__last time:__

if $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$, then $A^T = \begin{bmatrix} — & \vec{v}_1^T & — \\ — & \vec{v}_2^T & — \\ & \vdots & \\ — & \vec{v}_n^T & — \end{bmatrix}$

$$A^T \vec{w} = \begin{bmatrix} \vec{v}_1 \cdot \vec{w} \\ \vec{v}_2 \cdot \vec{w} \\ \vdots \\ \vec{v}_n \cdot \vec{w} \end{bmatrix}$$

If $Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}$ is an $n \times n$ matrix, where $\vec{u}_1, \ldots, \vec{u}_n$ is an ONB:

$$Q^T Q = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_1 & \cdots & \vec{u}_n \cdot \vec{u}_1 \\ \vec{u}_1 \cdot \vec{u}_2 & \vec{u}_2 \cdot \vec{u}_2 & \cdots & \vec{u}_n \cdot \vec{u}_2 \\ \vdots & & & \\ \vec{u}_1 \cdot \vec{u}_n & \vec{u}_2 \cdot \vec{u}_n & \cdots & \vec{u}_n \cdot \vec{u}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= I_n$$

__Def__ a $n \times n$ matrix M is __orthogonal__ if $M^T = M^{-1}$

$$\Rightarrow M^T M = M M^T = I_n$$

$\Leftrightarrow$ equivalently if the columns of M are an ONB of $\mathbb{R}^n$

eg orthonormal columns but square: $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$M^T M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M \cdot M^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$$

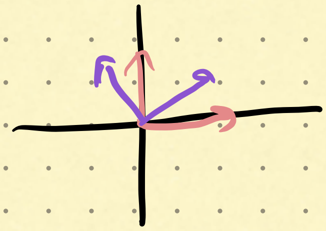
§ Orthogonal transformations

A linear transform $T: \mathbb{R}^n \to \mathbb{R}^n$ is called <u>orthogonal</u> if $\|T(\vec{v})\| = \|\vec{v}\|$ for all $\vec{v}$, and

angle between $T(\vec{v})$ and $T(\vec{w})$ = angle between $\vec{v}$ and $\vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$

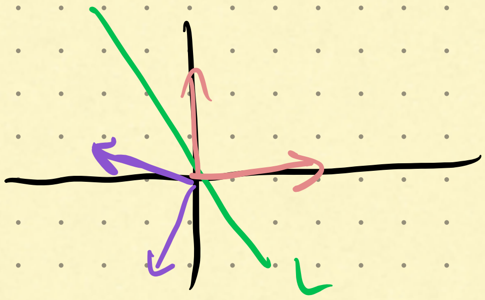
<u>eg.</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by θ:

matrix: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



det: $\cos\theta \cdot \cos\theta - (-\sin\theta \cdot \sin\theta) = \cos^2\theta + \sin^2\theta = 1$

<u>eg.</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$, reflection across a line



<u>matrix</u>: $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2+b^2=1$

det: $a \cdot (-a) - b \cdot b = -a^2 - b^2 = -1$

<u>FACT</u>: any orthogonal transformation is either a rotation, in which case det = 1

<u>OR</u> a reflection followed by rotation, in which case det = −1

<u>Fact</u>: A transformation T with matrix M is orthogonal $\Longleftrightarrow$ the matrix M is an orthogonal matrix

in other words, a matrix M satisfies $M^T = M^{-1}$

$\Longrightarrow$ the transformation $T(\vec{x}) = M\vec{x}$ preserves angles and lengths

5. Let

$$\vec{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix},$$

and let $V \subseteq \mathbb{R}^4$ be the subspace spanned by $\vec{u}_1$ and $\vec{u}_2$.

(a) Let $\vec{w} = \cancel{[2\ \ 3\ \ 4]}^T$. Compute $\text{proj}_V(\vec{w})$,

$[2\ 3\ 4\ 0]^T$

$\vec{u}_1, \vec{u}_2$ are orthonormal!

$\Rightarrow \text{proj}_V(\vec{w}) = (\vec{u}_1 \cdot \vec{w})\vec{u}_1 + (\vec{u}_2 \cdot \vec{w})\vec{u}_2$

$$= \begin{bmatrix} 7/4 \\ 11/4 \\ 9/4 \\ 9/4 \end{bmatrix}$$

(b) Let $Q = [\vec{u}_1\ \vec{u}_2]$ be the matrix whose columns are $\vec{u}_1$ and $\vec{u}_2$. Compute $QQ^T\vec{w}$. What do you notice?

(b) $$QQ^T\vec{w} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} — \vec{u}_1^T — \\ — \vec{u}_2^T — \end{bmatrix} \vec{w}$$

$$= \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \vec{u}_1 \cdot \vec{w} \\ \vec{u}_2 \cdot \vec{w} \end{bmatrix} = (\vec{u}_1 \cdot \vec{w})\vec{u}_1 + (\vec{u}_2 \cdot \vec{w})\vec{u}_2$$

Thm: if columns of Q are orthonormal, then the matrix for projecting onto $\text{im}\, Q$ is QQ^T

ex let $V = \text{span}\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Find the matrix for projection onto V.

$\vec{v}_1 \quad \vec{v}_2$

Not starting with an orthonormal set.

Run Gram-Schmidt:

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{v}_2^\perp = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \vec{v}_2 - \vec{u}_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^\perp\|}\vec{v}_2^\perp = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Now we can apply formula:

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$