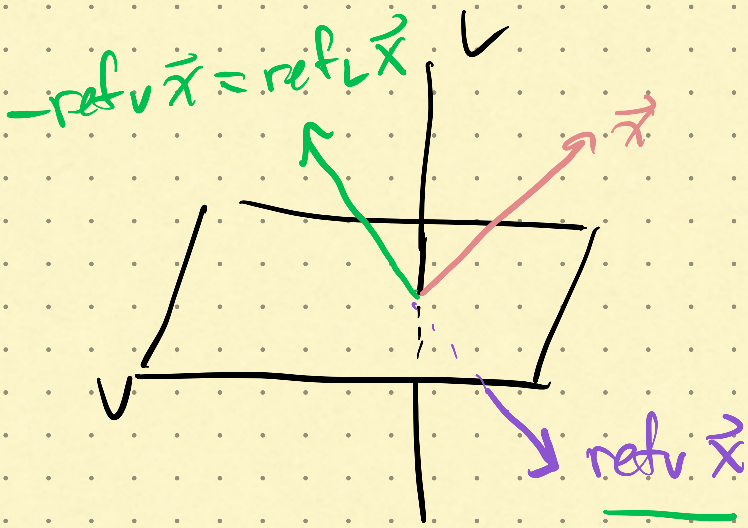
What about ref_V?



$\text{ref}_V\vec{x} = \vec{x} - 2\,\text{proj}_L\,\vec{x}$

In the above example, the matrix for this will be

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{6}\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-2&1\end{bmatrix}$$



Q Given some plane, how can we find a vector perpendicular to it?

eg. let V be the plane given by $2x - 3y + z = 0$

Recall: dot product. $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \cdot \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = v_1w_1 + v_2w_2 + v_3w_3$

Fact $\vec{v}\cdot\vec{w} = 0 \iff \vec{v}$ and $\vec{w}$ are perpendicular (or $\vec{v}=0$ or $\vec{w}=0$)

"$\vec{v}$ and $\vec{w}$ are orthogonal"

Note: $\begin{bmatrix}x\\y\\z\end{bmatrix} \in V \iff 2x-3y+z=0 \iff \begin{bmatrix}2\\-3\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$

$\iff \begin{bmatrix}2\\-3\\1\end{bmatrix}$ is perp. to $\begin{bmatrix}x\\y\\z\end{bmatrix}$

So the vector $\begin{bmatrix}2\\-3\\1\end{bmatrix}$ is perpendicular to every vector in V $\Rightarrow$ $\begin{bmatrix}2\\-3\\1\end{bmatrix}$ is perpendicular to the plane given by

$$2x-3y+z=0$$

Let $V=\operatorname{Span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}3\\0\\4\end{bmatrix}\right\}\subseteq\mathbb{R}^3$. Find a vector perpendicular to V.

A. "cross product" $\begin{bmatrix}1\\2\\-1\end{bmatrix}\times\begin{bmatrix}3\\0\\4\end{bmatrix}$

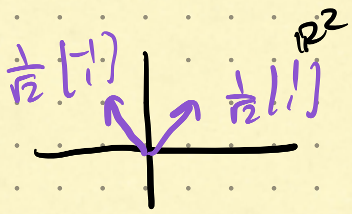
What about a plane $x-2y+z=4$?

This plane is <u>parallel</u> to the one given by $x-2y+z=0$

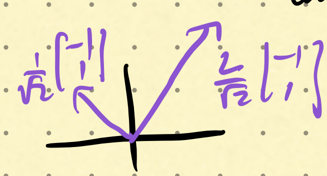
So $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is still perpendicular.

<u>Ch. 5</u> Orthogonal projection, orthonormal bases.

<u>Def</u> A set of vectors $\vec{u}_1,\dots,\vec{u}_d\in\mathbb{R}^n$ is called <u>orthonormal</u> if:
- each $\vec{u}_i$ has length 1
- each $\vec{u}_i$ is orthogonal to $\vec{u}_j$ whenever $i\neq j$

eg. 

these vectors are each length 1, and they're ortho to each other.

non-example: 

not orthonormal.

in $\mathbb{R}^3$:  e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ are orthonormal.

__Notice__: in first example: $\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\begin{bmatrix}-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}\right)$

$= \frac{1}{2}\cdot(-1\cdot 1 + 1\cdot 1) = 0$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{2}\cdot(1\cdot 1 + 1\cdot 1) = 1$$

__Recall__: for any vector $\vec{v}$, $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

$\|\vec{v}\|$ = "magnitude" of $\vec{v}$

= length of $\vec{v}$

In general, a set of vectors $\{\vec{u}_1, --, \vec{u}_d\}$ is orthonormal iff and only if:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

__Fact__ any orthonormal set of vectors is automatically lin. indep.

__Def__ a basis of a subspace V which is also orthonormal is called an "orthonormal basis" of V.

__eg__ Consider the vectors $\vec{u}_1 = \begin{bmatrix}2/\sqrt{6}\\1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$

in $\mathbb{R}^3$.

$\vec{u}_1\cdot\vec{u}_1 = 1$, $\vec{u}_2\cdot\vec{u}_2 = 1$, $\vec{u}_3\cdot\vec{u}_3 = 1$

$\vec{u}_1\cdot\vec{u}_2 = \vec{u}_2\cdot\vec{u}_3 = \vec{u}_1\cdot\vec{u}_3 = 0$) check!

$\Rightarrow$ $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are an orthonormal set of vectors.

$\Rightarrow$ they are lin independent. $\Rightarrow$ $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is an orthonormal basis of $\mathbb{R}^3$.

Continuing this example, let $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

Q What are the coords of $\vec{x}$ w.r.t. the basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$?

Old way: $\underbrace{\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}}_{\text{takes } \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \text{ to } \vec{u}_1}^{-1} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

New way: $[\vec{x}]_{\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \\ \vec{u}_3 \cdot \vec{x} \end{bmatrix}$

Why? We can write $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$ for some #s c_1, c_2, c_3.

We want to find c_1, c_2, c_3.

$$\vec{x} \cdot \vec{u}_1 = (c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3) \cdot \vec{u}_1 = c_1 \underbrace{\vec{u}_1 \cdot \vec{u}_1}_{1} + c_2 \underbrace{\vec{u}_2 \cdot \vec{u}_1}_{0} + c_3 \underbrace{\vec{u}_3 \cdot \vec{u}_1}_{0}$$

$$= c_1$$

Similarly, $\vec{x} \cdot \vec{u}_2 = c_2$, $\quad \vec{x} \cdot \vec{u}_3 = c_3$

In this case: $\vec{x} \cdot \vec{u}_1 = \frac{1}{\sqrt{6}}$, $\quad \vec{x} \cdot \vec{u}_2 = 3/\sqrt{2}$, $\quad \vec{x} \cdot \vec{u}_3 = 7/\sqrt{3}$

Check: $\vec{x} = \frac{1}{\sqrt{6}}\vec{u}_1 + 3/\sqrt{2}\,\vec{u}_2 + 7/\sqrt{3}\,\vec{u}_3$.