§5.2 - Agenda:
- Graham Schmidt algo
- QR factorization
- Application of ONB: Fourier series.

Ex: Find an ONB for span $\begin{bmatrix}4\\0\\0\\3\end{bmatrix}, \begin{bmatrix}5\\2\\14\\10\end{bmatrix}$ ($\vec{v}_1$, $\vec{v}_2$)

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$ $\qquad \|\vec{v}_1\| = \sqrt{4^2+0^2+0^2+3^2} = 5$

$= \frac{1}{5}\begin{bmatrix}4\\0\\0\\3\end{bmatrix}$

$\vec{u}_2 = ?$ First find $\vec{v}_2^{\perp} = \vec{v}_2 - \underbrace{(\vec{u}_1\cdot\vec{v}_2)\,\vec{u}_1}_{\text{proj}_{\vec{u}_1}\vec{v}_2}$, $\quad \vec{u}_1\cdot\vec{v}_2 = \frac{4}{5}\cdot 5 + 0\cdot 2 + 0\cdot 14 + \frac{3}{5}\cdot 10 = 10$

$\Rightarrow \vec{v}_2^{\perp} = \begin{bmatrix}5\\2\\14\\10\end{bmatrix} - 10\cdot\frac{1}{5}\begin{bmatrix}4\\0\\0\\3\end{bmatrix} = \begin{bmatrix}-3\\2\\14\\4\end{bmatrix}$ (ortho to $\vec{u}_1$)

$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp}$, $\qquad \|\vec{v}_2^{\perp}\| = \sqrt{3^2+2^2+14^2+4^2} = 15$

$= \frac{1}{15}\begin{bmatrix}-3\\2\\14\\4\end{bmatrix}$

Plane in $\mathbb{R}^4$

QR factorization: Let M be an $m \times n$ matrix with lin. indep columns, then a QR-factorization of M is an equation

$$M = Q \cdot R$$

$m \times n$ matrix, orthonormal columns

$n \times n$ matrix, "upper-triangular"

Def a square matrix A is "upper-triangular" if $a_{ij} = 0$ whenever $i > j$ (row > col)

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

So a QR factorization breaks down a kinda random matrix M into nicer matrices Q and R

Doing the Graham-Schmidt algo on the columns of M gives you the QR factorization.

eg. In the exercise from start of class, we know

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \quad \text{for some } r_{ij}$$

just because $\vec{v}_1, \vec{v}_2 \in \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$. $r_{ij} = ?$

$$\vec{v}_1 = r_{11}\vec{u}_1 + r_{21}\vec{u}_2$$
$$\vec{v}_2 = r_{12}\vec{u}_1 + r_{22}\vec{u}_2$$

Eqs from Graham-Schmidt algo:

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 \rightarrow \vec{v}_1 = \|\vec{v}_1\| \cdot \vec{u}_1 + 0 \cdot \vec{u}_2$$

$$\rightarrow \begin{bmatrix} r_{11} \\ r_{21} \end{bmatrix} = \begin{bmatrix} \|\vec{v}_1\| \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} \rightsquigarrow \vec{v}_2^{\perp} = \|\vec{v}_2^{\perp}\| \cdot \vec{u}_2$$

$$\rightsquigarrow \vec{v}_2 = \|\vec{v}_2^{\perp}\|\vec{u}_2 + (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

$$= 15\vec{u}_2 + 10\vec{u}_1$$ (we already computed these)

$$\Rightarrow \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$\Rightarrow [\vec{v}_1 \ \vec{v}_2] = [\vec{u}_1 \ \vec{u}_2] \cdot \begin{bmatrix} 5 & 10 \\ 0 & 15 \end{bmatrix}$$ upper triangular!

In general: R is given by: $r_{ij} = \vec{u}_i \cdot \vec{v}_j$

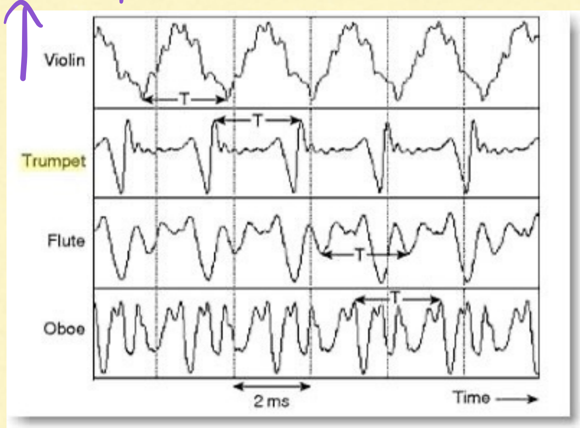
<u>Note</u>: • This $u_i \cdot v_j = 0$ when $i > j$

• Check: $\|v_i^{\perp}\| = u_i \cdot v_i$

Application: Fourier Series (§5.5)

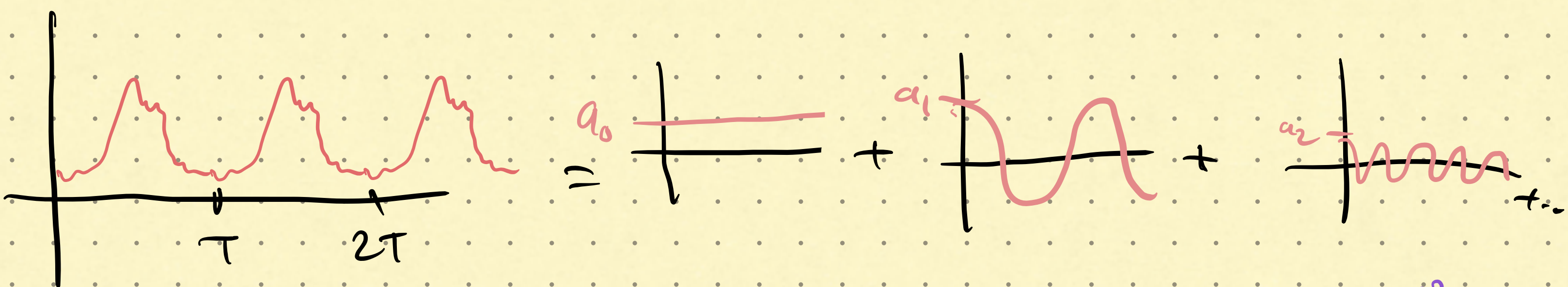
Sounds can be modeled by periodic functions

air pressure



Problem: these shapes are v. complicated!

Want to break them up in terms of simpler building blocks: sin and cos



$f(t)$ $= a_0 \cos\left(\frac{2\pi \cdot 0}{T} t\right) + \underbrace{a_1 \cos\left(\frac{2\pi \cdot 1}{T} t\right)}_{\text{period } T} + a_2 \cos\left(\frac{2\pi \cdot 2}{T} t\right)$

period $T/2$

Useful for:
- data compression
- Machine learning
- Audio processing.

Question: What are $a_0, a_1, a_2, \dots$?

→ Fourier series

Fact any (reasonable) function $f(t)$ w/ period 1 can be written as a combination of simple sin/cos waves

i.e. $$f(x) = \sum_{k=0}^{\infty} a_k \cos(2\pi k x) + b_k \sin(2\pi k x)$$

This is secretly a form of orthogonal projection!

Fact:

$$\int_0^1 \cos(2\pi m x)\cos(2\pi n x)\,dx = \begin{cases} 1/2, & \text{if } m=n \\ 0, & \text{if } m \neq n \end{cases}$$

(n, m integers)

Similarly for sin.

Also, $$\int_0^1 \cos(2\pi m x)\sin(2\pi n x)\,dx = 0$$

Thus:

choose some m

$$\int_0^1 f(x)\cos(2\pi m x)\,dx = \sum_{k=0}^{\infty} a_k \underbrace{\boxed{\int_0^1 \cos(2\pi k x)\cos(2\pi m x)\,dx}}_{=\frac{1}{2} \text{ when } k=m, \ 0 \text{ otherwise}} + \cancel{b_k \int_0^1 \sin(2\pi k x)\cos(2\pi m x)\,dx}_{\,0}$$

$$= a_m/2$$

Compare to formula: $$[\vec{v}]_{\mathcal{Q}} = \begin{bmatrix} \vec{v}\cdot\vec{u}_1 \\ \vdots \\ \vec{v}\cdot\vec{u}_n \end{bmatrix}$$

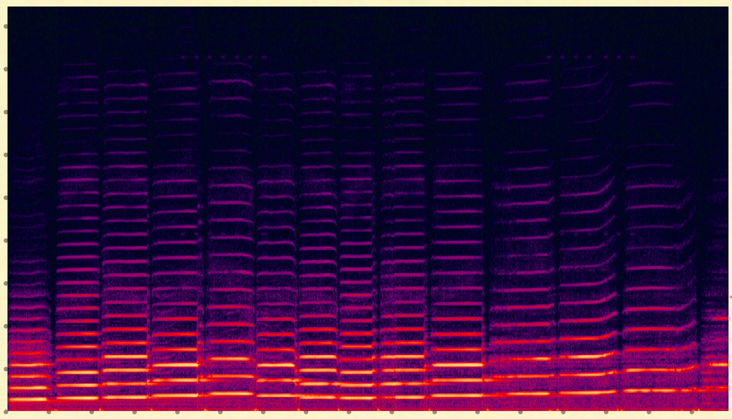
$\vec{u}_i \longleftrightarrow$ sin cos   dot product $\longleftrightarrow$ integral

So: to find a_k, b_k: just compute some integrals

$$a_k = 2\int_0^1 f(t)\cos(2\pi k t)\,dt,$$

$$b_k = 2\int_0^1 f(t)\sin(2\pi k t)\,dt$$

The a_k and b_k can be visualized in a "spectogram"



Violin playing

color = how large a_i, b_i

harmonics

fund. freq

i

time